Canad. Math. Bull. Vol. **54** (1), 2011 pp. 147–158
doi:10.4153/CMB-2010-090-x

Generalized Quandle Polynomials

Sam Nelson

Abstract. We define a family of generalizations of the two-variable quandle polynomial. These polynomial invariants generalize in a natural way to eight-variable polynomial invariants of finite biquandles. We use these polynomials to define a family of link invariants that further generalize the quandle counting invariant.

1 Introduction

A two-variable polynomial invariant of finite quandles, denoted $qp_Q(s,t)$, was introduced in [11]. This invariant was shown to distinguish all non-Latin quandles of order 5 and lower. A slight modification gives an invariant of subquandles embedded in larger quandles that is capable of distinguishing isomorphic subquandles embedded in different ways. This subquandle polynomial was used to augment the quandle counting invariant $|\mathrm{Hom}(Q(L),T)|$ to obtain a multiset-valued invariant that can distinguish knots and links with the same quandle counting invariant value.

In this paper we generalize the quandle polynomial in two ways. In Section 2 we define a family of two-variable polynomial invariants of finite quandles indexed by pairs of integers, denoted $qp_{m,n}(Q)$, such that $qp_Q(s,t) = qp_{1,1}(Q)$. In Section 3 we extend these generalized quandle polynomials in a natural way to obtain a family of eight-variable polynomial invariants of finite biquandles indexed by pairs of integers, denoted $bp_{m,n}(B)$. In Section 4 we define and give examples of link invariants defined using generalized quandle polynomials. In Section 5 we collect a few questions for future research.

2 The (m,n) Quandle Polynomial

We begin with a definition from [7].

Definition 2.1 A *quandle* is a set Q with a binary operation $\triangleright\colon Q\times Q \to Q$ satisfying

(i) for every $a \in Q$, $a \triangleright a = a$,
(ii) for every $a, b \in Q$, there is a unique $c \in Q$ such that $a = c \triangleright b$, and
(iii) for every $a, b, c \in Q$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

Axiom (ii) is equivalent to

(ii′) there is a second operation $\triangleright^{-1}\colon Q \times Q \to Q$ such that

$$(a \triangleright b) \triangleright^{-1} b = a = (a \triangleright^{-1} b) \triangleright b.$$

Received by the editors March 25, 2008; revised April 30, 2008.
Published electronically August 9, 2010.
AMS subject classification: **57M27**, 76D99.
Keywords: finite quandles, finite biquandles, link invariants.

As a useful abbreviation, let us denote the n-times repeated quandle operation as

$$x \triangleright^n y = (\cdots(x \triangleright y) \triangleright y \cdots) \triangleright y$$

and

$$x \triangleright^{-n} y = (\cdots(x \triangleright^{-1} y) \triangleright^{-1} y \cdots) \triangleright^{-1} y$$

where, as expected, n is the number of triangles. Note that $x \triangleright^0 y = x$ for all x, y.

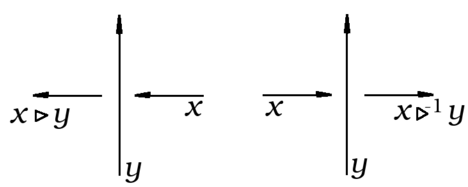
A standard example of a quandle is any group G, which has quandle structures given by $g \triangleright h = h^{-n} g h^n$ for $n \in \mathbb{Z}$ and $g \triangleright h = t(gh^{-1})h$ for any $t \in \mathrm{Aut}(G)$, $g, h \in G$. The special case of the latter where G is abelian is called an *Alexander quandle* and may be regarded as a module over $\mathbb{Z}[t^{\pm 1}]$ by thinking of $t \in \mathrm{Aut}(G)$ as a formal variable. In additive notation we have $x \triangleright y = tx + (1-t)y$.

Another standard example of a quandle structure is any module V over a commutative ring R with an antisymmetric bilinear form[1] $\langle \cdot, \cdot \rangle \colon V \times V \to R$ with $\mathbf{x} \triangleright \mathbf{y} = \mathbf{x} + \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}$. When R is a field and $\langle \cdot, \cdot \rangle$ is nondegenerate, V is a symplectic vector space, so this type of quandle is called a *symplectic quandle*. If $\langle \cdot, \cdot \rangle$ is instead a symmetric bilinear form, then the subset $S = \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{x} \rangle \neq 0\} \subset V$ is a quandle under

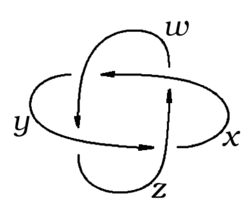
$$\mathbf{x} \triangleright \mathbf{y} = 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}\mathbf{y} - \mathbf{x}$$

called a *Coxeter quandle*.

