Can. J. Math., Vol. XXVII, No. 1, 1975, pp. 88–105

ON SOME RESULTS IN MORSE THEORY

GUDRUN KALMBACH

Introduction. The h-cobordism theorem in [**8**], the generalized Poincaré conjecture in higher dimensions in [**20**] and several other results in differential topology are proved by using the following theorems of Morse theory:

(1) the elimination of critical points;

(2) the existence of nondegenerate functions for which the descending and ascending bowls have normal intersection;

(3) the alteration of function values at critical points. (For the details see below.)

We shall give short and elementary proofs of these theorems together with some stronger statements than the ones given in [**8–13**] or [**19**].

The theorems are proved for noncompact manifolds rather than for compact manifolds since, by a trivial modification of the manifold (deleting the boundary or one point) the case of compact manifolds is included.

From now on M is a noncompact, C^∞-differentiable, connected, n-dimensional manifold with a Riemannian structure.

A Morse function on M is a nondegenerate, proper, realvalued C^∞-function on M.

A Morse function f on M is called a bowl function if and only if it has the property mentioned in (2) and fulfills: if A, B are two descending bowls of f then either A is contained in the closure ΓB of B in M or $A \cap \Gamma B = \emptyset$.

THEOREM 1. *Let f be a non-negative bowl function on M with the set of critical points $P(f)$. Then there exists a Riemannian metric on M such that for every constant $c \geqq 0$ the union $N_c = \bigcup_{p\in S} E_p$ of the descending bowls E_p associated with the critical points $p \in S = P(f) \cap f^{-1}([0, c])$ is a CW-complex.*

As a corollary we get that each mth homology and homotopy class of M has a representative in some CW-complex $N_c \subseteq M$ whose carrier is the union of finitely many cells of the m-dimensional skeleton of N_c.

In § 3, we prove the existence of "enough" bowl functions on M. For a suitable Riemannian metric on M the following theorem holds.

THEOREM 2. *Let f be a Morse function on M. For any constant $\delta > 0$ there exists a bowl function g on M such that*

(i) *f and g have the same set $P(f)$ of critical points on M (indices preserved);*

Received July 9, 1973 and in revised form, October 29, 1974.

(ii) *for every* $q \in P(f)$ *there exists an open neighborhood* V *of* q *in* M *such that* f *and* g *coincide except possibly on a compact subset of* $V - \{q\}$;

(iii) $|f(x) - g(x)| < \delta$ *for all* $x \in M$.

This includes that in studying topological properties of M we can use bowl functions instead of the frequently used Morse functions on M.

We prove in § 4 a structure theorem for the closure ΓE_p of the descending bowl E_p associated with a critical point p of a bowl function f on M. Let λ be the index of f at p and q be a critical point of f of index $\lambda - 1$ with $E_q \subseteq \Gamma E_p$. Let the neighborhood V_q of q in M and the local coordinates in V_q be suitably chosen. There exists a Riemannian structure on M such that for these pairs of critical points (p, q) of f the following theorem and corollary hold in every compact set $N \subseteq M$.

THEOREM 3. $\Gamma E_p \cap V_q$ *has* r *components which are linear halfsubspaces of* V_q. *The components intersect pairwise in their common boundary* E_q.

COROLLARY 4 (see also [**10**]). *Let* $\epsilon > 0$ *and* $r = 1$ *in Theorem* 3. *If* p, q *are the only critical points of* f *in* $A = \Gamma E_p \cap \{x \in M | f(x) > f(q) - \epsilon\}$ *then there exists an* λ*-dimensional submanifold* N *in* M *with* $A \subseteq$ interior N.

With respect to the alteration of function values at critical points, we prove in § 5: assume that d is a constant, f is a bowl function on M, the point $p \in M$ is a critical point of f and W is a (small) open neighborhood of $\Gamma E_p \cap \{x \in M | f(x) \geqq d\}$ in M for $d < f(p)$ or of $\Gamma I_p \cap \{x \in M | f(x) \leqq d\}$ for $f(p) < d$ where I_p is the ascending bowl at p.

THEOREM 5. *There exists a bowl function* g *on* M, *homotopic* (*via bowl functions*) *to* f *such that* g *and* f *have the same critical points* (*and indices*) *on* M, *that* $g(p) = d$ *and* g *differs from* f *only on* W.

For noncompact manifolds M the elimination of critical points of index n and of index 0 was formulated and proved in [**3**, § 4]. It appears (without proof) also as a theorem in [**14**, p. 195]. The claim in [**14**] that the proof follows from the compact case treated in [**9**] seems erroneous. In § 7 the following theorem is proved.

THEOREM 6. *Let* M *be a connected, noncompact, smooth,* n*-dimensional manifold and* f *be a bowl function on* M. *Then all critical points of* f *of index* 0 *and* n *can be eliminated except for one minimum point if* f *is bounded from below, or one maximum point if* f *is bounded from above.*

Similar as in Theorem 5, the construction of g is done such that g and f differ only on small neighborhoods of $\Gamma E_p \cap \{x \in M | f(x) \geqq d_p\}$ or $\Gamma I_p \cap \{x \in M | f(x) \leqq d_p\}$ (d_p are suitably chosen constants) for the critical points p of f of index n or 0.

In § 8 a proof of the elimination theorem of [**12**] is given: let f be a Morse function on M and p, q be critical points of f of index $\lambda, \lambda - 1$ such that E_p and

I_q intersect transversely in precisely one component. By § 5 we can assume that

$$\Gamma E_p \cap \{x \in M | f(x) > f(q)\} = A \subseteq E_p.$$

THEOREM 7. *There exists a Morse function F on M such that*

(i) *the set of critical points of f is the set of critical points of F (indices preserved) except for p, q which are not critical points of F;*

(ii) *F differs from f only on a small neighborhood of A in M.*

It should be noticed that the proof of this theorem given in [**12**] uses the results of [**1**; **2**; **9–11**] and is therefore very long. The short proof of a weaker version of the theorem is given in [**8**, pp. 48–66]; it does not include a construction of F, — the theorem is formulated in terms of "gradientlike vectorfields of f". Our proof of the elimination theorem is short, it uses only some of the results of § 4–7.

An application of Theorems 6 and 5 is the following normalform of a 2-dimensional manifold M. (See also [**6**, p. 172].) There exists a non-negative bowl function f on M and numbers $0 < c_1 < c_2 < \ldots < c_n < \ldots$ such that $f^{-1}([0, c_1])$ is a closed disc and for $n \geqq 2$ the set $A_n = f^{-1}([0, c_n])$ is obtained from A_{n-1} by attaching to each component of the boundary of A_{n-1} the lower boundary (l.b.) of disjoint copies of one of the following 2-dimensional manifolds with boundary:

$M_1 = \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 = 1, -1 \leqq z \leqq 1\}$, l.b.: $z = -1$

$M_2 = M_1 - D$ where D is an open disc in M_1 and $\Gamma D \subseteq \operatorname{int} M_1$

M_3: a torus T with two open discs D, E in T removed and with $\Gamma D \cap \Gamma E = \emptyset$

M_4: a Moebius strip S with an open disc D in S removed and with $\Gamma D \subseteq \operatorname{int} S$

l.b. of M_i: boundary of D for $2 \leqq i \leqq 4$.

The well-known characterization of compact 2-dimensional manifolds of [**18**, p. 141] is based on the "Normalformen" of M which are in a natural correspondence with our normalforms: we "cut through" M along all the 1-dimensional closed descending bowls of f, thus getting the interior (open 2-cell) of a "Normalform" N; the pairs of edges $\ldots a \ldots a^{-1} \ldots$ or $\ldots a \ldots a \ldots$ of N (see [**18**, pp. 135–140]) correspond in the obvious fashion to the 1-dimensional descending bowls of f. Note, that by [**18**, *6*, p. 139] we can replace an M_3 and M_4 attached to A_n by attaching three M_4 to A_n. Thus, if f has only a finite number of critical points on M we get the classification of "Polyederflächen" from [**6**, p. 149] by their orientability, genus and number of ends. The general classification theorem for open surfaces [**6**, p. 170] can be derived by using as the ends of M the sequences $(B_i)_{i \in \mathbf{N}}$ of components B_i of the sets $\Gamma(M - A_n)$ which satisfy B_i is a proper subset of B_j for $i > j$.

