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A MODULUS FOR THE 3-DIMENSIONAL WAVE EQUATION WITH NOISE: DEALING WITH A SINGULAR KERNEL

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ABSTRACT We give a modulus of continuity for solutions of the wave equation with a noise term

$$u_{tt} = \Delta u + a(u) + b(u)G, \quad x \in \mathbb{R}^3$$

where G is a Gaussian noise This case is more difficult than in lower dimensions because the fundamental solution of the wave equation is singular

1. **Introduction.** We give a modulus of continuity for solutions $u(t,x)$ to the wave equation with a noise term, for $x \in \mathbb{R}^3$.

$$(1.1) \qquad \begin{aligned} u_{tt} &= \Delta u + a(u) + b(u)G, \quad t \geq 0,\ x \in \mathbb{R}^3 \\ u(0,x) &= u_0(x) \\ u_t(0,x) &= u_1(x). \end{aligned}$$

We assume that $u_0(x), u_1(x)$ are C^∞ functions. Here, $G = G(t,x)$ is a generalized Gaussian field with covariance $E[G(t,x)G(s,y)] = \delta(t-s)R(|x-y|)$. We impose the following conditions on $R(x)$.

$$(1.2) \qquad \begin{aligned} |R(x) - R(y)| &\leq c|x-y| \\ |R(x)| &\leq c, \end{aligned}$$

for some constant c. Throughout the article, c will be a constant which may vary from line to line. As an example, $G(t,x)$ could be $B(t)$, that is, white noise in t. We further assume that $a(u), b(u)$ are Lipschitz functions:

$$(1.3) \qquad \begin{aligned} |a(u) - a(v)| &\leq c|u-v| \\ |b(u) - b(v)| &\leq c|u-v|. \end{aligned}$$

Also, assume

$$\begin{aligned} |a(u)| &\leq c(|u|+1) \\ |b(u)| &\leq c(|u|+1). \end{aligned}$$

We find that $u(t,x)$ is almost Holder continuous of order $\frac{1}{2}$. Let $\beta = \beta\big((t,x),(s,y)\big) = \Big|\log|(t,x)-(s,y)|\Big|^{\frac{1}{2}}$.

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THEOREM 1. *There exists a null set Ω_0 contained in the probability space, such that for each compact subset $R \subset \{(t,x) \in \mathbb{R}^4 : t \geq 0\}$, and for each $\omega \notin \Omega_0$, we may choose $c > 0$ such that for all $(t,x), (s,y) \in R$, we have*

$$|u(t,x) - u(s,y)| \leq c|(t,x) - (s,y)|^{\frac{1}{2}} \max(t,x)^{\beta}.$$

The study of moduli of continuity has played an important role in several recent papers on stochastic partial differential equations. Sometimes a probability estimate on the modulus can substitute for classical maximal inequalities for processes, which may be hard to prove in the multiparameter setting. The heat equation with noise was studied by the following authors. Walsh (1984) first derived a modulus. The modulus was used to study large deviations, limiting shape, support of solutions, and long-time existence, respectively, by Sowers (1990a,b), Mueller (1990a,b,c). For $x \in \mathbb{R}^d$, $d \leq 2$, Mueller (1991) used the modulus for the wave equation to show long-time existence. Finally, Carmona and Nualart (1988a,b) are also devoted to the wave equation with a nonlinear noise term.

Existence and uniqueness for all times t, for (1.1), follow from arguments similar to those given in Walsh (1984), Chapter 3. Our original hope was to combine the results of this paper and of Mueller (1991) to prove existence and uniqueness when $b(u)$ grows faster than linearly. The author believes he can prove such a theorem for $|b(u)| \leq c + c|u| \log\log(|u| + 3)^{\alpha}$ for the appropriate α, but such a weak result appears to have little interest.

J. Walsh and the author believe that both (1.1) and $u_{tt} = u_{xx} + b(u)\dot{W}(t,x)$ ($x \in \mathbb{R}$, $\dot{W}$ is 2-parameter white noise) have solutions which blow up, if $b(u) = |u|^{\alpha}$, $\alpha > 1$. But this is not proven.

The method in all of the above papers was based on the following idea. Re-express the differential equation as an integral equation of the form

$$u(t,x) = T(t,x) + \int_0^t \int_{\mathbb{R}^d} K(t-s, x-y) b\big(u(s,y)\big) G(dy\, ds) \tag{1.4}$$

where K is the fundamental solution of the heat or wave equation, and $T(t,x)$ represents unimportant terms. Then,

$$\begin{aligned} u(t_1,x_1) - u(t_2,x_2) &\\ = T(t,x_1) &- T(t_2,x_2) \\ &+ \int_0^{t_1 \vee t_2} \int_{\mathbb{R}^d} [K(t_1 - s, x_1 - y) - K(t_2 - s, x_2 - y)] b\big(u(s,y)\big) G(dy\, ds) \end{aligned}$$

where we assume $K(t,x) = 0$ if $t < 0$. Then, if K is smooth, $u(t_1,x_1) - u(t_2,x_2)$ is controlled by $K(t_1 - s, x_1 - y) - K(t_2 - s, x_2 - y)$. This technique breaks down if the fundamental solution $K(t,x)$ is not smooth. The wave equation

$$\bar{u}_{tt} = \Delta \bar{u}, \quad x \in \mathbb{R}^3$$

has fundamental solution

$$S(t,x) = \frac{1}{4\pi t}\delta(t - |x|)$$

which is certainly not smooth.

