Adv. Appl. Prob. **27**, 1186–1188 (1995)
Printed in N. Ireland

LETTER TO THE EDITOR

ON THE HNBUE PROPERTY IN A CLASS OF CORRELATED CUMULATIVE SHOCK MODELS

RAFAEL PÉREZ-OCÓN* AND
M. LUZ GÁMIZ-PÉREZ*, *University of Granada*

Abstract

Conditions for a correlated cumulative shock model under which the system failure time is HNBUE are given. It is shown that the proof of a theorem given by Sumita and Shanthikumar (1985) relative to this property is not correct and a correct proof of the theorem is given.

HARMONIC NEW BETTER THAN USED IN EXPECTATION; POISSON PROCESS; RENEWAL PROCESS

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60K10
SECONDARY 62N05

1. Introduction

Sumita and Shanthikumar (1985) have proved some interesting results on a correlated cumulative shock model. We use their notation and terminology. Let (X_n, Y_n), $n = 0, 1, 2, \cdots$ be a sequence of independently and identically distributed pairs of random variables. We assume the system to be new at time $t = 0$, and the magnitude X_n of the nth shock is correlated only with the time interval Y_n since the $(n-1)$th shock and does not affect future events. Theorem 3.A5 of their paper establishes conditions on the variables Y_n and X_n for the system failure time S_z, the time until the magnitude of a shock exceeds a prespecified level z, to belong to the HNBUE class. But the proof given by the authors is not correct. In Section 2 of this paper we give a proof of this theorem. In Section 3 we discuss the proof given by Sumita and Shanthikumar, and show that the inequality on which Theorem 3.A5 is based is not correct. Conditions on X_n are expressed in terms of the HNBUE property, so the conclusion is that S_z is HNBUE if both renewal processes Y_n and X_n are HNBUE.

2. The HNBUE property

In this section we give conditions on the shocks arrival and shock magnitudes processes, under which S_z satisfies the HNBUE property.

The renewal process $\{M_X(x), x > 0\}$ associated with the sequence (X_n) has a renewal function given by

$$\sum_{n=0}^{\infty} F_X^{(n)}(x) = 1 + H_X(x), \tag{2.1}$$

if at time $t = 0$ there is a renewal, see Çinlar (1975).

For $x > 0$, the random variable $M_X(x)$ is HNBUE if

$$\sum_{n=k}^{\infty} \boldsymbol{P}\{M_X(x) > n\} \leqq \{1 + H_X(x)\}\left\{\frac{H_X(x)}{1 + H_X(x)}\right\}^k. \tag{2.2}$$

Received 3 February 1994; revision received 24 April 1995.
* Postal address: Department of Statistics and OR, University of Granada, 18071 Granada, Spain.

The random variable X is said to be right tail better than new in Y, denoted by RTBN$(X \mid Y)$, if $\boldsymbol{P}(X > x \mid Y > y) \geqq \boldsymbol{P}(X > x)$ for all x and y.

We now state Theorem 3.A5 of Sumita and Shanthikumar (1985).

Theorem 1. Suppose

(i) Y_n is HNBUE, $n = 1, 2, \cdots$,

(ii) for $x > 0$, the random variable $M_X(x)$ is HNBUE, and

(iii) RTBN $(X_n \mid Y_n)$, $n = 1, 2, \cdots$.

Then S_z is HNBUE for all $z > 0$.

Proof. The proof follows on the lines given in [3], where the following inequality is obtained:

$$\int_t^\infty \bar{W}(z, \tau)\, d\tau \leqq \sum_{n=0}^\infty F_X^{(n)}(z) \int_t^\infty \{\bar{F}_Y^{(n+1)}(\tau) - \bar{F}_Y^{(n)}(\tau)\}\, d\tau. \tag{2.3}$$

To arrive here the hypothesis (iii) has been used.

The integrand can be rearranged to get

$$\int_t^\infty \bar{W}(z, \tau)\, d\tau \leqq \sum_{n=0}^\infty \{F_X^{(n)}(z) - F_X^{(n+1)}(z)\} \int_t^\infty \bar{F}_Y^{(n+1)}(\tau)\, d\tau. \tag{2.4}$$

Now we use the hypothesis (i). The HNBUE property implies

$$\int_t^\infty \bar{F}^{(n+1)}(\tau)\, d\tau \leqq \int_t^\infty \bar{G}^{(n+1)}(\tau)\, d\tau, \tag{2.5}$$

for all $t \geqq 0$ and $n \geqq 0$, where $G(u) = 1 - \exp\{-u/\eta_Y\}$ is the exponential distribution with mean η_Y.