The original version of this paper together with the papers [**4**; **5**] constitute the main part of my 1966 doctoral thesis written at the University of Göttingen.

S. S. Cairns helped me in 1968 in preparing an intermediate version of this paper. S. Lubkin has reawakened my interest in the subject in 1972. It was pointed out to me by J. Milnor in 1973 that Theorems 3 and 1 of this paper are actually needed to make the proofs of [**4**, Proposition 4] and of [**5**, p. 466] sound (or at least more easily intelligible). The referee has provided me with a thorough list of critical remarks which have improved the manuscript considerably. Finally, G. Bruns has contributed helpful comments. I am very grateful for all this support.

1. Notations. The notation introduced in this section is used throughout this paper.

ΓN is the closure of the subset N of a topological space X. For a Morse function f on M and for $a \in \mathbf{R}$ we define

$M^a = \{p \in M | f(p) \leqq a\}$,

$P(f)$ is the set of critical points of f on M,

V_p is an open neighborhood of $p \in P(f)$ with local coordinates x such that

$f(x) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$ where λ is the index of f at p.

The sets V_p are disjoint for different critical points p of f. The Riemannian metric on M is given on V_p by $\sum_{i=1}^{n} dx_i dx_i$ [**14**, p. 175].

$\varphi(q)$ is the (maximal) trajectory of f through $q \in M - P(f)$ and is locally given by $d\varphi(t, q)/dt = \text{grad} f(\varphi(t, q))$ and $\varphi(0, q) = q$.

In V_p these trajectories $\varphi(x)$ are given by

$$\text{(I)} \quad \varphi(t, x) = (e^{-2t}x_1, \ldots, e^{-2t}x_\lambda, e^{2t}x_{\lambda+1}, \ldots, e^{2t}x_n)$$

for $\alpha < t < \beta$ where $\alpha < 0 < \beta$ are constants. We set, for $p \in P(f)$,

$E_p = \{q \in M | q = p$ or $q \in M^{f(p)}$ and $p \in \Gamma\varphi(q)\}$,—the descending bowl associated with p

$I_p = \{q \in M | q = p$ or $q \in M - M^{f(p)}$ and $p \in \Gamma\varphi(q)\}$,

the ascending bowl associated with p. We have

$$E_p \cap V_p = \{x \in V_p | x_{\lambda+1} = \ldots = x_n = 0\}$$

and $I_p \cap V_p = \{x \in V_p | x_1 = \ldots = x_\lambda = 0\}$.

2. Alterations of bowls. Throughout this section f is a Morse function on M and p, q are critical points of f with $f(q) < f(p)$.

2.1 *Definition.* f is an (p, q)-bowl function if and only if either $E_q \cap \Gamma E_p = \emptyset$ or $E_q \subseteq \Gamma E_p$ and I_q and E_p intersect transversely [**8**, p. 45].

We prove below that by a small alteration of the function f we can construct a Morse function g on M which is an (p, q)-bowl function. This is the main result used in proving the existence of "enough" bowl functions. The function g is homotopic to f, and the homotopy is using only Morse functions. This remark and the fact, that g can be chosen arbitrarily close to f will be useful in other contexts too.

From now on we assume that f is not an (p, q)-bowl function, in particular that $E_q \not\subseteq \Gamma E_p$ and $E_q \cap \Gamma E_p \neq \emptyset$. The case that I_q and E_p do not intersect transversely is proved in the same way.

Since $E_q \not\subseteq E_p$ we can assume without loss of generality, that $(x_1, 0, \ldots, 0) \notin \Gamma E_p$ for $0 < x_1 \leqq c$ and some constant c. We use in the next lemma the particular choice of the following constants, sets and functions of (a)–(l).

(a) $f(q) < a$ such that $f^{-1}(a) \cap V_q$ is an open, nonempty set in $f^{-1}(a)$;

(b) $0 < d \leqq c$ with $d < (a - f(q))/2$ and an open neighborhood V of $(d, 0, \ldots, 0)$ in E_q such that $V_q \cap (V \times I_q) \cap \Gamma E_p = \emptyset$ and $\emptyset \neq (V \times I_q) \cap f^{-1}(a) = W$ is open in $f^{-1}(a)$. That such a set V exists follows from (I) of section 1 and the fact that $(c, 0, \ldots, 0)$ has an open neighborhood U in M with $U \cap \Gamma E_p = \emptyset$.

The set W is used in 2.2 as follows. We prove that the trajectories of f can be altered in M^a such that the new ascending bowl associated with q has its intersection with $f^{-1}(a)$ in W. Thus for the new descending bowl E_p^* associated with p we have $E_q \cap \Gamma E_p^* = \emptyset$.

(c) $H(x) = \sum_{i=2}^{\lambda} x_i^2$;

(d) $f(q) < b < b' < a' < a$ and $0 < \epsilon' < \epsilon$ are such that

$$(*) \quad \{x \in V_q | f(x) = a, H(x) < \epsilon, d + \epsilon' < x_1 < d + \epsilon\} \subseteq W \text{ and}$$

$$(**) \quad a' - 2(d + 2\epsilon) > b' \text{ hold.}$$

The inequality (**) will be used to show that the gradient of the function g which we are constructing in 2.2 is not zero.

(e) $A = \{x \in V_q | f(x) = a, H(x) < \epsilon, -\epsilon < x_1 < d + \epsilon\}$;

(f) $B = \{x \in V_q | \varphi(x) \cap A \neq \emptyset, b < f(x) < a\}$;

(g) $h(x)$ is the x_1-coordinate of $\varphi(x) \cap A$ for $x \in B$. Then h is smooth on B and satisfies $g_{\text{grad } f}(x) = 0$ and grad $h(x) \neq 0$. We write here $h_v(x)$ for the derivative of the function h in the direction of the tangent vector v to M at x. For the alteration of the trajectories of f which we want to do, the trajectories of h run in the right direction. Thus f and h will be put together in B to the function g, using the function G which we define now. The special choice of G is made such that it is easy to see that the derivative of g in the direction of either grad f or grad h in B is not zero, which implies then that grad g is not zero in B.

(h) $l(x) = H(\varphi(x) \cap A)$ for $x \in B$;

(k) m: $\mathbf{R} \to \mathbf{R}$ is a smooth function, bounded by 0 and 1 such that $m(t) = 1$ for $t \leqq 0$ and $m(t) = 0$ for $t \geqq 1$. For $0 < t < 1$ we have $m'(t) < 0$.

https://doi.org/10.4153/CJM-1975-011-0 Published online by Cambridge University Press

$$(l) \quad g_1(x) = \begin{cases} 1 - m((f(x) - b)/b' - b), & \text{for } f(x) \leqq b' \\ m((f(x) - a')/a - a'), & \text{for } f(x) \geqq b' \end{cases}$$

$$g_2(x) = \begin{cases} 1 - m((h(x) + \epsilon)/\epsilon - \epsilon'), & \text{for } x_1 \leqq 0 \\ m((h(x) - d - \epsilon')/\epsilon - \epsilon'), & \text{for } x_1 \geqq 0 \end{cases}$$

$$g_3(x) = m((l(x) - \epsilon')/\epsilon - \epsilon')$$

$$G(x) = \begin{cases} 1 - g_1(x)g_2(x)g_3(x), & \text{for } x \in B \\ 1, \quad \text{for } x \in V_q - B. \end{cases}$$

The function G is smooth on V_q.

2.2. LEMMA. *Assume f is not an (p, q)-bowl function. Then there exists a smooth function g on V_q such that*

(i) *$f(x) = g(x)$ for $x \in V_q - B$;*

(ii) *for the descending bowl $E_p{}^*$ of g associated with p, $E_q \cap \Gamma E_p{}^* = \emptyset$;*

(iii) grad $g(x) \neq 0$ *and* $g_{\text{grad } f}(x) \geqq 0$ *for* $x \in B$;

(iv) *g is an (p, q)-bowl function.*

Proof. Define $g(x) = (1 - G(x))(h(x) + e) + G(x)f(x)$ where $e = (a' + b')/2 + \epsilon$. Then (i) holds by the definition of g. To show (iii), observe that $f(x) - h(x) - e \leqq 0$ and $G_{\text{grad } f}(x) \leqq 0$ for $f(x) \leqq b'$ and that $G_{\text{grad } f}(x) \geqq 0$ and, using (d)(**), $f(x) - h(x) - e \geqq 0$ for $f(x) \geqq a'$. Hence $g_{\text{grad } f}(x) > 0$ if $G(x) \neq 0, 1$ and $g_{\text{grad } h}(x) \neq 0$ and $g_{\text{grad } f}(x) = 0$ if $G(x) = 0$. This shows (iii). Using (d)(*) and (b) it is straightforward to see that (ii) holds. The preceding proof shows also (iv).