The goal of this article is to give a new technique for deriving moduli, which works for (1.1). We again use the integral equation (1.4), but we express $u(t_1,x_1) - u(t_2,x_2)$ in terms of an integral of differences of u. A Gronwall-type argument finishes the proof.

Now we discuss the integral equation for $u(t,x)$. Indeed, we do not expect $u(t,x)$ to be differentiable, so (1.1) has no meaning. If G were smooth, (1.1) would be equivalent to the following integral equation. Since u is not smooth, we regard (1.1) as a shorthand for the integral equation.

Let $\bar{u}(t,x)$ satisfy

$$\begin{aligned}\bar{u}_{tt} &= \Delta\bar{u} \quad t \geq 0,\ x \in \mathbb{R}^3\\ \bar{u}(0,x) &= u_0(x)\\ \bar{u}_t(0,x) &= u_1(x).\end{aligned}$$

Let $u(t,x)$ satisfy

$$\begin{aligned}u(t,x) = \bar{u}(t,x) &+ \int_0^t \int_{\mathbb{R}^3} S(t-s,x-y)a\big(u(s,y)\big)\,dy\,ds\\ &+ \int_0^t \int_{\mathbb{R}^3} S(t-s,x-y)b\big(u(s,y)\big)G(dy\,ds).\end{aligned} \tag{1.6}$$

We interpret the second integral in (1.6) in terms of Walsh's (1984) theory of martingale measures. The reader can check that G is a martingale measure with nuclear covariance.

Define the light cone $C(t)$ by

$$C(t) = \big\{(s,x) \in \mathbb{R}^4 : 0 \leq s \leq t, |x| \leq t-s\big\}. \tag{1.7}$$

Our strategy is to show on each light cone $C(t)$, that $u(t,x)$ must be bounded and must satisfy the inequality appearing in Theorem 1.

Finally, the author wishes to thank J. Walsh and E. Perkins for their hospitality during the author's visit to Vancouver, and J. Walsh for his discussions of the wave equation with noise.

2. **Proof of Theorem 1.** Our first task is to stop the solution $u(t,x)$ when it becomes too irregular. We allow $u(t,x)$ to become more irregular as (t,x) gets further from the origin. Let $t_n = \sum_{k=\bar{n}}^{n} \frac{1}{k\log k}$, $\bar{n} \geq 3$, and let $\sigma_n = \sigma_n(\bar{n})$ be the first time $t \leq t_n$ that there exist $(s_1,x_1),(s_2,x_2) \in C(t)$ with $s_1 \leq s_2$ and either $|u(s_1,x_1)| > 2^n(s_1+2)^{\log n}$ or

$$\big|u(s_1,x_1) - u(s_2,x_2)\big| \geq 2^{n-1}\big|(s_1,x_1)-(s_2,x_2)\big|^{\frac{1}{2}}(s_2+2)^{|\log|(s_1,x_1)-(s_2,x_2)||^{\frac{1}{2}}}.$$

If there is no such t, let $\sigma_n = \infty$. Finally, let $\sigma = \sigma(\bar{n}) = \inf_{n\geq\bar{n}}\sigma_n(\bar{n})$.

Recall from the introduction that

$$\beta = \beta(s_1,x_1,s_2,x_2) = \Big|\log\big|(s_1,x_1)-(s_2,x_2)\big|\Big|^{\frac{1}{2}}. \tag{2.1}$$

Let $v(t,x)$ satisfy the equation

$$v_{tt}(t,x) = \Delta v(t,x) + a\big(u(t\wedge\sigma,x)\big) + b\big(u(t\wedge\sigma,x)\big)G(t,x)$$
$$v(0,x) = u_0(x) \tag{2.2}$$
$$v_t(0,x) = u_1(x)$$

for $t > 0$, $x \in \mathbb{R}^3$. This equation can be made rigorous via an integral equation similar to (1.6).

Let

$$N(t,x) = \int_0^t \int_{\mathbb{R}^3} S(t-s, x-y) b\big(u(s\wedge\sigma, y)\big) G(dy\,ds).$$

This is the noise term in the integral equation corresponding to (2.2).

Borrowing an idea from Kolmogorov's criterion for the continuity of processes and also from Walsh (1984), Corollary 3.4, we consider differences $N(t,x) - N(s,y)$ over adjacent dyadic rationals $(t,x),(s,y)$. This means that the entries of (t,x) and (s,y) have the form $k2^{-m}$ for some $m \geq 1$, and that $|(t,x)-(s,y)| = 2^{-m}$. We will use this idea to prove the following lemma.

