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SOME REMARKS TO ONO'S THEOREM ON A GENERALIZATION OF GAUSS' GENUS THEORY

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Let K/k be a finite Galois extension of finite algebraic number fields with Galois group $\mathfrak{g}$. We denote by G_m the multiplicative group defined over the rational number field $\boldsymbol{Q}$ and put

$$G_{m,K} = \operatorname{Spec}(K) \underset{\operatorname{Spec}(\boldsymbol{Q})}{\times} G_m\,, \qquad G_{m,k} = \operatorname{Spec}(k) \underset{\operatorname{Spec}(\boldsymbol{Q})}{\times} G_m\,.$$

Let $R^{(1)}_{K/k}(G_m)$ denote the kernel of the norm $N\colon R_{K/k}(G_{m,K}) \to G_{m,k}$, where $R_{K/k}$ is the Weil functor of restricting the field of definition from K to k; then we have an exact sequence of tori defined over k and split over K:

$$1 \longrightarrow R^{(1)}_{K,k}(G_m) \longrightarrow R_{K/k}(G_{m,K}) \longrightarrow G_{m,k} \longrightarrow 1\,.$$

In [3] T. Ono defined the class number $h(T)$ of an algebraic torus T intrinsically and it follows that $h(R_{K/k}(G_{m,K}))$ and $h(G_{m,k})$ coincide with the class numbers of algebraic number fields K and k, respectively. As a generalization of Gauss' genus theory, he investigated the alternating product

$$E(K/k) = \frac{h(K)}{h(k)h(R^{(1)}_{K/k}(G_m))}$$

and proved in [7] the following, using the class number formula and the Tamagawa number of tori established by himself (cf. [3, 4]),

$$E(K/k) = \frac{\operatorname{Card}(H^0(\mathfrak{g}, U_K))\operatorname{Card}(\operatorname{Ker}(H^0(\mathfrak{g}, K^\times) \longrightarrow H^0(\mathfrak{g}, K_A^\times))}{[\mathfrak{g}:\mathfrak{g}']\operatorname{Card}(H^0(\mathfrak{g}, \mathcal{O}_K^\times))}$$

where $K_A^\times$ and U_K are the idele group of K and its unit subgroup, $\mathfrak{g}'$ is the commutator subgroup of $\mathfrak{g}$ and $H^0(\mathfrak{g}, -)$ is the 0-th cohomology group modified by Tate.

The purpose of this paper is to give an analogous formula for class

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numbers in the *narrow* sense. Forgetting the total positivity, the proof of our formula becomes a simple proof of the Ono's theorem.

After we explain the tools we use in Section 1, we recall the definition of the class number of algebraic tori following [3] in Section 2. In the following Section 3, we shall prove our main theorem. Applying our formula to cyclic extensions, we shall obtain the formula for the number of ambigous classes in Section 4. In the last section we shall notice that

$$E(K_n/K_n^+) = 1$$

where K_n and K_n^+ are the n-th cyclotomic field and its maximal real subfield.

§ 1. Preliminaries

In this section we enumerate tools we use. For a group A, let $|A|$ be the order of A. We treat abelian groups multiplicatively. If a homomorphism f of abelian groups has finite kernel and cokernel, we put

$$q(f) = \frac{|\operatorname{Coker} f|}{|\operatorname{Ker} f|}$$

which is due to Tate.

SNAKE LEMMA. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 1 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 1 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 1 \end{array}$$

be a commutative diagram of abelian groups whose lines are exact; then we have an exact sequence:

$$1 \longrightarrow \operatorname{Ker} f' \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} f'' \longrightarrow \operatorname{Coker} f' \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} f'' \longrightarrow 1 .$$

LEMMA 1. *Let the notation be as in the above lemma. If two of $q(f')$, $q(f)$ and $q(f'')$ are defined, then the third one is defined and we have*

$$q(f) = q(f')q(f'') .$$

LEMMA 2. *Let $f\colon A \to B$ be a homomorphism of finite abelian groups; then we have*

$$q(f) = |B|/|A| .$$

LEMMA 3. *Let $f: A \to B$ and $g: B \to C$ be homomorphisms of abelian groups such that both of $q(f)$ and $q(g)$ are defined; then $q(g \cdot f)$ is defined and we have*

$$q(g \cdot f) = q(f)q(g) .$$

§2. Class number of algebraic tori

Following [3], we recall the class number of algebraic tori.