3. Bowl functions on M.

3.1 THEOREM. *Let f be a Morse function on M and $\delta > 0$ be a constant. Then there exists a bowl function g on M such that*

(i) $P(f) = P(g)$ (indices preserved);

(ii) $|f(x) - g(x)| < \delta$ for $x \in M$;

(iii) *g differs from f only on $\cup_{p \in P(f)} (V_p - W_p)$ where W_p is an open neighborhood of p in M with $\Gamma W_p \subseteq V_p$;*

(iv) *g is homotopic to f.*

Proof. Let $d \in \mathbf{R}$ be a constant and $a < d < b$. We say a Morse function H on M has property (A) for (a, b) if and only if H has properties (i)–(iv) of g from above and H is an (p, q)-bowl function for all $p, q \in P(f)$ with $a < H(q) < H(p) < b$. Assume now, h is a Morse function with property (A) for (a, b). If there exists a critical value of f less than a then choose the constant a^* such that for exactly one critical value c of f holds $a^* < c \leqq a$. Using, if necessary, 2.2 for the critical points q of h with $h(q) = c$, we can construct a Morse function h^* on M with property (A) for (a^*, b). The same construction can be done for $-h^*$ instead of h if there exists some critical value c of f greater than b. The existence of g follows then by using an inductive argument.

Theorem 2 of the introduction is an immediate consequence of 3.1.

The following remark is not used later in this paper. It shows that the alteration of the function f of 2.2 to get the function g can be "transported" along the trajectories of f from V_q to V_p.

3.2. *Remark.* Assume the Morse function f on M is not an (p, q)-bowl function for $p, q \in P(f)$ with $f(q) < f(p)$, but f is an (r, q)- and (p, r)-bowl function for all $r \in P(f)$ with $f(q) < f(r) < f(p)$. Then there exists a Morse function g^* on V_p and a Riemannian metric $\bar{g}$ on M which differs from the given Riemannian metric on M only on a compact subset C of V_p such that

(i) $f(y) = g^*(y)$ for $y \in V_p - C$;

(ii) for the descending bowl E_p^* (using the Riemannian metric $\bar{g}$) of g^* associated with p, $E_q \cap \Gamma E_p^* = \emptyset$;

(iii) $|f(y) - g^*(y)| < \delta$ for $y \in V_p$ and some constant $\delta > 0$;

(iv) g^* has no critical point on C;

(v) g^* is an (p, q)-bowl function.

Proof. Since f is an (r, q)- and (p, r)-bowl function for all $r \in P(f)$ with $f(q) < f(r) < f(p)$ there exists an open neighborhood N of ΓE_p in $M - M^{f(q)}$ with $\Gamma E_r \cap N \cap V_q = \emptyset$. We show in (a) that in 2.2 the alteration of f to an (p, q)-bowl function g can be modified such that g differs from f only on N. Using the diffeomorphism $\varphi_{\beta-\alpha}$ of [**7**, p. 13] from $\{x \in V_q | g(x) \neq f(x)\} = B$ onto $D \subseteq V_p$ (α, β suitable constants), we get as a new set of trajectories χ on D the images under $\varphi_{\beta-\alpha}$ of the trajectories of g in B. We have to pick (see (b)) another Riemannian metric $\bar{g}$ on D to show in (c) that there exists a Morse function g^* on D which has χ as its set of orthogonal trajectories and which fulfills also all the other requirements of the remark.

(a) By Proposition 2 of [**4**, p. 542] there exist neighborhoods W and W' of ΓE_p in $M - M^{f(q)}$ with $W' \subseteq W \subseteq N$ and a separation function (see [**4**]) H of $M - (W \cup M^{f(q)})$ and $W' - M^{f(q)}$. Some properties of H are: $H(x) = 1$ near the boundary of W and $H(x) = 0$ for $x \in W'$, the function H is smooth and $H_{\mathrm{grad}\, f}(x) = 0$ for $x \in W \cap V_q$. It is easy to see that the constants in (d) of section 2 can be chosen such that ΓE_p^* of 2.2 lies in W'. Thus the Morse function F on M defined by $F(x) = H(x)f(x) + (1 - H(x))g(x)$ for $x \in V_q$ and $F(x) = f(x)$ otherwise, has the properties of g in 2.2 and has the same descending bowl E_p^* associated with p as g.

(b) Let x, respectively y, be the preferred coordinate system in V_q respectively V_p. Note, that det $(\partial y(\varphi_{\beta-\alpha}(x))/\partial x) > 0$ throughout

$$V = \{x \in V_q | F(x) \neq f(x)\}.$$

Let $\tilde{g}$ be the Riemannian metric on M. Then there exists a new Riemannian metric $\bar{g}$ on V_p such that for $y \in \varphi_{\beta-\alpha}(V)$ holds $\bar{g}\ (y)(v, w) = \tilde{g}(\varphi_{-\beta+\alpha}(y))(d\varphi_{-\beta+\alpha}v, d\varphi_{-\beta+\alpha}w)$ and for $y \in V_p - V'$ holds $\bar{g}(y)(v, w) = \tilde{g}(y)(v, w)$ where V' is a neighborhood of $\Gamma\varphi_{\beta-\alpha}(V)$ in V_p not containing p and where $d\varphi_{-\beta+\alpha}$ is the map induced by $\varphi_{-\beta+\alpha}$ on the tangent bundle of $\varphi_{\beta-\alpha}(V)$. The existence of $\bar{g}$

https://doi.org/10.4153/CJM-1975-011-0 Published online by Cambridge University Press

is an easy consequence of the following facts. Let $A(y) = (\partial x(\varphi_{-\beta+\alpha}(y))/\partial y)$. Note, that $\tilde{g}(y) = \sum_{i=1}^{n} dy_i dy_i$ and $\det A(y) > 0$ for $y \in \varphi_{\beta-\alpha}(V)$. Let $\tilde{g}_{ik}(y)$ be the (i, k)-entry of the matrix $A(y)A^t(y)$. Then for $y \in V'$ we get

$$\tilde{g}(\varphi_{-\beta+\alpha}(y))(d\varphi_{-\beta+\alpha}\partial/\partial y_i, d\varphi_{-\beta+\alpha}\partial/\partial y_k) = \tilde{g}_{ik}(y).$$

(c) Set $g^*(y) = F(\varphi_{-\beta+\alpha}(y)) + \beta - \alpha$ for $y \in \varphi_{\beta-\alpha}(V)$ and set $g^*(y) = f(y)$ otherwise. The equation $d\varphi_{\beta-\alpha}$ grad F = grad g^* on $\varphi_{\beta-\alpha}(V)$ follows from

$$\begin{aligned}\bar{g}(y)(\partial/\partial y_i, d\varphi_{\beta-\alpha}\,\mathrm{grad}\,F) &= \tilde{g}(\varphi_{-\beta+\alpha}(y))\left(\sum_{j=1}^{n} \partial x_j(\varphi_{-\beta+\alpha}(y))/\partial y_i \cdot \partial/\partial x_j, \mathrm{grad} F\right) \\ &= \sum_{j=1}^{n} \partial F/\partial x_j \cdot \partial x_j(\varphi_{-\beta+\alpha}(y))/\partial y_i \\ &= \partial F \circ \varphi_{-\beta+\alpha}(y)/\partial y_i = \partial g^*(y)/\partial y_i.\end{aligned}$$

Then the set χ, the images under $\varphi_{\beta-\alpha}$ of the trajectories of F in V, is the set of trajectories of g^* on $\varphi_{\beta-\alpha}(V)$, since the vectors $d\,\varphi_{\beta-\alpha}$ grad F = grad g^* are the tangent vectors to the elements of χ.