Let A_n be the event that for all $(s_1,x_1),(s_2,x_2) \in C(t_n,0), s_1 \leq s_2$, and $|(s_1,x_1) - (s_2,x_2)| \leq \frac{1}{n}$, we have

$$|N(s_1,x_1) - N(s_2,x_2)| \leq \frac{1}{3}2^n |(s_1,x_1)-(s_2,x_2)|^{\frac{1}{2}}(s_2+2)^\beta,$$

where β was defined in (2.1).

LEMMA 2.1.

$$P\{A_n^c \mid A_{n-1}\cap\cdots\cap A_{\tilde{n}}\} \leq c_0 \exp[-c(\log n)^{\frac{3}{2}}(\log\log n + 2)^{-3}].$$

PROOF OF LEMMA 2.1. We let the reader check, using standard arguments related to Kolmogorov's criterion for continuity, (as in Walsh (1984), Corollary 1.2), that $N(t,x)$ has a continuous version. Thus, to decide whether A_n occurs, we need only consider points (s_i,x_i) with dyadic rational entries, that is $(s_i,x_i) = (k_1 2^{-m}, k_2 2^{-m}, k_3 2^{-m}, k_4 2^{-m})$ for some integer m.

Also, since we are conditioning on $A_{n-1}\cap\cdots\cap A_{\tilde{n}}$, we need only consider the case where one of the points $(s_1,x_1),(s_2,x_2)$ lies in $C(t_n,0)\setminus C(t_{n-1},0)$. Let $\mathcal{S}(t_n,t_{n-1},m)$ be the set of dyadic rationals $(k_1 2^{-m},\ldots,k_4 2^{-m})$ either lying in $\mathcal{C}(t_n,0)\setminus\mathcal{C}(t_{n-1},0)$ or lying in $\mathcal{C}(t_{n-1},0)$ and having a nearest neighbor in $\mathcal{C}(t_n,0)\setminus\mathcal{C}(t_{n-1},0)$. It is easy to see that for some c not depending on t_n, t_{n-1}, m, we have

$$\begin{aligned}|\mathcal{S}(t_n,t_{n-1},m)| &\leq c2^{4m}\text{volume}\,[\mathcal{C}(t_n,0)\setminus\mathcal{C}(t_{n-1},0)]\\ &\leq c\frac{1}{n}(\log n)^3 2^{4m}\end{aligned} \tag{2.3}$$

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where $|\mathcal{S}|$ is the number of points in $\mathcal{S}$. Let $E_{n,m}$ be the event that for all pairs (s_1, x_1), (s_2, x_2) which are nearest neighbors in $\mathcal{S}(t_n, t_{n-1}, m)$, with $s_1 \leq s_2$, we have

$$|N(s_1, x_1) - N(s_2, x_2)| \leq cK2^{n-1}2^{-m}(s_2+2)^{\beta}.$$

Let $E_n = \bigcap_{2^{m} \leq \frac{1}{n}} E_{n,m}$.

We claim that

$$P(E_{n,m}^c) \leq c2^{4m}\frac{1}{n}(\log\log n)^3 \frac{(\log\log n+2)^{\frac{3}{2}}}{m^{\frac{3}{4}}} \exp[-cm^{\frac{3}{2}}(\log\log n+2)^{-3}] \tag{2.4}$$

and thus

$$\begin{aligned} P(E_n^c) &\leq \sum_{m=[\log_2 n]}^{\infty} P(E_{n,m}^c) \\ &\leq c\exp[-c(\log n)^{\frac{3}{2}}(\log\log n+2)^{-3}]. \end{aligned} \tag{2.5}$$

Now we establish (2.4), by means of the following lemma. Note that

$$M(\hat{t}) \equiv \int_0^{\hat{t}} \int_{\mathbb{R}^3} S(t-s, x-y) b\big(u(s\wedge\sigma, y)\big) G(dy\, ds)$$

is a martingale, and that $M(t) = N(t,x)$. By an abuse of notation, we write $\langle N(t,x)\rangle$ for $\langle M\rangle_{\hat{t}}\big|_{\hat{t}=t}$. The effect is to hold the t in $S(t-s, x-y)$ constant when computing the square variation. We use the same notation for differences $\langle N(t,x) - N(s,y)\rangle$.

LEMMA 2.2. *If* $(t, x_1), (t, x_2), (t, x) \in \mathcal{C}(t_n, 0)$ *and* $s < t$, *and if* $|x_1 - x_2| \leq \frac{1}{n}$, $t - s \leq \frac{1}{n}$, *then*

$$\langle N(t,x_1) - N(t,x_2)\rangle \leq c2^{2n}|x_1 - x_2| \frac{(t+2)^{2\beta+3}}{\beta^3} \tag{A}$$

$$\langle N(t,x) - N(s,x)\rangle \leq c2^{2n}(t-s) \frac{(t+2)^{2\beta+3}}{\beta^3} \tag{B}$$

PROOF OF LEMMA 2.2, (A). We use the definition of $N(t,x)$ to split up $\langle N(t,x_1) - N(t,x_2)\rangle$ as follows. Without loss, we assume that $x_1 = x, x_2 = -x$, so that $|x_1 - x_2| = 2|x|$.