Let T be an algebraic torus defined over a finite algebraic number field k. For any place v of k, let k_v be the completion with respect to v, then the group $T(k_v)$ of k_v-valued points of T becomes a locally compact abelian group and if v is finite it contains the unique maximal compact subgroup $T(\mathcal{O}_v)$ where $\mathcal{O}_v$ is the ring of integers in k_v. The group $T(k_A)$ of the adele ring valued points of T can be identified with

$$\prod_v{}' \, T(k_v)$$

where v runs over the set of places of k and $'$ is the restricted direct product with respect to $\{T(\mathcal{O}_v)\}$. We define the unit group by

$$U_T = \prod_{\mathfrak{p}} T(\mathcal{O}_{\mathfrak{p}}) \times \prod_v T(k_v)$$

where $\mathfrak{p}$ runs over the set of finite places and v runs over the set of infinite places. We define the class number $h(T)$ of T by

$$h(T) = [T(k_A) \colon T(k) \cdot U_T]$$

where the group $T(k)$ of k-rational points of T is regarded as a subgroup of $T(k_A)$, and it is known that $h(T)$ is finite (cf. [3], Theorem 3.1.1).

Let K/k be a Galois extension of finite algebraic number fields. Let $G_{m,K}$ and $G_{m,k}$ be multiplicative groups defined over K and k, respectively. We define the norm torus $R^{(1)}_{K/k}(G_m)$ by the kernel of the norm homomorphism $N: R_{K/k}(G_{m,K}) \to G_{m,k}$, where $R_{K/k}$ is the Weil functor of restricting the field of definition (cf. [12]). Let $N_A: K_A^\times \to k_A^\times$, $N_U: U_K \to U_k$ and $N_{K/k}$: $K^\times \to k^\times$ be the norm maps, where U_K and U_k are unit groups of $K_A^\times$ and $k_A^\times$, and N_U is the restriction of N_A. We put $K_A^{(1)} = \operatorname{Ker} N_A$, $U_K^{(1)} = \operatorname{Ker} N_U$ and $K^{(1)} = \operatorname{Ker} N_{K/k}$. In general, for a commutative ring R, we denote by $R^\times$ the group of units.

PROPOSITION 1. *Notation being as above, we have*

$$h(R^{(1)}_{K/k}(G_m)) = [K_A^{(1)} \colon K^{(1)} \cdot U_K^{(1)}] .$$

Proof. Using some results proved by Weil (cf. [12], Chapter I), we have the following commutative diagrams:

$$(1)\qquad \begin{array}{ccc} R_{X/k}(G_{m,K})(k) & \xrightarrow{N(k)} & G_{m,k}(k) \\ \downarrow\wr & & \downarrow\wr \\ K^{\times} & \xrightarrow[N_{K/k}]{} & k^{\times} \end{array}$$

$$(2)\qquad \begin{array}{ccc} R_{K/k}(G_{m,K})(k_v) & \xrightarrow{N(k_v)} & G_{m,k}(k_v) \\ \downarrow\wr & & \downarrow\wr \\ \prod_{V\mid v} K_V^{\times} & \xrightarrow[N_v]{} & k_v^{\times} \end{array} \qquad (v\text{: any place})$$

where V runs over the set of places of K lying over v and $N_v((x_V)_V) = \prod_{V\mid v} N_{K_V/k_v}(x_v)$.