4. The local structure of ΓE_p. The next lemma is from [**8**, pp. 58–66]. We present a different proof here. A linear map l is called tangent to a diffeomorphism h at 0 if and only if the Jacobian of l and of h at 0 coincide. Let $(a_{ij})_{1 \leqq i, j \leqq n}$ be the Jacobian of l. Then l is called an elementary rotation if and only if there exist numbers η, p, q with $1 \leqq p, q \leqq n$ and $p \neq q$ such that $a_{ij} = \delta_{ij}$ (Kronecker symbol) for $\{i, j\} \not\subseteq \{p, q\}$, that $a_{pp} = \cos \eta = a_{qq}$ and $a_{qp} = \sin \eta = -a_{pq}$. For a definition of an intersection number see [**8**, p. 67]. For $1 \leqq m < n$ and $m + k = n$ the space $\mathbf{R}^m$, respectively $\mathbf{R}^k$, is the submanifold $\mathbf{R}^m \times 0$, respectively $0 \times \mathbf{R}^k$, of $\mathbf{R}^n$.

4.1 LEMMA. *Let h be an orientation preserving diffeomorphism from $\mathbf{R}^n$ into $\mathbf{R}^n$ with $h(0) = 0$ and $h(\mathbf{R}^m) \cap \mathbf{R}^k = \{0\}$ such that the intersection of $h(\mathbf{R}^m)$ and $\mathbf{R}^k$ is transverse with the intersection number* 1. *Then there exists a diffeomorphism $h^* : \mathbf{R}^n \to \mathbf{R}^n$ such that*

(i) *$h^*(x) = h(x)$ for $x \in \mathbf{R}^n - N$ where N is some neighborhood of* 0;

(ii) *$h^*(x) = x$ in a neighborhood of* 0;

(iii) *there exists an isotopy $(h_t)_{0 \leqq t \leqq 1}$ between h and h^* such that $h_t(\mathbf{R}^m) \cap \mathbf{R}^k = \{0\}$ and the intersection is transverse for every t* [**8**, p. 58].

Proof. We discuss two special cases first, in (a) that the identity is tangent to h and in (b) that h is an elementary rotation with $-\pi < \eta < \pi$. The general case is treated in (c). Let $h(x) = (h_1(x), \ldots, h_n(x))$.

(a) Let m be the function of (k) section 2 and $|m'(t)| < c$ for $0 \leqq t \leqq 1$. Define

$$h_j^*(x) = m(|x|^2 - a/b - a)x_j + (1 - m(|x|^2 - a/b - a)h_j(x)$$

https://doi.org/10.4153/CJM-1975-011-0 Published online by Cambridge University Press

where $0 < a = b/2$. We can choose a neighborhood V of 0 and a constant $\epsilon > 0$ such that the derivative of h_j with respect to x_j is larger than $1 - \epsilon$ and with respect to x_i for $i \neq j$ lies between $-\epsilon$ and ϵ. We ask also that $4c \cdot |h_j(x)/x_j - 1| < \epsilon$. Then $|\partial h_j^*(x)/\partial x_i| < 2\epsilon$ for $i \neq j$ and $\partial h_j^*(x)/\partial x_j > 1 - 2\epsilon$ which shows that for sufficiently small chosen b, ϵ and V, the Jacobian of h^*, is not zero. Using the Taylor formula for h_j^* it is straightforward to see that h^* is one-to-one, i.e. a diffeomorphism. (iii) is an easy consequence of the definition of h^*.

(b) Put $g(x) = \eta \cdot (1 - m(|x|^2 - a^*/b^* - a^*)$ for $0 < a^* < b^*$ and

$$H(x) = \left(\sum_{j=1}^{n} a_{ij}(x)x_j\right)_{1 \leqq i \leqq n}$$

where $a_{ij}(x) = \delta_{ij}$ for $\{i, j\} \not\subseteq \{p, q\}$ and $a_{pp}(x) = \cos\ (g(x)) = a_{qq}(x)$ and $a_{qp}(x) = \sin\ (g(x)) = -a_{pq}(x)$. The map H is one-to-one since $|H(x)| = |x|$. It is straightforward to check that the determinant of the Jacobian of H is always 1 and that (iii) holds.

(c) Let l be the linear map tangent to h at 0. Using (a) for the map $l^{-1} \circ h$ we can find a map h' with the properties (i), (iii) of 4.1 and with $h'(x) = l(x)$ in a neighborhood U of 0. The Jacobian A of l can be written as the product $C \cdot B$ of two matrices with det $B = 1$ and $C = (c_{ij})$ where $c_{11} = c > 0$ and $c_{ij} = \delta_{ij}$ otherwise. Since the map $r(x) = B \cdot x$ (matrix product) is a finite composition of elementary rotations we can use (b) to find a diffeomorphism f with the properties (i), (iii) of 4.1 in U (l replaces h) and which coincides with $s(x) = C \cdot x$ in a neighborhood W of 0. Finally, for s replacing h, we can find a function F with the properties of the Lemma. If we put h^* equal to F in W, equal to f in $U - W$ and equal to h' in $\mathbf{R}^n - U$ then h^* has the properties required.

4.2 *Definition.* Let p, q be critical points of a Morse function f on M of index λ, $\lambda - 1$ such that

(i) $f(q) < f(p)$ and $E_q \subseteq \Gamma E_p$, and

(ii) there exists a neighborhood W of E_p in $M^{f(p)+\epsilon} - M^{f(q)}$ where $\epsilon > 0$ such that $W \cap P(f) = \{p\}$.

Note that in this case there exist finitely many trajectories $\chi_1, \ldots, \chi_r$ of f in E_p such that the χ_i are exactly the trajectories of f which have p and q as limit points. We then call the pair (p, q) a preferred pair of critical points of type $(\chi_i)_{i=1,\ldots,r}$ (or of type r).

4.3 *Definition.* Let A be a linear k-dimensional subspace of $\mathbf{R}^n$ and L_i be halflines with the endpoint in A for $1 \leqq i \leqq r$. Asume P_i is the closed half-space in $\mathbf{R}^n$ spanned by A and L_i and $P_i \cap P_j = A$ for $i \neq j$. Then $B = \bigcup_{i=1}^{r} P_i$, is a $((k + 1)$-dimensional) book with r pages having spine A.

4.4 THEOREM. *Let f be a bowl function on M and $N \subseteq M$ compact. There exists a neighborhood W_p of $p \in P(f)$ with $\Gamma W_p \subseteq V_p$ and a Riemannian metric on M which differs from the original Riemannian metric only on $\bigcup_{p \in P(f) \cap N} V_p - W_p$*

such that with respect to the new Riemannian metric on M the following holds: for every preferred pair of critical points (p, q) (where $p, q \in N$) of type r the intersection $\Gamma E_p \cap W_q$ is a book with r pages having the spine $E_q \cap W_q$.

Proof. Assume that (p, q) is of type $(\chi_i)_{i=1,\ldots,r}$ and (II) holds;

(II) we constructed a new Riemannian metric in a neighborhood N_i of a point $x_0 \in \chi_i$ (i fixed) in V_q such that

(i) dist $(\Gamma N_i, q) > 0$;

(ii) χ_i remains a trajectory of f;

(iii) there is a neighborhood A_i of $\Gamma\chi_i$ in ΓE_p (taken in the new Riemannian metric) and a neighborhood U_q of q in M such that $A_i \cap U_q$ is a closed λ-dimensional linear halfsubspace of U_q, (λ is the index at p).

Then the theorem follows by using disjoint sets N_i and, if necessary, the diffeomorphism from $M^{f(p)-\epsilon} \cap W$ to $M^{f(q)+\epsilon} \cap W$ induced by f, together with some inductive argument similar to the one used in 3.1. Here $\epsilon > 0$, and the neighborhood W of $E_p - M^{f(q)}$ has the properties of W in 4.2. The proof of (II) is in three parts. In (a) we define an isotopy h_t on $A = f^{-1}(f(q) + \epsilon) \cap N_i$ with the analogue to the properties of h_t in 4.1. The isotopy h_t is used in (b) to define on N_i a new set of trajectories ψ with the following properties:

(iv) the elements of ψ and the trajectories of f coincide as sets close to the boundary of N_i (and on χ_i), thus ψ can be extended to a set of trajectories on M;

(v) using ψ, the new descending bowl associated with p is, close to χ_i, a linear subspace of some open neighborhood U_q of q in M. To get the trajectories ψ to be the orthogonal trajectories of f we change in (c) the Riemannian metric on N_i. We are using the diffeomorphisms $\varphi_{\beta-\alpha}$ of [**7**, p. 13].