$$\begin{aligned} \langle N(t,x) - N(t,-x)\rangle \leq{}& 2\Big\langle \int_0^t \int_{\mathbb{R}^3} S(t-s,y)\big[b\big(u(s\wedge\sigma, x-y)\big) - b\big(u(s\wedge\sigma, -x-y)\big)\big] \\ & G\big(d(x-y)\, ds\big)\Big\rangle \\ & + 2\Big\langle \int_0^t \int_{\mathbb{R}^3} S(t-s,y) b\big(u(s\wedge\sigma, -x-y)\big)\big[G\big(d(x-y)\, ds\big) \\ & - G\big(d(-x-y) ds\big)\big]\Big\rangle \\ ={}& (I) + (II). \end{aligned}$$

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First, consider term (I). By the stochastic calculus of martingale measures, as given in Walsh (1984), we may determine (I) by squaring the integral, and using the rule

$$G\big(d(x-y_1)\,ds_1\big)G\big(d(x-y_2)\,ds_2\big) = \delta(s_1-s_2)R(y_1-y_2)\,dy_1\,dy_2\,ds_1.$$

Thus we find, using a constant c which may vary from line to line

$$\begin{aligned}(I) &= 2\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} S(t-s,y_1)S(t-s,y_2)R(y_1-y_2)\\ &\qquad \Big[b\big(u(s\wedge\sigma,x-y_1)\big)-b\big(u(s\wedge\sigma,-x-y_1)\big)\Big]\\ &\qquad \Big[b\big(u(s\wedge\sigma,x-y_2)\big)-b\big(u(s\wedge\sigma,-x-y_2)\big)\Big]\,dy_2\,dy_1\,ds\\ &\le c|x|\int_0^t(s+2)^{2\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}S(t-s,y_1)S(t-s,y_2)\,dy_1\,dy_2\\ &\le c|x|\int_0^t(s+2)^{2\beta}(t-s)^2\,ds.\end{aligned}$$

Now

$$\begin{aligned}\int_0^t(s+2)^{2\beta}(t-s)^2\,ds &= \int_0^t(s+2)^{2\beta}[(t+2)-(s+2)]^2\,ds\\ &= \int_0^t[(s+2)^{2\beta+2}-2(s+2)^{2\beta+1}(t+2)+(s+2)^{2\beta}(t+2)^2]\,ds\\ &\le (t+2)^{2\beta+3}\Big[\frac{1}{2\beta+3}-\frac{2}{2\beta+2}+\frac{1}{2\beta+1}\Big]\\ &= (t+2)^{2\beta+3}\frac{(4\beta^2+6\beta+2)-2(4\beta^2+8\beta+3)+(4\beta^2+10\beta+6)}{(2\beta+3)(2\beta+2)(2\beta+1)}\\ &\le \frac{c(t+2)^{2\beta+3}}{\beta^3}.\end{aligned}$$

Thus,

$$(I) \le c|x|\frac{(t+2)^{2\beta+3}}{\beta^3}.$$

Here we have used the fact that $|R(y_1-y_2)|\le 1$ and that $|b(x)-b(y)|\le c|x-y|$.

Using similar ideas, we deal with (II). In particular, using the covariance of G,

$$\begin{aligned}(II) &= 2\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} S(t-s,y_1)S(t-s,y_2)b\big(u(s\wedge\sigma,-x-y_1)\big)b\big(u(s\wedge\sigma,-x-y_2)\big)\\ &\qquad 2[R(y_1-y_2)-R(y_1-y_2-2x)]\,dy_1\,dy_2\,ds.\end{aligned}$$

Thus, using condition (1.2) and the fact that

$$\Big|b\big(u(s\wedge\sigma,-x-y)\big)\Big| \le c2^n(s+2)^{\sqrt{\log n}} \quad \text{(by 1.3)}$$

we find that

$$\begin{aligned}(II) &\le c2^{2n}x\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}S(t-s,y_1)S(t-s,y_2)(s+1)^{2\sqrt{\log n}}\,dy_2\,dy_1\,ds\\ &\le c2^{2n}xt^2\int_0^t(s+2)^{2\sqrt{\log n}}\,ds\\ &\le c2^{2n}xt^2\frac{(t+2)^{2\sqrt{\log n}+1}}{2\sqrt{\log n}+1}\\ &\le c2^{2n}x\frac{(t+2)^{2\beta+3}}{\beta}.\end{aligned}$$