$$(3)\qquad \begin{array}{ccc} R_{K/k}(G_{m,K})(\mathcal{O}_{\mathfrak{p}}) & \xrightarrow{N(\mathcal{O}_{\mathfrak{p}})} & G_{m,k}(\mathcal{O}_{\mathfrak{p}}) \\ \downarrow\wr & & \downarrow\wr \\ \prod_{\mathfrak{P}\mid\mathfrak{p}} \mathcal{O}_{\mathfrak{P}}^{\times} & \xrightarrow[N_{\mathfrak{p}}]{} & \mathcal{O}_{\mathfrak{p}}^{\times} \end{array} \qquad (\mathfrak{p}\text{: finite place}).$$

$$(4)\qquad \begin{array}{ccc} R_{K/k}(G_{m,K})(k_A) & \xrightarrow{N(k_A)} & G_{m,k}(k_A) \\ \downarrow\wr & & \downarrow\wr \\ K_A^{\times} & \xrightarrow[N_A]{} & k_A^{\times} \end{array} .$$

By (1) and (4), we have

$$R^{(1)}_{K/k}(G_m)(k) = K^{(1)}$$

and

$$R^{(1)}_{K/k}(G_m)(k_A) = K^{(1)}_A .$$

Moreover by the maximal compactness we have

$$R^{(1)}_{K/k}(G_m)(\mathcal{O}_{\mathfrak{p}}) = \operatorname{Ker} N_{\mathfrak{p}} \cap \prod_{\mathfrak{P}\mid\mathfrak{p}} \mathcal{O}_{\mathfrak{P}}^{\times}$$

and

$$U_{R^{(1)}_{K/k}(G_m)} = U^{(1)}_K . \qquad \text{Q.E.D.}$$

§ 3. Ono invariants $E(K/k)$ and $E^{+}(K/k)$

If F is a finite algebraic number field, denote by P_F, I_F and H_F the group of principal ideals, the group of fractional ideals, and the ideal class group of F. For a subgroup A of F we denote by A^{+} the subgroup of A consisting of totally positive elements. Let K/k be a finite Galois extension of finite algebraic number fields with Galois group $\mathfrak{g}$. We define relative class numbers $h(K/k)$ and $h^{+}(K/k)$ by

$$h(K/k) = [K_A^{(1)} : K^{(1)} \cdot U_K^{(1)}]$$

and

$$h^{+}(K/k) = [K_A^{(1)} : K^{(1)+} \cdot U_K^{(1)}] .$$

Let $I_K^{(1)}$ be the kernel of $N_{K/k}: I_K \to I_k$, $P_K^{(1)} = P_K \cap I_K^{(1)}$ and $P_K^{(1)+}$ the subgroup of $P_K^{(1)}$ consisting of principal ideals generated by totally positive elements.

PROPOSITION 2. *Notation being as above, we have*

$$h(K/k) = \frac{[I_K^{(1)} : P_K^{(1)}][\mathcal{O}_k^{\times} \cap N_{K/k}K^{\times} : N_{K/k}\mathcal{O}_K^{\times}]}{[U_k \cap N_A K_A^{\times} : N_U U_K]}$$

and

$$h^{+}(K/k) = \frac{[I_K^{(1)} : P_K^{(1)+}][\mathcal{O}_k^{\times +} \cap N_{K/k}K^{\times +} : N_{K/k}\mathcal{O}_K^{\times +}]}{[U_k \cap N_A K_A^{\times} : N_U U_K]} .$$

Proof. We shall prove the second equality only, because a similar argument without "+" yields the first one. Consider the following commutative diagrams:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_K & \longrightarrow & K_A & \longrightarrow & I_K & \longrightarrow & 1 \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \\ 1 & \longrightarrow & U_k & \longrightarrow & k_A & \longrightarrow & I_k & \longrightarrow & 1 \end{array}$$

and

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_K^{\times +} & \longrightarrow & K^{\times +} & \longrightarrow & P_K^{+} & \longrightarrow & 1 \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \\ 1 & \longrightarrow & \mathcal{O}_k^{\times +} & \longrightarrow & k^{\times +} & \longrightarrow & P_k^{+} & \longrightarrow & 1 . \end{array}$$

Applying the snake lemma to these diagrams, we have two exact sequences:

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$$1 \longrightarrow U_K^{(1)} \longrightarrow K_A^{(1)} \longrightarrow I_K^{(1)} \xrightarrow{\delta_1} U_k/NU_K \longrightarrow k_A^\times/NK_A^\times$$

and

$$1 \longrightarrow \mathcal{O}_K^{(1)+} \longrightarrow K^{(1)+} \longrightarrow P_K^{(1)+} \xrightarrow{\delta_2} \mathcal{O}_k^{\times+}/N\mathcal{O}_K^{\times+} \longrightarrow k^{\times+}/K^{\times+} ,$$

where $\operatorname{Im}\delta_1 = U_k \cap NK_A^\times/NU_K$, $\operatorname{Im}\delta_2 = \mathcal{O}_k^{\times+} \cap NK^{\times+}/N\mathcal{O}_K^{\times+}$ and these are finite abelian groups. Therefore we have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^{(1)+}/\mathcal{O}_K^{(1)+} & \longrightarrow & P_K^{(1)+} & \longrightarrow & \mathcal{O}_k^{\times+} \cap NK^{\times+}/N\mathcal{O}_K^{\times+} & \longrightarrow & 1 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 1 & \longrightarrow & K_A^{(1)}/U_K^{(1)} & \longrightarrow & I_K^{(1)} & \longrightarrow & U_k \cap NK_A^\times/NU_K & \longrightarrow & 1 \end{array};$$

hence, by Lemma 1, we have

$$\begin{aligned}[I_K^{(1)}: P_K^{(1)+}] &= |\operatorname{Coker} f| = q(f) = q(f')q(f'') \\ &= [K_A^{(1)}: K^{(1)+}\cdot U_K^{(1)}]\frac{[U_k \cap NK_A^\times : NU_K]}{[\mathcal{O}_k^{\times+} \cap NK^{\times+} : N\mathcal{O}_K^{\times+}]} .\end{aligned}$$

Q.E.D.

We define Ono invariants by

$$E(K/k) = \frac{h(K)}{h(k)\cdot h(K/k)}$$

and

$$E^+(K/k) = \frac{h^+(K)}{h^+(k)\cdot h^+(K/k)}$$

where $h^+(k)$ is the class number in the narrow sense, i.e., the order of the group $H_k^+ = I_k/P_k^+$ and so on. By Propositions 1 and 2, we see that $E(K/k)$ is nothing but the original one defined by Ono (cf. [5, 7]). The first part of the following Theorem 1 is due to T. Ono and the second one is our main theorem, whose proof becomes a simpler one than the proof given by Ono (cf. [5, 7]), which is our motivation to write this paper.

THEOREM 1. *Let K/k be a finite Galois extension of finite algebraic number fields with Galois group $\mathfrak{g}$; then we have*

$$E(K/k) = \frac{|H^0(\mathfrak{g}, U_K)|\,|\operatorname{Ker}(H^0(\mathfrak{g}, K^\times) \longrightarrow H^0(\mathfrak{g}, K_A^\times))|}{[\mathfrak{g}:\mathfrak{g}']\cdot|H^0(\mathfrak{g}, \mathcal{O}_K^\times)|}$$

and

$$E^+(K/k) = \frac{|H^0(\mathfrak{g}, U_K)|\,|\operatorname{Ker}(H^0(\mathfrak{g}, K^\times) \longrightarrow H^0(\mathfrak{g}, K_A^\times))|}{[\mathfrak{g}:\mathfrak{g}']\cdot[\mathcal{O}_k^{\times+}:\mathcal{O}_K^{\times+}]q(\varphi)}$$

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where $\varphi: k^{\times +}/NK^{\times +} \to k^{\times}/NK^{\times}$ *is the canonical homomorphism.*