Another important example is the *knot quandle* $Q(L)$ defined in [7], which associates a quandle generator to every arc in a link diagram L and a relation at every crossing. The elements of a knot quandle are equivalence classes of quandle words in the generators modulo the equivalence relation generated by the quandle axioms and the relations imposed by the crossings.



Example 2.2 The two-component link pictured below has knot quandle given by the listed presentation.



$$Q(L) = \langle x, y, z, w \mid x \triangleright z = y, z \triangleright y = w, y \triangleright w = x, w \triangleright x = z \rangle.$$

[1] If the characteristic of R is 2, then we require that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$.

https://doi.org/10.4153/CMB-2010-090-x Published online by Cambridge University Press

For symbolic computation, it is convenient to represent a finite quandle $Q = \{x_1, x_2, \dots, x_n\}$ with an $n \times n$ matrix M that encodes the operation table of Q, *i.e.*, $M_{ij} = k$, where $x_k = x_i \triangleright x_j$. For example, the finite Alexander quandle

$$Q = \mathbb{Z}[t^{\pm 1}]/(2, t^2 + 1) = \{x_1 = 0, x_2 = 1, x_3 = t, x_4 = 1 + t\}$$

has quandle operation matrix

$$M_Q = \begin{bmatrix} 1 & 4 & 4 & 1 \\ 3 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix}.$$

Quandles have been much studied in recent years; see [1,3,4,7,10,15], etc.. Finite quandles are of particular interest, since they can be used to define an easily computable invariant of knots and links, the *quandle counting invariant* $|\mathrm{Hom}(Q(L), T)|$. Each quandle homomorphism $f \in \mathrm{Hom}(Q(L), T)$ can be pictured as a coloring of a link diagram representing the link L with a quandle element $f(x) \in T$ attached to each arc x satisfying the crossing condition pictured above.

It is well known that knot quandles (and hence quandle counting invariants) are stronger invariants than knot groups (and knot group counting invariants). For example, Joyce [7] shows that the square knot and the granny knot have nonisomorphic knot quandles despite having isomorphic knot groups.[2]

In [11] we have the following definition.

Definition 2.3 Let Q be a finite quandle. For any element $x \in Q$, let

$$C(x) = \{y \in Q : y \triangleright x = y\} \quad \text{and} \quad R(x) = \{y \in Q : x \triangleright y = x\}$$

and set $r(x) = |R(x)|$ and $c(x) = |C(x)|$. Then the *quandle polynomial of* Q, $qp_Q(s,t)$, is

$$qp_Q(s,t) = \sum_{x \in Q} s^{r(x)} t^{c(x)}.$$

An isomorphism $\phi \colon Q \to Q'$ induces bijections

$$\phi_r \colon R(x) \to R\big(\phi(x)\big) \quad \text{and} \quad \phi_c \colon C(x) \to C\big(\phi(x)\big),$$

so $qp_Q(Q) = qp_Q(Q')$ and qp_Q is an invariant of isomorphism type for finite quandles.

We can now make our first new definition.

Definition 2.4 Let Q be a finite quandle. For any element $x \in Q$, let

$$C_n(x) = \{y \in Q : y \triangleright^n x = y\} \quad \text{and let} \quad R_m(x) = \{y \in Q : x \triangleright^m y = x\}$$

[2] However, there are group counting invariants that do distinguish the granny knot from the square knot, using generalized knot groups derived from the knot quandle. See [14] and [12].

https://doi.org/10.4153/CMB-2010-090-x Published online by Cambridge University Press

and set $r_m(x) = |R_m(x)|$ and $c_n(x) = |C_n(x)|$. Then the *(m, n)-quandle polynomial of Q*, $qp_{m,n}(Q)$, is

$$qp_{m,n}(Q) = \sum_{x \in Q} s^{r_m(x)} t^{c_n(x)}.$$

An isomorphism $\phi\colon Q \to Q'$ then induces for each $x \in Q$ bijections

$$\phi_{r,m}\colon R_m(x) \to R_m\big(\phi(x)\big) \quad \text{and} \quad \psi_{c,n}\colon C_n(x) \to C_n\big(\phi(x)\big).$$

It follows that $qp_{m,n}(Q) = qp_{m,n}(Q')$ and qp_Q is an invariant of quandle isomorphism for finite quandles.

Example 2.5 From the definition, we have $qp_{0,0}(Q) = |Q| s^{|Q|} t^{|Q|}$ and $qp_{1,1}(Q) = qp_Q(Q)$.