Here we have used the fact that since $|(t,x_1)-(t,x_2)| \leq \frac{1}{n}$, we have $\sqrt{\log n} \leq \big|\log|(t,x_1)-(t,x_2)|\big|^{\frac{1}{2}} = \beta$. Also, if $t \geq 0$ and $x > 2$, then $\frac{(t+2)^x}{x}$ is nondecreasing in x, since

$$\frac{d}{dx}\frac{(t+2)^x}{x} = \frac{(t+2)^x}{x^2}[x\log(t+2)-1] \geq 0.$$

PROOF OF LEMMA 2.2, PART (B). For ease of notation, we give the proof for $x=0$. Then

$$\begin{aligned}\langle N(t,0)-N(s,0)\rangle &\leq 2\Big\langle \int_s^t \int_{\mathbb{R}^3} S(t-r,y)b\big(u(r\wedge\sigma,y)\big)G(dy\,ds)\Big\rangle \\ &\quad + 2\Big\langle \int_0^s [S(t-r,y)-S(s-r,y)]b\big(u(r\wedge\sigma,y)\big)G(dy\,dr)\Big\rangle \\ &= (III)+(IV).\end{aligned}$$

Using the same techniques as for (I) and (II), we find

$$\begin{aligned}(III) &\leq c2^{2n}\int_s^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} S(t-r,y_1)S(t-r,y_2)(r+2)^{2\sqrt{\log n}}R(y_1-y_2)\,dy_2\,dy_1 \\ &\leq c2^{2n}(t-s)^2\int_s^t (r+2)^{2\sqrt{\log n}}\,dr \\ &\leq c2^{2n}(t-s)^2\frac{(t+2)^{2\beta+3}}{\beta}.\end{aligned}$$

Here we have used $R(y_1-y_2)\leq 1$.

Term (IV) is a bit more complicated. First we give a lemma. Here, $\delta(y)$ means the vector of length δ pointing in the same direction as y.

LEMMA 2.3. *If*

$$\bar{S}(t,\delta,y) = S(t+\delta,y) - S\big(t,y+\delta(y)\big)\left[\frac{|y|}{|y|+\delta}\right]^2$$

then $\int_{\mathbb{R}^3}|\bar{S}(t,\delta,y)|\,dy = \delta$.

PROOF. Recall that $S(t,y)$ is a generalized function corresponding to uniform measure of mass t on the sphere of radius t about 0. Projecting this measure radially onto the sphere of radius $t+\delta$, we must include the expansion factor $[\frac{|y|}{|y|+\delta}]^2$. Thus, $\int_{\mathbb{R}^3}|\bar{S}|$ is just the difference in mass between $S(t+\delta,\cdot)$ and $S(t,\cdot)$, which is $(t+\delta)-t=\delta$.

Now we deal with term (IV), splitting it up as we did with term (A). Let $\Delta = t-s$. We use some of the same methods as for the terms (I) and (II).

$$\begin{aligned}(IV) &\leq 4\Big\langle \int_0^s \int_{\mathbb{R}^3} \bar{S}(s-r,\Delta,y)b\big(u(r\wedge\sigma,y)\big)G(dy\,dr)\Big\rangle \\ &\quad + 8\Big\langle \int_0^s \int_{\mathbb{R}^3} S(s-r,y)\Big[b\big(u(r\wedge\sigma,y)\big) - b\Big(u\big(r\wedge\sigma,y+\delta(y)\big)\Big)\Big]G(dy\,dr)\Big\rangle \\ &\quad + 8\Big\langle \int_0^s \int_{\mathbb{R}^3} S(s-r,y)b\Big(u\big(r\wedge\sigma,y+\delta(y)\big)\Big)\Big[G(dy\,dr) - G\Big(d\big(y+\delta(y)\big)dr\Big)\Big]\Big\rangle \\ &= (IVa)+(IVb)+(IVc).\end{aligned}$$

Arguing as before, we have (using $R(x) \leq 1$)

$$(IV\,a) \leq c2^{2n} \int_0^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \bar{S}(s-r,\Delta,y_1)\bar{S}(s-r,\Delta,y_2)(r+2)^{2\sqrt{\log n}}\,dy_2\,dy_1\,dr$$
$$\leq c2^{2n}\Delta\frac{(s+2)^{2\beta+3}}{\beta}.$$

Arguing as for term (I), we have, since $\Delta = \delta(y)$,

$$(IV\,b) \leq c2^{2n}\Delta\frac{(s+2)^{2\beta+3}}{\beta}.$$

Term (IV c) is similar to term (II).

$$(IV\,c) \leq c2^{2n}\Delta\int_0^S (s+2)^{2\sqrt{\log n}}$$
$$\leq c2^{2n}\Delta\frac{(s+2)^{2\beta+3}}{\beta}.$$

Putting all these together and choosing a different constant c, we get Lemma 2.2 (B).

Now we finish the proof of inequality (2.4), using Lemma 2.2. To get a probability estimate, we can regard $N(t,x_1)-N(t,x_2)$ as a time-changed Brownian motion, with time scale $\langle N(t,x_1)-N(t,x_2)\rangle$ bounded by $T_1 = c2^{2n}|x_1-x_2|\frac{(t+2)^{2\beta+3}}{\beta}$.