Proof. We shall prove the second formula only. Let $\tilde{a}: k^{\times}/NK^{\times} \to k_A^{\times}/NK_A^{\times}$ be a canonical homomorphism; then, by Lemma 3, we have $q(\tilde{a}\cdot\varphi) = q(\varphi)\cdot q(\tilde{a})$ and we put $\tilde{a}\cdot\varphi = a$. Consider the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_k^{\times +}/\mathcal{O}_k^{\times +} \cap NK^{\times +} & \longrightarrow & k^{\times +}/NK^{\times +} & \longrightarrow & k^{\times +}/\mathcal{O}_k^{\times +}\cdot NK^{\times +} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a''} & & \\ 1 & \longrightarrow & U_k/U_k \cap NK_A^{\times} & \longrightarrow & k_A/NK_A & \longrightarrow & k_A^{\times}/U_k\cdot NK_A^{\times} & \longrightarrow & 1\,; \end{array}$$

then we have

$$(0) \qquad q(a) = q(\varphi)\cdot q(\tilde{a}) = q(a')q(a'')\,.$$

First we shall compute $q(a')$. Applying Lemmas 1 and 2 to the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}_k^{\times +} \cap NK^{\times +}/N\mathcal{O}_K^{\times +} & \longrightarrow & \mathcal{O}_k^{\times +}/N\mathcal{O}_K^{\times +} & \longrightarrow & \mathcal{O}_k^{\times +}/\mathcal{O}_k^{\times +} \cap NK^{\times +} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a'} & & \\ 1 & \longrightarrow & U_k \cap NK_A^{\times}/NU_K & \longrightarrow & U_k/NU_K & \longrightarrow & U_k/U_k \cap NK_A^{\times} & \longrightarrow & 1\,, \end{array}$$

we have

$$(1) \qquad q(a') = \frac{q(b)}{q(b')} = \frac{[U_k : NU_K]}{[\mathcal{O}_k^{\times +} : N\mathcal{O}_K^{\times +}]}\cdot\frac{[\mathcal{O}_k^{\times +} \cap NK^{\times +} : N\mathcal{O}_K^{+}]}{[U_k \cap NK_A^{\times} : NU_K]}$$

where $[U_k : NU_k] = |H^0(\mathfrak{g}, U_k)|$, because $U_k = U_K^{\mathfrak{g}}$.

Secondly we shall compute $q(a'')$. Applying the snake lemma to the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_K^{+} & \longrightarrow & I_K & \longrightarrow & H_K^{+} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N'} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N''} & & \\ 1 & \longrightarrow & P_k^{+} & \longrightarrow & I_k & \longrightarrow & H_k^{+} & \longrightarrow & 1\,, \end{array}$$

we have an exact sequence:

$$\begin{array}{l} 1 \longrightarrow P_K^{(1)+} \longrightarrow I_K^{(1)} \longrightarrow \operatorname{Ker} N'' \\ \qquad \xrightarrow{\ \delta\ } P_k^{+}/NP_K^{+} \xrightarrow{\ c\ } I_k/NI_K \longrightarrow \operatorname{Coker} N'' \longrightarrow 1 \\ \qquad\qquad \downarrow\wr \qquad\quad \circlearrowleft \qquad\quad \downarrow\wr \\ \qquad k^{\times +}/\mathcal{O}^{\times +}\cdot NK^{\times +} \xrightarrow[a'']{} k_A^{\times}/U_K\cdot NK_A^{\times}\;. \end{array}$$

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Therefore we have

$$(2)\quad \begin{aligned} q(a'') &= q(c) = |\mathrm{Coker}\, N''|/|\mathrm{Ker}\, c| = |\mathrm{Coker}\, N''|\cdot[I_K^{(1)}\colon P_K^{(1)+}]/\mathrm{Ker}\, N'' \\ &= q(N'')\cdot[I_K^{(1)}\colon P_K^{(1)+}] = [I_K^{(1)}\colon P_K^{(1)}]h^+(k)/h^+(K)\,. \end{aligned}$$