For any element $y \in Q$, the second quandle axiom implies that y acts on Q by a permutation $\rho_y \in S_{|Q|}$ (where $S_{|Q|}$ is the symmetric group on $|Q|$ letters) given by the column corresponding to y in the matrix of Q. Let $\operatorname{ord}(\rho)$ denote the order of ρ in $S_{|Q|}$, *i.e.,* the cardinality of the cyclic subgroup of $S_{|Q|}$ generated by ρ. Then if $n \equiv n' \bmod \operatorname{ord}(\rho_y)$, we have $x \triangleright^n y = x \triangleright^{n'} y$. It follows that for any finite quandle Q, there are at most N^2 distinct generalized quandle polynomials where $N = \operatorname{lcm}\{\operatorname{ord}(\rho_y) : y \in Q\}$. In particular, to find all generalized quandle polynomials it suffices to consider the subset $\{qp_{m,n} \mid 0 \le m, n \le N-1\}$ of the $\mathbb{Z}^2$-lattice of generalized quandle polynomials. For ease of comparison, we can write these entries in an $N \times N$ matrix $M_{qp}(Q)$ whose (i, j) entry is $qp_{i-1,j-1}(Q)$, which we will call the *generalized quandle polynomial matrix* of Q. Both the size and the entries of this matrix are invariants of quandle isomorphism type.

Example 2.6 The Alexander quandle

$$Q = \mathbb{Z}[t^{\pm 1}]/(2, t^2 + 1) = \{x_1 = 0, x_2 = 1, x_3 = t, x_4 = 1 + t\}$$

has $\operatorname{ord}(\rho_y) = 2$ for all $y \in Q$, so $N = 2$; we compute the generalized quandle polynomials $qp_{0,0} = 4s^4t^4$, $qp_{0,1} = 4s^2t^4$, $qp_{1,0} = 4s^4t^2$, and $qp_{1,1} = 4s^2t^2$. Thus, Q has generalized quandle polynomial matrix

$$M_{qp}(Q) = \begin{bmatrix} 4s^4t^4 & 4s^2t^4 \\ 4s^4t^2 & 4s^2t^2 \end{bmatrix}.$$

Example 2.7 A quandle is *strongly connected* or *Latin* if its operation matrix is a Latin square, that is, if its rows as well as its columns are permutations (see [6]). In [11], Maple computations showed that $qp_{1,1}(Q)$ distinguishes all non-Latin quandles of cardinality up to 5, while Latin quandles always have $qp_{1,1}(Q) = |Q|st$. Our Maple computations show that of the three Latin quandles with five elements, two have the same generalized quandle polynomial matrix while one has a different matrix. This shows that the generalized quandle polynomials contain additional information about quandle isomorphism type not contained in $qp_Q(s, t)$, while still not determining the quandle's isomorphism type for Latin quandles. See Table 1.

Quandle matrix	Generalized quandle polynomial matrix
$\begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 3 & 2 & 1 & 5 & 4 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 5 & 4 & 3 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}$	$\begin{bmatrix} 5s^5t^5 & 5st^5 & 5st^5 & 5st^5 \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \end{bmatrix}$
$\begin{bmatrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 2 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 5 \end{bmatrix}$	$\begin{bmatrix} 5s^5t^5 & 5st^5 & 5st^5 & 5st^5 \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \\ 5s^5t & 5st & 5st & 5st \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$	$\begin{bmatrix} 5s^5t^5 & 5st^5 \\ 5s^5t & 5st \end{bmatrix}$

Table 1: Generalized quandle polynomial matrices of Latin quandles of cardinality 5.

3 Biquandle Polynomials

In this section we define the analog of generalized quandle polynomials for finite biquandles. We start with the definition of a biquandle, also known as a type of *switch* or *Yang-Baxter Set*; see [8].

Definition 3.1 A *biquandle* is a set B with four binary operations $B \times B \to B$ denoted by $(a, b) \mapsto a^b, a^{\overline{b}}, a_b$, and $a_{\overline{b}}$, respectively, satisfying the following axioms:

1. For every pair of elements $a, b \in B$, we have

$$\text{(i) } a = a^{b\overline{b_a}}, \quad \text{(ii) } b = b_{\overline{aa^b}}, \quad \text{(iii) } a = a^{\overline{b}b_{\overline{a}}}, \quad \text{and} \quad \text{(iv) } b = b_{\overline{a}a^{\overline{b}}}.$$

2. Given elements $a, b \in B$, there are unique elements $x, y \in B$ such that

$$\begin{aligned} &\text{(i) } x = a^{b_{\overline{x}}}, && \text{(ii) } a = x^{\overline{b}}, && \text{(iii) } b = b_{\overline{x}a}, \\ &\text{(iv) } y = a^{\overline{b_y}}, && \text{(v) } a = y^b, && \text{(vi) } b = b_{y\overline{a}}. \end{aligned}$$