Therefore, using the reflection principle for Brownian motion, and standard estimates for the normal,

$$(2.6)\qquad \begin{aligned} P\{|N(t,x_1)-N(t,x_2)| > R\} &\leq P\Big\{\sup_{t\leq T_1}|B_t| > R\Big\} \\ &\leq 4P\{B_{T_1} > R\} \\ &\leq c\frac{T_1^{\frac{1}{2}}}{R}\exp\Big[-\frac{R^2}{2T_1}\Big]. \end{aligned}$$

Using the same reasoning, with $T_2 = c2^{2n}(t-s)\frac{(t+2)^{2\beta+3}}{\beta}$, we have

$$(2.7)\qquad P\{|N(t,x)-N(s,x)| > R\} \leq c\frac{T_2^{\frac{1}{2}}}{R}\exp\Big[-\frac{R^2}{2T_2}\Big].$$

We can use estimates (2.6) and (2.7) to bound $P(E^c_{n,m})$, since there are at most $c2^{4m}\frac{1}{n}(\log\log n)^3$ differences of the above form involved in the event $E_{n,m}$. Such differences have $|x_1-x_2| = 2^{-m}$ or $(t-s) = 2^{-m}$. Thus, if $t = t_n \approx \log\log n$ and

$$T_3 = c2^{2n}2^{-m}\frac{(\log\log n+2)^{2\beta+3}}{\beta^3},$$
$$R = \frac{1}{3}2^n2^{-\frac{m}{2}}(\log\log n+2)^{\beta}$$

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and K is large, we have, recalling $\beta = \sqrt{-\log(t-s)} = \sqrt{m}$ or $\beta = \sqrt{-\log|x_1 - x_2|} = \sqrt{m}$, that

$$P\{E^c_{n,m}\} \leq c2^{4m}\frac{1}{n}(\log\log n)^3 \frac{T_3^{\frac{1}{2}}}{R}\exp\Big[-\frac{R^2}{2T_3}\Big]$$
$$\leq c2^{4m}\frac{1}{n}(\log\log n)^3 \frac{(\log\log n+2)^{\frac{3}{2}}}{m^{\frac{3}{4}}}\exp[-cm^{\frac{3}{2}}(\log\log n+2)^{-3}].$$

This verifies (2.4). Summing from $m = [\log_2 n]$ to ∞, and making some obvious estimates, we arrive at inequality (2.5).

Finally, we use inequality (2.5) to prove Lemma 2.1. We claim that, for the right choice of c in the definition of $E_{n,m}$, that

$$P(A^c_n|A_{n-1}\cap\cdots\cap A_{\bar{n}}) \leq P(E^c_n) + P(E^c_{n-1}). \tag{2.8}$$

In light of (2.5), we see that (2.8) would imply Lemma 2.1. To estimate $P(A^c_n \mid A_{n-1}\cap\cdots\cap A_{\bar{n}})$, we must consider differences $N(s_1,x_1) - N(s_2,x_2)$ such that $|(s_1,x_1)-(s_2,x_2)| \leq \frac{1}{n}$, and both points lie in $\mathcal{C}(t_n,0)$. Because of the conditioning, the difference of the N's is well controlled if both points (s_1,x_1) and (s_2,x_2) lie in $\mathcal{C}(t_{n-1},0)$. Since the distance between points is less than or equal to $\frac{1}{n}$, we need only consider points in $\mathcal{C}(t_n,0)\setminus\mathcal{C}(t_{n-2},0)$. Finally, using the continuity of $N(t,x)$, we focus on the points $(s_1,x_1),(s_1,x_2) \in \mathcal{C}(t_n,0)\setminus \mathcal{C}(t_{n-2},0)$, $s_1 \leq s_2$, whose coordinates are dyadic rationales.

We claim that, connecting (s_1,x_1) and (s_2,x_2), there is a path $(p_0,p_1,\ldots,p_N)$ with $p_0 = (s_1,x_1)$, $p_N = (s_2,x_2)$ such that

(2.9)
- (a) All points p_ι lie in $\mathcal{C}(t_n,0)\setminus\mathcal{C}(t_{n-1},0)$.
- (b) At most eight of the steps $|p_\iota - p_{\iota+1}|$ have the same length.
- (c) Each step $p_\iota,p_{\iota+1}$ satisfies, for some m, that $|p_\iota - p_{\iota+1}| = 2^{-m}$ and the components of the vectors $p_\iota,p_{\iota+1}$ are dyadic rationales of the form $\frac{k}{2^m}$.
- (d) All steps satisfy
$$|p_\iota - p_{\iota+1}| \leq |(s_1,x_1)-(s_2,x_2)|$$
- (e) The s-coordinate of p_ι is nondecreasing with i.

Note that if (2.9) is satisfied, and the events E_n and E_{n-1} occur, (and the c in the definition of $E_{n,m}$ is small enough) then we may sum the differences $|N(p_\iota)-N(p_{\iota+1})|$ to conclude

$$|N(s_1,x_1)-N(s_2,x_2)| \leq \frac{1}{3}2^n|(s_1,x_1)-(s_2,x_2)|^{\frac{1}{2}}(s_2+2)^\beta.$$

Thus, to show (2.8) and hence prove Lemma 2.1, we need to verify (2.9). We explain how to choose such a path, satisfying (2.9) (b)–(e) in one dimension, and let the reader check the 3-dimensional case. Start with $s_1 < s_2$. Let $\big(k2^{-m},(k+1)2^{-m}\big)$ be the largest dyadic interval contained in (s_1,s_2). Then use the binary expansions of s_1 and s_2 to choose the remaining dyadic intervals $(p_\iota,p_{\iota+1})$ on either side of the first interval. This path satisfies (b)–(e).