Thirdly we shall compute $q(a)$. From the exact sequence

$$1 \longrightarrow K^\times \longrightarrow K_A^\times \longrightarrow C_K \longrightarrow 1$$

where C_K is the idele class group, we have a long exact sequence:

$$\begin{array}{l} \cdots \longrightarrow H^{-1}(\mathfrak{g}, C_K) \longrightarrow H^0(\mathfrak{g}, K^\times) \xrightarrow{\tilde{a}} H^0(\mathfrak{g}, K_A^\times) \longrightarrow \\ \qquad\qquad\qquad\qquad\quad\; \wr\| \qquad\qquad\qquad \wr\| \\ \qquad\qquad\qquad\qquad k^\times/NK^\times \qquad\quad k^\times/NK_A^\times \\ \qquad\qquad\qquad\qquad\qquad \longrightarrow H^0(\mathfrak{g}, C_K) \longrightarrow H^1(\mathfrak{g}, K^\times) \longrightarrow \cdots \end{array}$$

where $H^1(\mathfrak{g}, K^\times) = \{1\}$ and $H^0(\mathfrak{g}, C_K) \cong \mathfrak{g}/\mathfrak{g}'$ (cf. [1]). Thus we have

$$(3)\qquad q(a) = q(\varphi)\cdot q(\tilde{a}) = q(\varphi)\,\frac{[\mathfrak{g}\colon \mathfrak{g}']}{[\mathrm{Ker}\,\tilde{a}]}\,.$$

Combining (0), (1), (2), (3), we have the formula. Q.E.D.

§ 4. Cyclic extensions

Let K/k be a finite cyclic extension of finite algebraic number fields with Galois group $\mathfrak{g} = \langle\sigma\rangle$ and $|\mathfrak{g}| = n$. We denote by ρ the number of real infinite places ramified in K/k. We put

$$e = \prod_{\mathfrak{p}} e_{\mathfrak{P}}$$

where $\mathfrak{p}$ runs over all finite places and $e_{\mathfrak{P}}$ is the ramification index of any place $\mathfrak{P}$ lying over $\mathfrak{p}$.

LEMMA 4. *Let K/k be a finite cyclic extension; then we have*

$$[U_k \cap NK_A^\times\colon NU_K] = 1\,.$$

Proof. We use the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_K & \longrightarrow & K_A^\times & \xrightarrow{\pi_K} & I_K & \longrightarrow & 1 \\ & & \downarrow N & & \downarrow N & & \downarrow N & & \\ 1 & \longrightarrow & U_k & \longrightarrow & k_A^\times & \xrightarrow[\pi_k]{} & I_k & \longrightarrow & 1\,. \end{array}$$

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Let a be an idele in $K_A^\times$ such that $Na \in U_k$. Set $\pi_K(a) = \mathfrak{a}$, then $(1) = \pi_k(Na) = N(\pi_K(a)) = N\mathfrak{a}$. By Hilbert's Theorem 90 for ideals, we get an ideal $\mathfrak{q}$ in I_K such that $\mathfrak{a} = \mathfrak{q}^{1-\sigma}$. If b is an idele in $K_A^\times$ such that $\pi_K(b) = \mathfrak{q}$, then we have $\pi_K(a \cdot (b^{1-\sigma})^{-1}) = \mathfrak{a} \cdot \mathfrak{a}^{-1} = (1)$; hence there exists $u \in U_K$ such that $a = b^{1-\sigma} \cdot u$. Therefore we have $Na = N(b^{1-\sigma})N(u) = Nu \in NU_K$.

Q.E.D.

LEMMA 5. *Let φ: $k^{\times +}/NK^{\times +} \to k^\times/NK^\times$ be a canonical homomorphism for a finite cyclic extension K/k; then we have*

$$q(\varphi) = 2^\rho .$$

Proof. We denote by μ_2 the group consisting of $+1$ and -1. Let $\{\sigma_1, \cdots, \sigma_r\}$ be the set of real imbedding of k and $\{\sigma_i^{(j)} | 1 \leqq i \leqq r - \rho, 1 \leqq j \leqq n = |\mathfrak{g}|\}$ the set of real imbedding of K where $\{\sigma_i^{(j)} | 1 \leqq j \leqq n\}$ is the set of extensions of σ_i to K. Define S_K: $K \to \mu_2^R$, $R = n(r - \rho)$, by

$$S_K(\alpha) = (\operatorname{sgn} \sigma_1^{(1)}(\alpha), \cdots, \operatorname{sgn} \sigma_{r-\rho}^{(n)}(\alpha)) ,$$