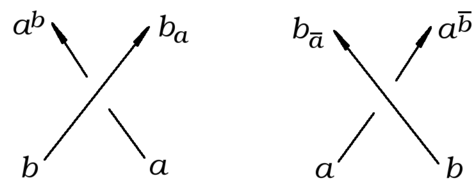
3. For every triple $a, b, c \in B$ we have:

$$\begin{aligned} &\text{(i) } a^{bc} = a^{c_b b^c}, && \text{(ii) } c_{ba} = c_{a^b b_a}, && \text{(iii) } (b_a)^{c_{a^b}} = (b^c)_{a^{c_b}}, \\ &\text{(iv) } a^{\overline{b}\overline{c}} = a^{\overline{c_{\overline{b}}}\,\overline{b^{\overline{c}}}}, && \text{(v) } c_{\overline{b}\overline{a}} = c_{\overline{a^{\overline{b}}}\,\overline{b_{\overline{a}}}}, && \text{(vi) } (b_{\overline{a}})^{\overline{c_{\overline{a^{\overline{b}}}}}} = (b^{\overline{c}})_{\overline{a^{\overline{c_{\overline{b}}}}}}. \end{aligned}$$

https://doi.org/10.4153/CMB-2010-090-x Published online by Cambridge University Press

4. Given an element $a \in B$, there are unique elements $x, y \in B$ such that

$$\text{(i) } x = a_x, \quad \text{(ii) } a = x^a, \quad \text{(iii) } y = a^{\overline{y}}, \quad \text{and} \quad \text{(iv) } a = y_{\overline{a}}.$$



The biquandle axioms come from dividing an oriented link diagram into semiarcs at every over and under crossing point; then both inbound semiarcs act on each other at both positive and negative crossings, for a total of four binary operations $(a, b) \mapsto a^b, a^{\overline{b}}, a_b$ and $a_{\overline{b}}$. The axioms are then the result of transcribing a minimal set of oriented Reidemeister moves. See [8].

One standard example of a biquandle is any quandle, which is a biquandle under

$$a^b = a \triangleright b, \ \ a^{\overline{b}} = a \triangleright^{-1} b, \ \ a_b = a_{\overline{b}} = a$$

as well as under

$$a^b = a \triangleright^{-1} b, \ \ a^{\overline{b}} = a \triangleright b, \ \ a_b = a_{\overline{b}} = a,$$

$$a_b = a \triangleright b, \ \ a_{\overline{b}} = a \triangleright^{-1} b, \ \ a^b = a^{\overline{b}} = a$$

and

$$a_b = a \triangleright^{-1} b, \ \ a_{\overline{b}} = a \triangleright b, \ \ a^b = a^{\overline{b}} = a.$$

Another standard example of a biquandle structure is any module over $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ with

$$a^b = ta + (1 - st)b, \quad a^{\overline{b}} = t^{-1}a + (1 - s^{-1}t^{-1})b, \quad a_b = sa, \quad \text{and} \quad a_{\overline{b}} = s^{-1}b;$$

biquandles of this type are called *Alexander biquandles*. For a concrete example, take $B = \mathbb{Z}_n$ and let $s, t \in B$ be any two invertible elements. See [8] and [9].

Much as with finite quandles, we can represent a finite biquandle $B = \{x_1, \dots, x_n\}$ with a block matrix encoding the operation tables of the four operations:

$$M_B = \left[\begin{array}{c|c} B^1 & B^2 \\ \hline B^3 & B^4 \end{array} \right] \quad B^k_{ij} = m, \quad \text{where} \quad x_m = \begin{cases} (x_i)^{\overline{(x_j)}} & k = 1 \\ (x_i)^{(x_j)} & k = 2 \\ (x_i)_{\overline{(x_j)}} & k = 3 \\ (x_i)_{(x_j)} & k = 4. \end{cases}$$

Example 3.2 The Alexander biquandle $B = \mathbb{Z}_4$ with $s = 3$ and $t = 1$ has biquandle matrix

$$M_B = \left[\begin{array}{cccc|cccc} 3 & 1 & 3 & 1 & 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 \\ 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 \\ \hline 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{array}\right],$$

where $B = \{x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0\}$.

In what follows, we will find it convenient to use the notation

$$\mathrm{op}_1(x, y) = x^{\overline{y}}, \quad \mathrm{op}_2(x, y) = x^y, \quad \mathrm{op}_3(x, y) = x_{\overline{y}} \quad \text{and} \quad \mathrm{op}_4(x, y) = x_y,$$

and as before $\mathrm{op}_i^n(x, y) = \mathrm{op}_i(\dots \mathrm{op}_i(\mathrm{op}_i(x, y), y) \dots y)$, where n is the number of "$\mathrm{op_i}$"s.

We can extend the generalized quandle polynomials in the obvious way to obtain an invariant of biquandles up to isomorphism.