Thus, Lemma 2.1 is proved, and we can control differences $|N(s_1,x_1) - N(s_2,x_2)|$. Next, let

$$A(t,x) = \int_0^t \int_{\mathbb{R}^3} S(t-s,x-y)a\big(u(s\wedge\sigma,y)\big)\,dy\,ds. \tag{2.10}$$

Our goal is to control $|A(s_1,x_1) - A(s_2,x_2)|$. Assume that $(t,x_1),(t,x_2) \in \mathcal{C}(t_n,0)$, and $t_n < T$ for some $T > 0$.

Observe that

(2.11)

$$\begin{aligned}
|A(t,x_1) - A(t,x_2)| &\leq \int_0^t \int_{\mathbb{R}^3} S(t-s,x-y)c|u(s\wedge\sigma,x_1-y) - u(s\wedge\sigma,x_2-y)|\,dy\,ds \\
&\leq c\int_0^t (t-s)2^n|x_1-x_2|^{\frac{1}{2}}(s+2)^{\beta(t,x_1,t,x_2)} \\
&\leq c2^n|x_1-x_2|^{\frac{1}{2}}\frac{(t+2)^{\beta+2}}{\beta} \\
&\leq \frac{1}{6}2^n|x_1-x_2|^{\frac{1}{2}}(t+2)^{\beta}
\end{aligned}$$

if $\bar{n}$, and hence β are large enough. Note that the lower bound for $\bar{n}$ will depend on T.

Next, we deal with $|A(t,x) - A(s,x)|$ for $s \leq t \leq T$. There are some similarities to the proof of Lemma 2.2 part B. For simplicity, we let $x = 0$. Assume that $(t,0) \in \mathcal{C}(t_n,0)$, and $t - s \leq \frac{1}{n}$. We have

$$\begin{aligned}
|A(t,0) - A(s,0)| &\leq \int_s^t \int_{\mathbb{R}^3} S(t-r,y)\Big|a\big(u(r\wedge\sigma,y)\big)\Big|\,dy\,dr \\
&\quad + \left|\int_0^s [S(t-r,y) - S(s-r,y)]a\big(u(r\wedge\sigma,y)\big)\,dy\,dr\right| \\
&= (I) + (II).
\end{aligned}$$

Because of the bound $|u(r\wedge\sigma,y)| \leq c2^n(r+2)^{\sqrt{\log n}}$ we have

$$\begin{aligned}
(I) &\leq \int_s^t (t-r)c2^n(r+2)^{\sqrt{\log n}}\,dr \\
&\leq c(t-s)^{\frac{1}{2}}2^n\frac{(t+2)^{\beta+1}}{\beta} \\
&\leq \frac{1}{12}2^n(t-s)(t+2)^{\beta}
\end{aligned}$$

if $\bar{n}$ is large enough as in the previous case. Using Lemma 2.3, we find,

$$\begin{aligned}
(II) &\leq \int_0^s \int_{\mathbb{R}^3} \bar{S}(s-r,t-s,y)\Big|a\big(u(r\wedge\sigma,y)\big)\Big|\,dy\,dr \\
&\quad + \int_0^s \int_{\mathbb{R}^3} S(s-r,y)\Big|a\big(u(r\wedge\sigma,y)\big) - a\Big(u\big(r\wedge\sigma,y+\delta(y)\big)\Big)\Big|\,dy\,dr \\
&\leq c2^{n-1}\int_0^s (t-s)(r+2)^{\sqrt{\log n}}dr + c2^n\int_0^s (s-r)(t-s)^{\frac{1}{2}}(r+2)^{\sqrt{\log n}}dr \\
&\leq c2^n(t-s)^{\frac{1}{2}}\frac{(t+2)^{\beta+2}}{\beta} \\
&\leq \frac{1}{12}2^n(t-s)^{\frac{1}{2}}(t+2)^{\beta}
\end{aligned}$$

if $\bar{n}$ is large enough, as before. Thus we have shown, (in the case $x = 0$),

$$|A(t,x) - A(s,x)| \le \frac{1}{6}2^n(t-s)^{\frac{1}{2}}(t+2)^\beta. \tag{2.12}$$

From (2.11) and (2.12) we conclude, with the usual restrictions on (t,x), (s,y), and n, that

$$|A(t,x) - A(s,y)| \le \frac{1}{3}2^n|(t,x)-(s,y)|^{\frac{1}{2}}(t+2)^\beta. \tag{2.13}$$

Finally, we are ready to finish the proof of Theorem 1. We first cite a theorem about the deterministic wave equation.