(a) In A (respectively $\varphi_{f(p)-\epsilon-f(q)-\epsilon}(N_i) \cap f^{-1}(f(p) - \epsilon)$ we choose the intersection with χ_i as the origin and the coordinates u (respectively v) as the translation induced by the preferred coordinate system in V_q (respectively V_p). Put $h(v) = \varphi_{-f(p)+\epsilon+f(q)+\epsilon}(v)$. The map h is a diffeomorphism of A onto A if we change the coordinate system u on A to either $v = u$ or ($v_i = u_i$ for $i \neq \lambda$ and $v_\lambda = -u_\lambda$) or ($v_i = u_i$ for $i \neq \lambda, \lambda + 1, v_\lambda = -u_\lambda, v_{\lambda+1} = -u_{\lambda+1}$) whatever is necessary to make h also orientation preserving and $h(E_p)$ and $I_q \cap A$ intersect transversely with the intersection number 1. Take h_t as the isotopy of 4.1.

(b) Define in N_i the curves $c(w)$ for $w = h_0(v) \in N_i \cap f^{-1}(f(q) + \epsilon) = N^*$ by $c(t, w) = \varphi_{a-f(q)-\epsilon}(h_t(v))$ where $a = f(q) + \epsilon - t \cdot b$ for $0 \leqq t \leqq 1$ and some small constant $b > 0$. Then $w \in E_p \cap N^*$ if and only if $c(1, w) \in N_i \cap \{x \in f^{-1}(f(q) + \epsilon - b) | x_\lambda = \ldots = x_{n-1} = 0, x_n < 0\} = N'$. Here we assume that x is the preferred coordinate system in V_q and χ_i is the negative x_n-axis in V_q. We extend the curves $c(w)$ by the trajectories of f through $c(0, w)$ respectively $c(1, w)$ in the direction of increasing respectively decreasing values of f. For points of M which do not lie on one of these curves we take the trajectories of f, thus getting a set c of disjoint curves on $M - P(f)$. These curves may fail to be smooth on $N^* \cap N'$. A lemma of Munkres [**15**, p. 26] applies

https://doi.org/10.4153/CJM-1975-011-0 Published online by Cambridge University Press

in this case and proves the existence of a smooth set of curves ψ which coincides with c outside of a small neighborhood of $N^* \cap N'$.

(c) We choose a parametrization of the trajectories of ψ such that the vectorfield $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ of tangents of these trajectories on V_q coincide with grad f outside of N_i. Define the new Riemannian metric

$$g^*(x) = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j \text{ on } N_i$$

by $g_{ij}(x) = \delta_{ij}$ (Kronecker symbol) if $i, j \neq n$, by $g_{in}(x) = g_{ni}(x) = -1/\nu_n(x)$ $(\nu_i(x) + \gamma_i 2x_i)$ for $1 \leqq i \leqq n - 1$ with $\gamma_i = 1$ if $i < \lambda$ and $\gamma_i = -1$ if $\lambda \leqq i$ and by

$$g_{nn}(x) = \frac{1}{\nu_n(x)^2} \cdot \left(2x_n \cdot \nu_n(x) + \sum_{i=1}^{n-1} \nu_i(x)(\nu_i(x) + \gamma_i 2x_i)\right).$$

We can assume that $|\nu_i(x)|$ and $|x_i|$ are small in comparison with $|\nu_n(x)|$ and $|x_n|$ for $i = 1, \ldots, (n - 1)$. Then it is easy to show that g^* is a Riemannian metric on V_q. Note, that $g^*(x) = \sum_{i=1}^{n} dx_i dx_i$ outside of N_i and that ψ is the set of orthogonal trajectories of f with respect to the Riemannian metric g^* on N_i.

Theorem 3 of the introduction follows immediately from 4.4. The proof of the corollary to Theorem 3: using the notation of 4.4(b) recall, that $\Gamma E_p \cap I_q$ respectively ΓE_q was given in a neighborhood U_q of q by

$$\{x \in U_q | x = (0, \ldots, 0, x_n), x_n \leqq 0\}$$

respectively $\{x \in U_q | x_\lambda = \ldots = x_{n-1} = 0, x_n \leqq 0\}$. We define with some suitable constants $\alpha > 0$ and $\beta > 0$ the set N by

$$N = (\Gamma E_p - M^{f(q)-\beta}) \cup \{x \in V_q | x_\lambda = \ldots = x_{n-1} = 0,$$
$$0 < x_n < \alpha, f(x) > f(q) - \beta\}.$$

Note, that the proof of 4.4 can be modified for a bowl function f on M such that in Theorem 4.4 not all requirements of 4.2, but only 4.2(i) is needed as an assumption for the pair of critical points (p, q), to show that the intersection $\Gamma E_p \cap W_q$ is the union of a finite number of closed linear submanifolds of W_q which pairwise intersect in $E_q \cap W_q$. In particular the indices at p and q may differ by more than 1. We use this fact below in 4.5, a correction to the proof of Proposition 4 of [**4**, p. 543]; this Proposition 4 itself is now used in the

Proof of Theorem 1 of the introduction: Let f be a non-negative bowl function on M satisfying the modified version of 4.4 and $a > 0$ be a noncritical value of f. We want to show that the union $K^a \subseteq M$ of the descending bowls E_p associated with $p \in M^a \cap P(f)$ is a CW-complex. Clearly K^a with the induced topology is a Hausdorff space and is the union of disjoint (open) cells. Assume $p, q \in M^a$ are distinct critical points of f of index λ, τ and $E_q \subseteq \Gamma E_p$. Since $I_q \cap \Gamma E_p \neq \emptyset$ and I_q and ΓE_p intersect transversely we have $\lambda + (n - \tau) -$

$1 \geqq n$, i.e., $\lambda > \tau$. Hence $\Gamma E_p - E_p$ lies in the $(\lambda - 1)$-dimensional skeleton of K^a. It remains to show that there exists an attaching map from the closed λ-dimensional unit ball onto ΓE_p. We prove this by proving that for a sufficiently small constant $\epsilon > 0$ the cell $A = \Gamma(E_p - M^{f(p)-2\epsilon})$ is attached to $K^{f(p)-\epsilon}$. Without loss of generality, $f(p)$ is the only critical value of f between $f(p) - 3\epsilon$ and $f(p) + \epsilon$. Let m be the function of (k) section 2 and $g(x) = m((f(x) - f(p) + 2\epsilon)/\epsilon)$ for $x \in M^{f(p)-\epsilon}$. If r_t for $0 \leqq t \leqq 1$ is the deformation retraction of $M^{f(p)-\epsilon}$ onto $K^{f(p)-\epsilon}$ from [**4**, Proposition 4] then,

$$r(x) = \begin{cases} x, & \text{for } x \in \Gamma E_p - M^{f(p)-\epsilon} \\ r_{g(x)}(x), & \text{for } x \in M^{f(p)-\epsilon} \cap \Gamma E_p \end{cases}$$

is a map attaching A to $K^{f(p)-\epsilon}$. Using [**23**, p. 129] we have as an easy consequence of Theorem 1 the corollary to Theorem 1.

The preceding proof of Theorem 1 and 4.5 is added to the original version of this paper due to questions of J. Milnor and the referee of this paper.