Definition 3.3 Let B be a finite biquandle. For every $x \in B$, define

$$C_n^i(x) = \{y \in B \mid \mathrm{op}_i^n(y, x) = y\} \quad \text{and} \quad R_m^i(x) = \{y \in B \mid \mathrm{op}_i^m(x, y) = x\}$$

where $m, n \in \mathbb{Z}$. Let $c_n^i(x) = |C_n^i(x)|$ and $r_m^i(x) = |R_m^i(x)|$ for $i = 1, \dots, 4$. Then the (m, n) *biquandle polynomial* of B is

$$bp_{m,n}(B) = \sum_{x \in B} s_1^{r_m^1(x)} s_2^{r_m^2(x)} s_3^{r_m^3(x)} s_4^{r_m^4(x)} t_1^{c_n^1(x)} t_2^{c_n^2(x)} t_3^{c_n^3(x)} t_4^{c_n^4(x)}.$$

Example 3.4 As in the quandle case, we have

$$bp_{0,0}(B) = |B| s_1^{|B|} t_1^{|B|} s_2^{|B|} t_2^{|B|} s_3^{|B|} t_3^{|B|} s_4^{|B|} t_4^{|B|}$$

for every finite biquandle. Also as in the quandle case, specializing $s_i = t_i = 1$ for all $i = 1, \dots, 4$ yields $|B|$ for all $m, n \in \mathbb{Z}$.

Example 3.5 Every quandle Q is a biquandle with $a^b = a \triangleright b$, $a^{\overline{b}} = a \triangleright^{-1} b$ and $a_b = a_{\overline{b}} = a$. Specializing $s_2 = s, t_2 = t$, and $s_i = t_i = 1$ for $i = 1, 3, 4$ in $bp_{m,n}(Q)$ yields $qp_{m,n}(Q)$.

Example 3.6 The Alexander biquandle $B = \mathbb{Z}_3$ with $s = 2$, $t = 1$ has biquandle matrix

$$M_B = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 & 1 & 3 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right].$$

https://doi.org/10.4153/CMB-2010-090-x Published online by Cambridge University Press

We can compute $bp_{1,1}(B)$ by counting the number of times a row number appears in each column and row in each of the operation block matrices. Here we see that $bp_{1,1}(B) = 2s_1s_2t_3t_4 + s_1t_1^3s_2t_2^3s_3^3t_3s_4^3t_4$.

As before, a biquandle isomorphism $\phi\colon B \to B'$ induces bijections $\phi_{i,n}\colon C_n^i(x) \to C_n^i(\phi(x))$ and $\psi_{i,m}\colon R_m^i(x) \to R_m^i(\phi(x))$ for each $i = 1,\dots,4$, and $n, m \in \mathbb{Z}$, so $B \cong B'$ implies $bp_{m,n}(B) = bp_{m,n}(B')$ and each $bp_{m,n}(B)$ is an invariant of biquandle isomorphism.

The columns in a finite biquandle matrix, like those in a quandle matrix, are permutations of $\{1, 2, \dots, |B|\}$. Let $\rho_1(y) \in S_{|B|}$, $\rho_2(y) \in S_{|B|}$, $\rho_3(y) \in S_{|B|}$, and $\rho_4(y) \in S_{|B|}$ be the permutations corresponding to the actions of y on B given by $\mathrm{op}_i(\_, y)\colon B \to B$. Then as in the quandle case, the fact that $n \equiv n' \bmod \mathrm{ord}(\rho_i(y))$ implies $\mathrm{op}_i^n(x, y) = \mathrm{op}_i^{n'}(x, y)$ then implies that $bp_{m,n}(B) = bp_{m',n'}(B)$ if $n \equiv n' \mathrm{mod} N$ and $m \equiv m' \mathrm{mod} N$, where $N = \mathrm{lcm}\{\mathrm{ord}(\rho_i(y)) : y \in B,\ i = 1,\dots,4\}$. Hence, as before, there are at most N^2 distinct biquandle polynomials for a biquandle B.

Thus, for every finite biquandle B, the *biquandle polynomial matrix* of B is the $N \times N$ matrix whose m, n entry is $bp_{m,n}(B)$. Continuing with Example 3.5, if a biquandle B is a quandle with $a \triangleright b = a^b$, $a \triangleright^{-1} b = a^{\bar{b}}$ and $a_b = a_{\bar{b}} = a$, then specializing $s_1 = t_1 = s_3 = t_3 = s_4 = t_4 = 1$ and $s_2 = s,\ t_2 = t$ in the biquandle polynomial matrix of B yields the generalized quandle polynomial matrix $M_{qp}(B)$.

Example 3.7 The Alexander biquandle in example 3.2 has biquandle polynomial matrix

$$\begin{bmatrix} 4s_1^4t_1^4s_2^4t_2^4s_3^4t_3^4s_4^4t_4^4 & 2s_1^2t_1^4s_2^2t_2^4t_3^4t_4^4 + 2s_1^2t_1^4s_2^2t_2^4s_3^4t_3^4s_4^4t_4^4 \\ 2s_1^4s_2^4s_3^4t_3^2s_4^4t_4^2 + 2s_1^4t_1^4s_2^4t_2^4s_3^4t_3^2s_4^4t_4^2 & 2s_1^2s_2^2t_3^2t_4^2 + 2s_1^2t_1^4s_2^2t_2^4s_3^4t_3^2s_4^4t_4^2 \end{bmatrix}$$

according to our Maple computations.