LEMMA 2.4 (SEE TREVES (1975), THEOREM 13.2.). *Let u_0, u_1 be C^∞ functions. If $\bar{u}(t,x)$ is the solution of*

$$\bar{u}_{tt} = \Delta\bar{u}$$
$$\bar{u}(0,x) = u_0(x)$$
$$\bar{u}_t(0,x) = u_1(x)$$

then $\bar{u}(t,x)$ is a C^∞ function of (t,x).

Since the assumptions of Lemma 2.4 are satisfies, $\bar{u}(t,x)$ is a C^∞ function, and hence a bounded Lipschitz function on $\mathcal{C}(T,0)$ for each $T > 0$.

The following lemma will finish the proof of Theorem 1.

LEMMA 2.5. *Fix $T > 0$. Given $\epsilon > 0$, there exists $\bar{n} > 0$ such that $P\{\sigma(\bar{n}) \le T\} < \epsilon$.*

Suppose Lemma 2.5 is valid. The event that the solution $u(t,x)$ exists for all (t,x) is contained in the event $\bigcap_{T=1}^\infty \bigcup_{K=1}^\infty \{\sigma(K) > T\}$. Thus, Lemma 2.5 implies Theorem 1.

PROOF OF LEMMA 2.5. Rewriting (2.2), we have

$$v(t,x) = \bar{u}(t,x) + A(t,x) + N(t,x). \tag{2.14}$$

A sufficient condition for $\sigma > T$ is that for all $(t,x), (s,y) \in \mathcal{C}(t_n, 0)$, $s < t$, and for all $n \ge \bar{n}$ such that $t_n \le T+1$, we have

(i)

$$|\bar{u}(t,x)| \le \frac{1}{3}2^n(t+2)^{\sqrt{\log n}} \tag{2.15}$$
$$|A(t,x)| \le \frac{1}{3}2^n(t+2)^{\sqrt{\log n}}$$
$$|N(t,x)| \le \frac{1}{3}2^n(t+2)^{\sqrt{\log n}}$$

and (ii)

$$|\bar{u}(t,x) - \bar{u}(s,y)| \le \frac{1}{3}2^n|(t,x)-(s,y)|^{\frac{1}{2}}(t+2)^\beta$$
$$|A(t,x) - A(s,y)| \le \frac{1}{3}2^n|(t,x)-(s,y)|^{\frac{1}{2}}(t+2)^\beta$$
$$|N(t,x) - N(s,y)| \le \frac{1}{3}2^n|(t,x)-(s,y)|^{\frac{1}{2}}(t+2)^\beta$$

provided

$$|(t,x)-(s,y)| \leq \frac{1}{n}.$$

Let $F(T,\bar{n})$ be the event specified by (2.15). Suppose that $\bigcap_{n=\bar{n}}^{\infty} A_n$ occurs. Then by (2.12) and Lemma 2.4, (2.15) (ii) holds provided $\bar{n}$ is large enough.

We claim that $\bar{n}$ is large enough, and (2.15) (ii) holds, then (2.15) (i) holds. Certainly the bound on $|\bar{u}(t,x)|$ holds if $n \geq \bar{n}$ is large enough. Assuming (2.15) (ii), we will prove the bound on $|N(t,x)|$ in (2.15) (i). The bound on $|A(t,x)|$ is similar. Suppose $(t,x) \in \mathcal{C}(t_n)$. We claim there is a path $(p_{\bar{n}-1}, p_{\bar{n}}, \ldots, p_n)$ such that $p_k \in \mathcal{C}(t_k)$, $p_{\bar{n}-1}$ is on the surface $\{t=0\}$, the t coordinates of the p_k are nondecreasing, and $|p_k - p_{k-1}| \leq \frac{1}{k}$. We leave this construction to the readers, reminding them that the width of $\mathcal{C}(t_k) \setminus \mathcal{C}(t_{k-1})$ is $\frac{1}{k}$. Thus

$$\begin{aligned} |N(t,x)| &\leq \sum_{k=\bar{n}}^{n} |N(p_k) - N(p_{k-1})| \\ &\leq \sum_{k=\bar{n}}^{n} \frac{1}{3} 2^k \frac{1}{k} (t+2)^{\beta} \\ &\leq \frac{1}{3} 2^n (t+2)^{\beta}. \end{aligned}$$

Thus, to prove Lemma 2.5, we need only show that $P\{F(T,\bar{n})^c\} < \epsilon$ for $\bar{n}$ sufficiently large. But, by the above comments and by Lemma 2.1, we have

$$\begin{aligned} P\{F(T,\bar{n})^c\} &\leq P\Big\{ \Big(\bigcap_{n=\bar{n}}^{\infty} A_n \Big)^c \Big\} \\ &= \sum_{n=\bar{n}}^{\infty} P\{A_n^c \mid A_{n-1} \cap \cdots \cap A_{\bar{n}}\} \\ &\leq c_0 \sum_{n=\bar{n}}^{\infty} \exp[-c(\log n)^{\frac{3}{2}} (\log\log n)^{-3}] \\ &< \epsilon \end{aligned}$$

if $\bar{n}$ is large enough.

This proves Lemma 2.5, and finishes the proof of Theorem 1.

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