4.5 *Correction to the proof of Proposition* 4 *of* [**4**, p. 543]: We complete the proof of the statement of [**4**, p. 543, line 9-11]: $\chi(p)$ has exactly one limit point $p' \in K^c$. By 3.1 and the note to 4.4 we can assume that the function f is a bowl function which satisfies: if $E_q \subseteq \Gamma E_r$ ($q, r \in M^c$) then $\Gamma E_r \cap W_q$ is the union of a finite number of closed linear submanifolds of W_q which pairwise intersect in $E_q \cap W_q$. This construction can be done such that also the level submanifolds h_{ik} = const. of the function h_{ik} of Proposition 4 are linear submanifolds of W_q intersecting I_q transversely. Thus in the notation of [**4**], (A) if $u \in (V_{i_v}' - (W_{i_v}' \cup E_{i_v})) \cap I_{ik-1}$ then $\chi(u) \subseteq I_{ik-1}$ holds. Here I_{ik-1}, E_{i_v} are the sets I_q, E_r and V_{i_v}', W_{i_v}' are neighborhoods of E_{i_v} in M. But (A) implies that for $v \in W_q$ the trajectory $\chi(v)$ does not have a limit point in both, E_r and E_q. Thus $\chi(v)$ has exactly one limit point in K^c.

5. The alteration of critical values. Theorem 5 of the introduction is an easy consequence of the following Lemma.

5.1 LEMMA. *Let f be a Morse function on M and $p \in P(f)$ such that for some constant $d < f(p)$ the set $B = \Gamma E_p - M^d$ contains no critical point of f other than p. Then there exists a neighborhood N of B in M and a Morse function F on M such that*

(i) *F coincides with f in $M - N$;*
(ii) *$F(p) = d$;*
(iii) *p is the only critical point of F in N;*
(iv) *f and F are homotopic* (via Morse functions).

Proof. We define in (a) the auxiliary function G which is used in (b) to define F. We also use in (b) the fact that by Proposition 1 of [**4**, p. 541] there exists a differentiable function h on N satisfying $h(x) = 0$ for $x \in B$ and $h(x) > 0$ for $x \in N - B$. For the gradient of h holds grad $h(x) \neq 0$ for $x \in N - B$

and grad $h(x) = 0$ for $x \in B$. The derivative $h_{\text{grad } f}(x)$ of h in the direction of grad f is not negative. Each trajectory $\psi(x)$ of h through $x \in N - B$ has exactly one limit point $x' \in B$. The function m is taken from (k) of section 2. Assume that $0 < a < f(p) - d$ and $P(f) \cap (\Gamma E_p - M^{d-3a}) = \{p\}$.

(a) Define $g^*(x) = (d - a) + a \cdot (f(x) - d + a)/(f(p) - d + a)$ and $G(x) = m((f(x) - d + 2a)/a)(f(x) - g^*(x)) + g^*(x)$ for $x \in E_p - M^{d-3a}$. Then $G(p) = d$. We have $G(x) < f(x)$ for all $x \in E_p - M^{d-a}$ and $G(x) > f(x)$ for $x \in E_p$ with $d - 2a \leqq f(x) < d - a$. The function G satisfies $G(x) = f(x)$ for $x \in E_p$ with $d - 3a \leqq f(x) \leqq d - 2a$ and G and f have the same level manifolds. Moreover, $G_{\text{grad } f}(x) > 0$ for $x \in E_p$ with $d - 3a \leqq f(x) < f(p)$.

(b) Let $\epsilon > 0$ be a constant with $A = \{x \in M | h(x) \leqq 2\epsilon\} \subseteq N$. Define $F(x) = f(x) - m(h(x)/\epsilon) \cdot (f(x') - G(x'))$ for $x \in N$ with $h(x) \leqq 2\epsilon$ and $F(x) = f(x)$ otherwise. Here x' is the limit point in B of the trajectory $\psi(x)$ mentioned above. Then $F(p) = d$. The point p is the only critical point of F in A since $F_{\text{grad } h}(x) \neq 0$ if $h(x) \leqq 2\epsilon$ and $x \notin E_p$ and $f(x) \neq d - a$. If $h(x) \leqq 2\epsilon$ and $x \notin E_p$, but $f(x) = d - a$ then $F_{\text{grad } f}(x) > 0$. The existence of a homotopy between f and F follows from the definition of F.

6. Functions on submanifolds of M. This section contains the auxiliary Lemma 6.1 which is used in the proof of Theorem 6.

6.1 Lemma. *If (p, q) is a preferred pair of critical points of f of type (χ) and the index at p is n then there exists a Morse function f' on M and neighborhoods Z respectively Z_p of $\Gamma\chi$ respectively $\Gamma E_p - M^{f(q)}$ such that f and f' have the same level manifolds on $M - Z$, that* grad $f'(x) \neq 0$ *for $x \in Z$ and f coincides with f' on $M - Z_p$.*

Proof. In our first step (a) of the proof we delete a neighborhood L of q in E_q to show that on $M - L$ there exists a smooth function G with the same levels as f such that $G(x) \leqq f(q) - a$ for $x \in \Gamma E_p - (L \cup M^{f(q)-3a})$ and for some constant $a > 0$. G and f coincide on $M^{f(q)-3a}$. The set L is deleted since G would be discontinuous on L and for the purpose G serves later, G does not have to be defined on L. In (b) we define a function F on Z without critical points on Z which is used in (f) to replace the function f on a neighborhood U of χ contained in Z. The function G replaces f on $\Gamma E_p - (Z \cup M^{f(q)-3a})$. To put the functions G and F smoothly and without critical points on $Z - U$ together we use the function H, g_0 and f_0 constructed in (c), (d) and (e). Finally, in (f) we construct the function f' with the properties listed in 6.1. The function m is taken from (k) section 2. The local coordinates in V_q respectively V_p are x respectively y such that $\chi \cap V_q = \{(0, \ldots, 0, x_n) | x_n < 0\}$ and $\chi \cap V_p = \{(y_1, 0, \ldots, 0) | y_1 > 0\}$. We can choose the constants $d, e \in \mathbf{R}$ such that $f(y) - y_1 - d \leqq 0$ for $y \in V_p$ with $y_1 > 0$ and $f(x) + x_n + e - d \geqq 0$ for $x \in V_q$ with $x_n < 0$. The special choice of d, e is used in (b) to show that the gradient of F on Z is not zero.

(a) Take $a > 0$ such that

$$L = \left\{ x \in V_q \,\middle|\, \sum_{i=1}^{n-1} x_i^2 \leqq 3a, x_n = 0 \right\}$$

is closed in V_q. Put $\bar{g}(z) = (f(q) - 2a) + a \cdot (f(z) - f(q) + 2a)/(f(p) - f(q) + 2a)$ and define $G(z) = \bar{g}(z) + m((f(z) - f(q) + 3a)/a) \cdot (f(z) - \bar{g}(z))$ for $z \in \Gamma E_p - (L \cup M^{f(q)-3a})$ and $G(z) = f(z)$ otherwise. Then G has the properties listed above.

(b) Put $h'(y) = y_1$ for $y \in V_p$ and $h'(x) = x_n + e$ for $x \in V_q$. Let $0 < a' < b'$ and $c' < d' < 0$ be small constants. Define the function g^* on Z by $g^*(y) = m((h'(y) - a')/b' - a')$ for $y \in V_p$, for $x \in V_q$ by $g^*(x) = 1 - m((h'(x) - e - c')/d' - c')$ and $g^*(z) = 0$ for $z \in Z - (V_p \cup V_q)$. Then $F'(z) = g^*(z) \cdot h'(z) + (1 - g^*(z)) \cdot (-f(z) + d)$ is a smooth, bounded function on Z. Let $\eta > 0$ be such that $|F'(z) - F'(u)| < \eta$ for all z, $u \in Z$. Define $F(z) = a \cdot F'(z)/\eta$ for $z \in Z$. We list some properties of F: if $g^*(z) = 0$ then $F_{\text{grad } f}(z) \neq 0$. If $g^*(z) \neq 0$ then $F_{\text{grad } h'}(z) > 0$, using the special choice of the constants d, e from above. In $Z - (V_p \cup V_q)$ we have $F(z) = a \cdot (-f(z) + d)/\eta$ and for $z \in \{y \in V_p | y_1 \leqq 0\} \cup \{x \in V_q | x_n \geqq 0\}$, we have $F(z) = a \cdot h'(z)/\eta$.