Maple code for computing quandle and biquandle polynomials is available for download at www.esotericka.org/quandles. Computations with this code reveal that all isomorphism classes of biquandles with up to four elements are distinguished by $bp_{1,1}(B)$ alone, using the list of biquandle isomorphism classes from [13].[3]

4 Link Invariants from Generalized Quandle Polynomials

In this section we extend the subquandle polynomial defined in [11] to the $qp_{m,n}(Q)$ and $bp_{m,n}(B)$ settings and exhibit some examples of the resulting link invariants.

Definition 4.1 Let Q be a finite quandle and $S \subset Q$ a subquandle. Then for any $m, n \in \mathbb{Z}$, the *generalized subquandle polynomial* is

$$sqp_{m,n}(S \subset Q) = \sum_{x \in S} s^{r_m(x)} t^{c_n(x)}.$$

[3]Note that the published list contains a few small typographical errors, but one can regenerate the correct list with the Maple code.

https://doi.org/10.4153/CMB-2010-090-x Published online by Cambridge University Press

Similarly, for any finite biquandle B with subbiquandle $S \subset B$ and $m, n \in \mathbb{Z}$ the *subbiquandle polynomial* is

$$sbp_{m,n}(S \subset B) = \sum_{x \in S} s_1^{r_m^1(x)} s_2^{r_m^2(x)} s_3^{r_m^3(x)} s_4^{r_m^4(x)} t_1^{c_n^1(x)} t_2^{c_n^2(x)} t_3^{c_n^3(x)} t_4^{c_n^4(x)}.$$

Thus, the subquandle and subbiquandle polynomials are the contributions to the quandle and biquandle polynomials coming from the subquandle or subbiquandle in question. The $(1, 1)$ subquandle polynomial was shown in [11] to encode information about how the subquandle S is embedded in Q; indeed, $sqp_{1,1}$ can distinguish isomorphic subquandles embedded in different ways.

Since the image of a homomorphism of a knot quandle into a target quandle T is a subquandle of T, we can modify the quandle counting invariant to obtain a multiset-valued link invariant by counting the subquandle polynomial of $\mathrm{Im}(f)$ for each $f \in \mathrm{Hom}(Q(L), T)$. The cardinality of this multiset is then the usual counting invariant. This generalizes the specialized subquandle polynomial invariant that was shown in [11] to distinguish some links that have the same quandle counting invariant.

Definition 4.2 Let L be a link, T a finite quandle and $m, n \in \mathbb{Z}$. Then the multiset

$$\Phi_{sqp_{m,n}}(L, T) = \{sqp_{m,n}(\mathrm{Im}(f) \subset T) : f \in \mathrm{Hom}(Q(L), T)\}$$

is the (m, n)-*subquandle polynomial invariant* of L with respect to T. We can rewrite the multiset in a polynomial-style form by converting the multiset elements to exponents of a dummy variable q and converting their multiplicities to coefficients:

$$\phi_{sqp_{m,n}}(L, T) = \sum_{f \in \mathrm{Hom}(Q(L),T)} q^{sqp_{m,n}(\mathrm{Im}(f) \subset T)}.$$

If T is a finite biquandle, we similarly define the (m, n)-*subbiquandle polynomial invariant* to be the multiset

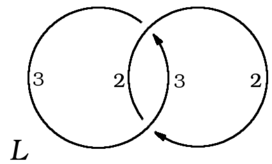
$$\Phi_{sbp_{m,n}}(L, T) = \{sbp_{m,n}(\mathrm{Im}(f) \subset T) : f \in \mathrm{Hom}(B(L), T)\}$$

or in polynomial form

$$\phi_{sbp_{m,n}}(L, T) = \sum_{f \in \mathrm{Hom}(B(L),T)} q^{sbp_{m,n}(\mathrm{Im}(f) \subset T)}.$$

Collecting all of the subquandle or subbiquandle polynomial invariants into an $N \times N$ matrix whose (m, n) entry is $sqp_{m,n}(\mathrm{Im}(f) \subset T)$ or $sbp_{m,n}(\mathrm{Im}(f) \subset T)$ yields an invariant of links that includes information from all of the subquandle or subbiquandle polynomials. Specializing $s_i = t_i = 0$ for $i = 1, \dots, 4$ or specializing $q = 1$ in any entry of the matrix yields the appropriate counting invariant.