Substituting (2.5) in (2.4), using the property that the sequence $(F_X^{(n)}(z))$, $n = 0, 1, \cdots$ is decreasing in n for all $z \geqq 0$, and rearranging the right-hand side we get

$$\begin{aligned} \int_t^\infty \bar{W}(z, \tau)\, d\tau &\leqq \sum_{n=0}^\infty \left(\int_t^\infty \{\bar{G}^{(n+1)}(\tau) - \bar{G}^{(n)}(\tau)\}\, d\tau \right) F_X^{(n)}(z) \\ &\leqq \sum_{n=0}^\infty \sum_{k=0}^n \eta_Y \exp(-t/\eta_Y) \left(\frac{t}{\eta_Y}\right)^k \frac{1}{k!} F_X^{(n)}(z) \\ &= \eta_Y \exp(-t/\eta_Y) \sum_{k=0}^\infty \left(\frac{t}{\eta_Y}\right)^k \frac{1}{k!} \sum_{n=k}^\infty F_X^{(n)}(z). \end{aligned} \tag{2.6}$$

Now using the hypothesis (ii) and the equality $\boldsymbol{E}(S_z) = \eta_Y\{1 + H_X(z)\}$, we obtain

$$\int_t^\infty \bar{W}(z, \tau)\, d\tau \leqq \boldsymbol{E}(S_z) \exp(-t/E(S_z)),$$

completing the proof.

If the arrival process is dominated in the sense of the following corollary, then conditions (ii) and (iii) of the theorem are sufficient for S_z to be HNBUE. This condition substitutes the HNBUE property on Y_n to get that S_z be HNBUE.

Corollary. If $N_1(t)$ is a renewal process of arrivals and $N(t)$ is a Poisson process with the same mean, and the following inequality is satisfied:

$$\int_t^\infty \boldsymbol{P}\{N_1(x) = n\}\, dx \leqq \int_t^\infty \boldsymbol{P}\{N(x) = n\}\, dx \qquad \text{for all } t \geqq 0, \tag{2.7}$$

then S_z is HNBUE.

3. Comments

1. The inequality (3.11) in [3] is not true in general. If F_Y is a gamma distribution with index $\alpha = 2$ and parameter $\beta = 2$, then the survival function can be expressed as

$$\bar{F}_Y(t) = e^{-\beta t} \sum_{i=0}^{\alpha-1} \frac{(\beta t)^i}{i!} = e^{-\beta t}(1 + \beta t). \tag{3.1}$$

As $\alpha > 1$, $\bar{F}$ is IFR and therefore HNBUE. But this distribution does not satisfy the equation (3.11) in [3]. For example, take $n = 1$ in this expression to get

$$\int_t^\infty \{\bar{F}_Y^{(2)}(x) - \bar{F}_Y^{(1)}(x)\}\, dx \leqq e^{-t}(1 + t) \tag{3.2}$$

where $F^{(2)}$ is a gamma distribution with index 4 and parameter 2, using equation (3.1) the expression (3.2) is

$$e^{-2t}(1 + 2t + 2t^2 + 2t^3/3) \leqq e^{-t}(1 + t) \tag{3.3}$$

and this inequality is not true for $t = 1$. The value of the left-hand side of equation (3.3) is 0.766899938 while the value on the right-hand side is 0.7357588824.

2. We have interpreted condition (ii) of Theorem 3.A5 in [3] in terms of the ageing property of the renewal process associated to the sequence X_n, that must be HNBUE too.

3. The inequality (3.11) in [3] cannot be obtained from [2]. Klefsjö proved the inequality

$$\int_t^\infty \bar{H}(x)\, dx \leqq \int_t^\infty \bar{S}(x)\, dx \tag{3.4}$$

where H and S are the distribution functions of a shock model with arrivals according to a counting process $M(t)$ and a homogeneous pure birth process $M_1(t)$, respectively.

The inequality (3.4) is equivalent to

$$\sum_{k=0}^\infty \bar{P}_k \left[\int_t^\infty \boldsymbol{P}\{M(x) = k\}\, dx - \int_t^\infty \boldsymbol{P}\{M_1(x) = k\}\, dx \right] \leqq 0, \tag{3.5}$$

but obviously this does not imply (3.11) in [3].

4. If it is supposed that Y_n has an exponential distribution with mean $1/\lambda$, $n = 1, 2, \cdots$ then the proof given in [3] is valid.

Acknowledgements

The authors wish to thank Professor R. P. Gupta for helpful comments on the final version of this paper, and the referee for his careful reading and suggestions.

References

[1] Çinlar, E. (1975) *Introduction to Stochastic Processes.* Prentice-Hall, Englewood Cliffs, NJ.

[2] Klefsjö, B. (1981) HNBUE survival under some shock models. *Scand. J. Statist.* **8**, 39–47.

[3] Sumita, U. and Shanthikumar, J. G. (1985) A class of correlated cumulative shock models. *Adv. Appl. Prob.* **17**, 347–366.