(c) For a suitable neighborhood Z'' of $\Gamma\chi$ in M with $\Gamma Z'' \subseteq Z$ we have that the orthogonal trajectory $\psi(z)$ of F through $z \in Z''$ has its intersection with boundary Z in $V_p \cup V_q$. Thus for $z \in Z''$ and $x' = \psi(z) \cap \{x \in V_q | x_n = 0\}$ we can define $H(z) = H(x')$ where $H(x) = \sum_{i=1}^{n-1} x_i^2$ for $x = (x_1, \ldots, x_{n-1}, 0) \in V_q$. Then H is smooth and $H_{\text{grad } F}(z) = 0$ for $z \in Z$. Define $Z' = \{z \in Z | H(z) \leqq 4a\}$.

(d) Let a^*, b^*, c^*, d^*, be constants with $0 < a^* < b^*$ and $c^* < d^* < 0$ such that $a^{*2} = 3a$ and $b^* - a^* = a = d^* - c^*$. The property $a^{*2} = 3a$ is used in (f) to show on a subset E' of $V_p \cup V_q$ defined below, that on E' the derivative of the function f' in the direction of grad H is negative. Put $g_1(z) = 1 - m((h'(z) - c^*)/a)$ for $z \in V_p$ and $g_1(z) = m((h'(z) - e - a^*)/a)$ for $z \in V_q$. In Z', set $g_2(z) = 1 - m((H(z) - a)/a)$ and $g_3(z) = m((H(z) - 3a)/a)$. Define $g_0(z) = g_1(z)g_2(z)g_3(z)$ in Z' and $g_0(z) = 0$ in $M - Z'$. Then $g_{0 \text{ grad } F}(z) = 0$ for $z \in Z' - A'$ since $g_1(z) = 1$ and $H_{\text{grad } F}(z) = 0$. Here $A' = \{x \in V_q | x_n \geqq 0\} \cup \{y \in V_p | y_1 \leqq 0\}$. Other properties of g_0 are: g_0 is bounded by 0 and 1 and is zero on $(M - Z') \cup \Gamma\chi$ and 1 on $\{u \in Z' | 2a \leqq H(u) \leqq 3a, u \notin A'\}$. The function g_0 is smooth and $g_{0 \text{ grad } F}(z) \geqq 0$ in V_p respectively $g_{0 \text{ grad } F}(z) \leqq 0$ in V_q.

(e) We need the following constants, functions and sets: $d^* < a_1 < a_2 < 0$ and $0 < b_1 < b_2 < a^*$; define $G'(z) = m((h'(z) - e - b_1)/b_2 - b_1)$ in V_q and $G'(z) = 1 - m((h'(z) - a_1)/a_2 - a_1)$ in V_p. For $z \in C - (V_p \cup V_q)$ put $G'(z) = 1$. Here $C = \{z \in Z' | g_2(z) < 1\}$. Finally, put $G'(z) = 0$ for $z \in M - C$ and $c' = \inf_{z \in Z'} F(z)$. Let $A = \{z \in Z' | g_0(z) = 1\}$. Define

$$f_0(z) = G'(z) \cdot (F(z) + f(q) - a - c') + (1 - G'(z))G(z)$$
for $z \in M - A$.

If $G'(z) \neq 0$ then $a^{*2} = 3a$ implies that $f_{0\,\mathrm{grad}\,F}(z) > 0$. This holds for $z \in Z' - A$ and $H(z) \leqq 3a$. For $z \in C - \{z \in C | G'(z) < 1\}$ holds $f_0(z) = F(z) + f(q) - a - c'$, on $M - Z'$ the functions f_0 and G coincide and grad $f_0(z) \neq 0$ holds for $z \in Z' - A$.

(f) Put $E = \{z \in M | 0 < g_0(z) < 1\}$. We also use the sets

$$D' = \{z \in V_p \cup V_q | 0 < g_1(z) < 1\} \cup (\{z \in Z' | H(z) \leqq 3a\} - A),$$
$$D'' = \{z \in Z' | H(z) \geqq 3a\} - (\{y \in V_p | y_1 \leqq 0\} \cup \{x \in V_q | x_n \geqq 0\}), \text{ and}$$
$$E' = \{z \in V_p \cup V_q | g_1(z) = 1, H(z) > 3a\} \cap (\{y \in V_p | y_1 \leqq 0\} \cup \{x \in V_q | x_n \geqq 0\}).$$

Then $E \subseteq D' \cup D'' \cup E'$. Define

$$f'(z) = (1 - g_0(z))f_0(z) + g_0(z)(-H(z) + f(q) + 3a).$$

For $z \in D''$ holds $f'_{\mathrm{grad}\,F}(z) = G_{\mathrm{grad}\,F}(z) < 0$ and for $z \in D'$ we have $f'_{\mathrm{grad}\,F}(z) > 0$. Since $a^{*2} = 3a$ we get $f'_{\mathrm{grad}\,H}(z) < 0$ if $z \in E'$. If $z \in A$ and $g_0(z) = 0$ or 1 then grad $H(z) \neq 0$ implies that grad $f'(z) \neq 0$. The other properties of f' listed in 6.1 follow from the definition of f'.

7. Elimination of critical points of index n and 0.

Proof of Theorem 6 *of the introduction*: In 7.1 below we show that for every critical point p of f of index n which is not a maximum point we have a preferred pair (p, q) of critical points of f (see Definition 4.2). By 6.1 we can eliminate p and q as critical points. This shows the part of Theorem 6 for the critical points of f of index n if only finitely many such points exist on M. If infinitely many critical points of f of index n exist then we have to alter the function values of f at critical points of index n and $n - 1$ such that for every point $x \in M$ only finitely many alterations of f by 6.1 reach x. To do this, we alter the function values of f as follows: there exists a sequence

$$\ldots < c_{-m} < c_{-(m-1)} < \ldots < c_0 < c_1 < \ldots < c_{m-1} < c_m < \ldots$$

of real numbers such that f has at most one critical point p of index n on $f^{-1}(c_i)$ and for every such critical point of index n holds $f(p) = c_i$ for some i. In 7.2 below we show that for every critical point p of index n with $f(p) = c_i$ there exists a preferred pair of critical points (p, q) with $c_{i-2} < f(q)$. Using these preferred pairs of critical points we can eliminate by 6.1 inductively all superficial critical points of index n and only finitely many such alterations of f reach a given compact subset of M. Using this construction for $-f$ instead of f we can also eliminate all superficial points of index 0.

7.1 *Remark*. Let f be a bowl function on M and p be a critical point of f of index n which is not a maximum point. Then there exists $q \in P(f)$ such that (p, q) is a preferred pair of critical points of f.

Proof. We show in (a) that there exists a critical point $q \in \Gamma E_p$ such that ΓE_p does not contain an n-dimensional neighborhood of q. We use this fact in (b) to show 7.1.

(a) If ΓE_p contains for every $q \in P(f) \cap \Gamma E_p$ an n-dimensional neighborhood then ΓE_p is an n-dimensional manifold. Since p is not a maximum point we have $\Gamma E_p \neq M$, contradicting M connected.

(b) Take the point $q \in P(f) \cap \Gamma E_p$ of (a) with a maximal function value. If $E_r \subseteq \Gamma E_p$ for $r \in P(f)$ with $f(q) < f(r)$ then $\Gamma E_r \cap E_q = \emptyset$ since the index of f at r is less or equal to $n - 1$, the point r is an interior point of ΓE_p and the index of f at q has to be $(n - 1)$ because $I_q \cap f^{-1}(f(q) + \epsilon)$ for $\epsilon > 0$ has to be disconnected. Hence E_p and I_q intersect in exactly one trajectory of f.

7.2 *Remark.* The assumptions are as in the proof of Theorem 6. Then for every critical point p of index n with $f(p) = c_i$ there exists a preferred pair of critical points (p, q) with $c_{i-2} < f(q)$.

Proof. For every constant c the set $f^{-1}(c)$ meets with at most a finite number of sets ΓE_p in disconnected components of $f^{-1}(c)$ where p is a critical point of f of index n. Hence for at most a finite number of such points p the critical point q of the preferred pair (p, q) of critical points of 7.1 satisfies $f(q) < c$. Therefore, to get the result of 7.2 we have to use at most a finite number of alterations of f according to § 5 which meet $f^{-1}(c)$.