Example 4.3



$$M_T = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 & 2 & 2 \\ 2 & 3 & 3 & 2 & 3 & 3 \end{array}\right]$$

The Hopf link L has nine biquandle colorings by the three-element biquandle T below. In particular, the pictured coloring has image subbiquandle $\mathrm{Im}(f) = \{2,3\} \subset T$. The element $2 \in T$ has $r_1^i = 2$ and $c_1^i = 3$ for $i = 1,2,3,4$, so $2 \in \mathrm{Im}(f)$ contributes $s_1^2 s_2^2 s_3^2 s_4^2 t_1^3 t_2^3 t_3^3 t_4^3$ to the exponent of q for this homomorphism. Indeed, the contribution from $3 \in T$ is the same, so we have a contribution of $q^{2 s_1^2 s_2^2 s_3^2 s_4^2 t_1^3 t_2^3 t_3^3 t_4^3}$ from this homomorphism. Repeating with the other homomorphisms, we have

$$\begin{aligned}\phi_{sbp_{1,1}}(L,T) = q^{s_1^3 t_1 s_2^3 t_2 s_3^3 t_3 s_4^3 t_4} + 4q^{s_1^3 t_1 s_2^3 t_2 s_3^3 t_3 s_4^3 t_4 + 2 s_1^2 t_1^3 s_2^2 t_2^3 s_3^2 t_3^3 s_4^2 t_4^3} \\ + 2q^{s_1^2 t_1^3 s_2^2 t_2^3 s_3^2 t_3^3 s_4^2 t_4^3} + 2q^{2 s_1^2 t_1^3 s_2^2 t_2^3 s_3^2 t_3^3 s_4^2 t_4^3}.\end{aligned}$$

Since the least common multiple of the orders of the columns of T is 2, the full invariant is a 2×2 matrix, of which the above value is one entry.

If K is a single-component link, that is, a knot, then the knot quandle of K is connected, and hence the image of any quandle homomorphism $f\colon Q(K) \to T$ must lie inside a single orbit subquandle of T. In particular, we have

$$|\mathrm{Hom}(Q(K),T)| = \Big|\mathrm{Hom}\Big(Q(K), \bigcup_{i=1}^{n} T_i\Big)\Big| = \sum_{i=1}^{n} |\mathrm{Hom}(Q(K),T_i)|,$$

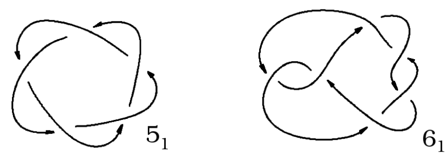
where T_i are the orbit subquandles of T. In practice, this has meant that multi-orbit quandles have been largely ignored in favor of single-orbit ("connected") quandles. The next example demonstrates that by using generalized subquandle polynomials, non-connected quandles can still be used to distinguish knots whose counting invariants are the same.

Example 4.4 The quandle T' with operation matrix

$$M_{T'} = \begin{bmatrix} 1 & 3 & 5 & 2 & 4 & 3 & 1 & 4 & 2 & 5 \\ 5 & 2 & 4 & 1 & 3 & 5 & 3 & 1 & 4 & 2 \\ 4 & 1 & 3 & 5 & 2 & 2 & 5 & 3 & 1 & 4 \\ 3 & 5 & 2 & 4 & 1 & 4 & 2 & 5 & 3 & 1 \\ 2 & 4 & 1 & 3 & 5 & 1 & 4 & 2 & 5 & 3 \\ 8 & 9 & 10 & 6 & 7 & 6 & 10 & 9 & 8 & 7 \\ 7 & 8 & 9 & 10 & 6 & 8 & 7 & 6 & 10 & 9 \\ 6 & 7 & 8 & 9 & 10 & 10 & 9 & 8 & 7 & 6 \\ 10 & 6 & 7 & 8 & 9 & 7 & 6 & 10 & 9 & 8 \\ 9 & 10 & 6 & 7 & 8 & 9 & 8 & 7 & 6 & 10 \end{bmatrix}$$

https://doi.org/10.4153/CMB-2010-090-x Published online by Cambridge University Press

has two 5-element orbit subquandles. The two knots 5_1 and 6_1 pictured below both have counting invariant $|\mathrm{Hom}(Q(5_1),T')| = 30 = |\mathrm{Hom}(Q(6_1),T')|$ with respect to T'. Indeed, the two knots have the same $\phi_{sqp_{1,1}}(K,T')$ value. However, the generalized subquandle polynomial invariants with $m=2$ distinguish the knots, detecting the fact that the sets of homomorphisms are different despite having the same cardinality.



n	$\phi_{sqp_{2,n}}(5_1,T')$	$\phi_{sqp_{2,n}}(6_1,T')$
0	$5q^{s^{10}t^{10}} + 20q^{5s^{10}t^{10}} + 5q^{s^{10}t^{2}}$	$5q^{s^{10}t^{2}} + 20q^{5s^{10}t^{2}} + 5q^{s^{10}t^{10}}$
1	$5q^{s^{2}t^{10}} + 20q^{5s^{2}t^{10}} + 5q^{s^{2}t^{2}}$	$5q^{s^{2}t^{2}} + 20q^{5s^{2}t^{2}} + 5q^{s^{2}t^{10}}$
2	$5q^{s^{6}t^{10}} + 20q^{5s^{6}t^{10}} + 5q^{s^{6}t^{2}}$	$5q^{s^{6}t^{2}} + 20q^{5s^{6}t^{2}} + 5q^{s^{6}t^{10}}$
3	$5q^{s^{2}t^{10}} + 20q^{5s^{2}t^{10}} + 5q^{s^{2}t^{2}}$	$5q^{s^{2}t^{2}} + 20q^{5s^{2}t^{2}} + 5q^{s^{2}t^{10}}$