8. The elimination theorem. A short proof of the elimination theorem (Theorem 7) is completed in this section. It uses 8.1 and 8.2. The proof of Theorem 7 itself is the original one given by M. Morse [**10**, p. 270, 271, 304–307, 312–316] and is repeated in this section for the convenience of the reader. The length of the proof of Theorem 7 given by M. Morse is due to the fact that M. Morse needs all the results of [**1**; **2**] and [**9**–**11**] to fill in the part of the proof of Theorem 7 in which we are using 8.1 and 8.2.

8.1 Let (p, q) be a preferred pair of critical points of the Morse function f on M of type 1 and λ be the index of f at p. Let N be the λ-dimensional submanifold of M given by the Corollary 4 of the introduction. Then there exists a smooth function f^* on N such that

(i) f^* has no critical points on N;

(ii) f^* and f have the same levels outside of a neighborhood of $E_p \cap I_q$;

(iii) $f^*(x) < f(x)$ for $x \in A$ where $\{x \in \Gamma E_p | f(x) \geqq f(q)\} \subseteq A$ and A is a compact subset of N;

(iv) $f^*(x) \leqq f(x)$ for $x \in N$ and $f^*(x) = f(x)$ for all $x \in N$ near the boundary of N.

Proof. From Theorem 6 we have the existence of a function f' on N with all properties of f^* asked in 8.1 except perhaps $f'(x) \leqq f(x)$ for all $x \in N$. Let the constants a, b^*, e and the function h' be taken as in Lemma 6.1. Choose the constant e' such that $f(x) = f'(x)$ for $f'(x) \geqq e'$. Since the minimum value

$k(t)$ of f on the level $f'(x) = t$ for $f(q) - 4a \leqq t \leqq e'$ is a strictly monotonic increasing function there exists a smooth strictly monotonic increasing function $b(t)$ on $f(q) - 4a < t < e'$ with $b(t) < k(t)$ and with $b(t) = t$ for $t \leqq f(q) - 4a$ or $t \geqq e'$. Let $b^* < c_1 < c_2$ with $c_2 - c_1 = a$. Then 8.1 holds for the function

$$f^*(x) = m((h'(x) - e - c_1)/a) \cdot b(f'(x)) + (1 - m((h'(x) - e - c_1)/a)f(x).$$

8.2 Assumptions as in 8.1. There exists a smooth function h on a neighborhood W of N such that

(i) $h(x) = 0$ for $x \in N$ and $h(x) > 0$ for $x \in W - N$;

(ii) $h_{\text{grad } f}(x) \geqq 0$, grad $h(x) \neq 0$ for $x \in W - N$ and grad $h(x) = 0$ for $x \in N$;

(iii) each trajectory $\psi(x)$ of h for $x \in W - N$ has exactly one limit point x' in N.

Proof. By [**4**, p. 541] there exists a function h_p on a suitable neighborhood W_p of $A = E_p - M^{f(q)-\epsilon}(\epsilon > 0)$ with the properties of 8.2 for (A, W_p) replacing (N, W) and such that $h_{p \text{ grad } f}(x) = 0$ for $x \in W_p - V_p$. Let $h^*(x) = \sum_{i=\lambda}^{n-1} x_i^2$ in V_q. There exists a function $\bar{h}$ on V_q such that h^* and $\bar{h}$ have the same levels in V_q and $\bar{h}(x) \leqq h_p(x)$ on

$$B' = \{x \in V_q | \epsilon' \leqq h_q(x) \leqq 2\epsilon', h^*(x) \leqq \epsilon_1\}$$

for some $\epsilon_1 > 0$ and $\epsilon' > 0$ where $h_q(x) = h^*(x) + x_n^2$. Define $h(x) = h_p(x) + m((h_q(x) - \epsilon')/\epsilon')(\bar{h}(x) - h_p(x))$ in V_q and $h(x) = h_p(x)$ in $W_p - V_q$. We have then $h_{\text{grad } f}(x) > 0$ if $x \in B' - N$ since $h_{p \text{ grad } f}(x) = 0$. The other properties of 8.2 hold trivially.

Proof of Theorem 7. This proof is now exactly the proof of Morse given in [**12**, p. 313] but uses only 8.1 and 8.2. For the convenience of the reader we repeat this proof: Let $\eta > 0$. If $x \notin N$ then let $\psi(x)$ be the trajectory of h through x and $\Gamma\psi(x) \cap N = \{x'\}$. Define $F(x) = f(x)$ for $x \in M - W$ or $x \in W$ with $h(x) \geqq 2\eta$; for $x \in W$ with $h(x) < 2\eta$ let $F(x) = f(x) - m(h(x)/\eta)(f(x') - f^*(x'))$. Then F is smooth and $F(x) = f(x)$ for $x \in W$ with $\eta \leqq h(x) \leqq 2\eta$. Since $F(x) = f^*(x)$ for $x \in N$ we have grad $F(x) \neq 0$ for $x \in N$. Let $x \in W$ and $0 < h(x) < \eta$. Then $F_{\text{grad } h}(x) = f_{\text{grad } h}(x) - h_{\text{grad } h}(x)m'(h(x)/\eta) \cdot 1/\eta \cdot (f(x') - f^*(x'))$. Now, $f^*(x') \leqq f(x')$ implies $F_{\text{grad } h}(x) \geqq 0$ and $f^*(x') < f(x')$ implies $F_{\text{grad } h}(x) > 0$. Hence F has no critical points in W. The other properties of F are easily checked.

References

1. Huebsch-Morse, *Conditioned differentiable isotopies*, Diff. Analysis, Bombay Coll. (1964), 1–25.

2. ——— *A model nondegenerate function*, Rev. Roumaine Math. Pures Appl. *10* (1965), 691–722.

3. G. Kalmbach, *Über niederdimensionale CW-Komplexe in nichtkompakten Mannigfaltigkeiten*, Ph.D. Dissertation Göttingen, 1966.

4. ——— *Deformation retracts and weak deformation retracts of noncompact manifolds*, Proc. Amer. Math. Soc. *20* (1969), 539–544.

5. ——— *On smooth bounded manifolds*, Proc. Amer. Math. Soc. *22* (1969), 466–469.

6. B. v. Kérekjarto, *Vorlesungen über Topologie, I*, Berlin, Grundlehren d. Math. Wiss. in Einzeldarst. VIII.

7. J. Milnor, *Morse theory*, Ann. of Math. Studies *51*, 1963.

8. ——— *Lectures on the h-cobordism theorem*, Princeton, N.J., 1965.

9. M. Morse, *The existence of polar nondegenerate functions on differentiable manifolds*, Ann. of Math. *71* (1960), 352–383.

10. ——— *Quadratic forms θ and θ-fibre bundles*, Ann. of Math. *81* (1965), 303–340.

11. ——— *Bowls, f-fibre bundles and the alteration of critical values*, An. Acad. Brasil. Ci. *36* (1964), 245–259.

12. ——— *The elimination of critical points of a nondegenerate function on a diff. manifold*, J. Analyse Math. *13* (1964), 257–316.

13. ——— *Bowls of a nondegenerate function on a compact diff. manifold*, Diff. and Combin. Topol., Proc. of a symposium in honour of M. Morse, Princeton, N.J., 1964, 81–103.

14. M. Morse and S. S. Cairns, *Critical point theory in global analysis and differential topology* (Academic Press, N.Y., 1969).

15. J. Munkres, *Obstructions to the smoothing of piecewise differentiable homeomorphisms*, Ann. of Math. *72* (1960), 521–554.

16. ——— *Elementary differential topology*, Ann. of Math. Studies *54*, 1966.

17. J. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. *106* (1963), 259–269.

18. Seifert-Threlfall, *Lehrbuch d. Topologie*, N.Y., 1934.

19. S. Smale, *On gradient dynamical systems*, Ann. of Math. *74* (1961), 199–206.

20. ——— *The generalized Poincaré conjecture in dimension greater than* 4, Ann. of Math. *74* (1961), 391–406.

21. E. H. Spanier, *Algebraic topology* (McGraw Hill, New York, 1966).

22. J. H. C. Whitehead, *Combinatorial homotopy, I*, Math. works of J. H. C. Whitehead, Voi. III, 85–117.

23. ——— *Combinatorial homotopy, II*, Math. works of J. H. C. Whitehead, Vol. III, 119–162.

Pennsylvania State University,
University Park, Pennsylvania