5 Questions for Future Research

The quandle and biquandle polynomials as currently defined only make sense for finite quandles. Consequently, to use these polynomials for defining invariants of knot and link quandles and biquandles, which are typically infinite, we must first convert to finite quandles in some way. If a version of the quandle polynomial could be be defined for arbitrary quandles, or perhaps just for finitely generated quandles such as knot quandles, we might use such a polynomial (or series?) to obtain link invariants more directly.

If two knots or links have distinct quandles or biquandles, let the *sub(bi)quandle polynomial matrix index* of the pair be the cardinality of the smallest finite (bi)quandle whose polynomial matrix invariant distinguishes the pair, or ∞ if there is no such finite (bi)quandle. Is there a pair of knots or links whose sub(bi)quandle polynomial matrix index is infinite?

Each of the entries in a quandle or biquandle polynomial matrix has total coefficient equal to the cardinality of the (bi)quandle. What other relationships, if any, exist among the entries in a (bi)quandle polynomial matrix? In particular, what is the minimal subset of the entries that determines the other entries?

Acknowledgments The author would like to thank Jozef Przytycki for his comments and observations at the Knots in Washington conference that directly inspired Definition 2.4, as well as Thao-Nhi Luu, whose conversations with the author also influenced the direction of this paper, and the referee, whose comments and observations improved the paper.

References

[1] J. S. Carter, M. Elhamdadi, M. Graña, and M. Saito, *Cocycle knot invariants from quandle modules and generalized quandle homology.* Osaka J. Math. **42**(2005), no. 3, 499–541.

[2] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces.* Trans. Amer. Math. Soc. **355**(2003), no. 10, 3947–3989. doi:10.1090/S0002-9947-03-03046-0

[3] M. Eisermann. *Quandle coverings and their Galois correspondence.* http://arxiv.org/abs/math/0612459.

[4] R. Fenn and C. Rourke, *Racks and links in codimension two.* J. Knot Theory Ramifications **1**(1992), no. 4, 343–406. doi:10.1142/S0218216592000203

[5] R. Henderson, T. Macedo, and S. Nelson, *Symbolic computation with finite quandles.* J. Symbolic Comput. **41**(2006) 811–817. doi:10.1016/j.jsc.2006.03.002

[6] B. Ho and S. Nelson, *Matrices and finite quandles.* Homology Homotopy Appl. **7**(2005), no. 1, 197–208.

[7] D. Joyce, *A classifying invariant of knots, the knot quandle.* J. Pure Appl. Algebra **23**(1982), no. 1, 37–65. doi:10.1016/0022-4049(82)90077-9

[8] L. H. Kauffman and D. Radford, *Bi-oriented quantum algebras, and a generalized Alexander polynomial for virtual links.* In: Diagrammatic morphisms and applications (San Francisco, CA, 2000), Contemp. Math., 318, American Mathematical Society, Providence, RI, 2003, pp. 113–140.

[9] D. Lam and S. Nelson, *An isomorphism theorem for Alexander biquandles.* Internat. J. Math. **20**(2009), no. 1, 97–107.

[10] S. V. Matveev, *Distributive groupoids in knot theory.* (Russian) Mat. Sb. (N.S.) **119(161)**(1982), no. 1, 78–88, 160.

[11] S. Nelson, *A polynomial invariant of finite quandles.* J. Algebra Appl. **7**(2008), no. 2, 263–273. doi:10.1142/S0219498808002801

[12] S. Nelson and W. Neumann, *The 2-generalized knot group determines the knot.* Commun. Contemp. Math. **10**(2008), suppl. 1, 843–847. doi:10.1142/S0219199708003058

[13] S. Nelson and J. Vo, *Matrices and finite biquandles.* Homology, Homotopy Appl. **8**(2006), no. 2, 51–73.

[14] C. Tuffley, *Generalised knot groups distinguish the square and granny knots.* J. Knot Theory Ramifications **18**(2009), no. 8, 1129–1157. doi:10.1142/S0218216509007385

[15] J. Zablow. *Intersections of curves on surfaces with disk families in handlebodies.* J. Knot Theory Ramifications **15**(2006), no. 5, 631–649. doi:10.1142/S0218216506004671

Department of Mathematical Sciences, Claremont McKenna College, Claremont, CA 91711, U.S.A.
e-mail: knots@esotericka.org