and S_k: $k \to \mu_2^r$ by

$$S_k(\alpha) = (\operatorname{sgn} \sigma_1(\alpha), \cdots, \operatorname{sgn} \sigma_r(\alpha)) ,$$

then we have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^{\times +} & \longrightarrow & K^\times & \xrightarrow{S_K} & \mu_2^R & \longrightarrow & 1 \\ & & \downarrow N & & \downarrow N & & \downarrow N'' & & \\ 1 & \longrightarrow & k^{\times +} & \longrightarrow & k^\times & \xrightarrow[S_k]{} & \mu_2^r & \longrightarrow & 1 \end{array}$$

where N'' is the homomorphism defined by

$$N''(\varepsilon_1^{(1)}, \cdots, \varepsilon_1^{(n)}, \cdots, \varepsilon_{r-\rho}^{(1)}, \cdots, \varepsilon_{r-\rho}^{(n)}) = (\varepsilon_1^{(1)} \cdots \varepsilon_1^{(n)}, \cdots, \varepsilon_{r-\rho}^{(1)} \cdots \varepsilon_{r-\rho}^{(n)}, \overbrace{1, \cdots, 1}^{\rho\text{-times}}) .$$

Applying the snake lemma to this, we have an exact sequence:

$$\cdots \longrightarrow \operatorname{Ker} N'' \longrightarrow k^{\times +}/NK^{\times +} \xrightarrow{\varphi} k^\times/NK^\times \longrightarrow \operatorname{Coper} N'' \longrightarrow 1 .$$

By Proposition 1.1 in [2], we have $|\operatorname{Ker} \varphi| = 1$. Therefore we have

$$q(\varphi) = |\operatorname{Coker} N''| = 2^\rho .$$

Q.E.D.

THEOREM 2. *Let K/k be a finite cyclic extension; then we have*

$$E(K/k) = \frac{2^{\rho}}{n\cdot|H^0(\mathfrak{g}, \mathcal{O}_K^\times)|} = \frac{e}{|H^1(\mathfrak{g}, \mathcal{O}_K^\times)|} = \frac{e}{[\mathcal{O}_K^{(1)} : (\mathcal{O}_K^\times)^{-\sigma}]}$$

and

$$E^+(K/k) = \frac{e}{n\cdot[\mathcal{O}_k^{\times +} : N\mathcal{O}_K^{\times +}]}\,.$$

Proof. Since K/k is cyclic, we have

$$|\mathrm{Ker}\,(H^0(\mathfrak{g}, K^\times) \longrightarrow H^0(\mathfrak{g}, K_A^\times))| = 1$$

and

$$H^0(\mathfrak{g}, U_K) = e\cdot 2^{\rho}$$

(cf. [1]). As is well known, we have

$$H^1(\mathfrak{g}, \mathcal{O}_K^\times) = n\cdot 2^{-\rho} H^0(\mathfrak{g}, \mathcal{O}_K^\times)$$

(cf. [10] CH. 13). Therefore, by Theorem 1, we have the first two equalities in the first formula. Since the following two homomorphisms are isomorphisms:

$$\mathcal{O}_K^{(1)} = \{u \in \mathcal{O}_K^\times | Nu = 1\} \longrightarrow Z^1(\mathfrak{g}, \mathcal{O}_K^\times) \qquad (u \longmapsto (u, u^{1+\sigma}, \cdots, Nu))$$

and

$$(\mathcal{O}_k^\times)^{1-\sigma} \longrightarrow B^1(\mathfrak{g}, \mathcal{O}_K) \qquad (u^{1-\sigma} \longmapsto (u^{1-\sigma}, (u^{1-\sigma})^{1+\sigma}, \cdots, N(u^{1-\sigma})))\,,$$

we have the third equality.

Now we shall prove the second formula. By Theorem 1, Lemma 5 and the above argument, we have

$$E^+(K/k) = \frac{e\cdot 2^{\rho}}{n\cdot[\mathcal{O}_k^{\times +} : N\mathcal{O}_K^{\times +}]\cdot q(\varphi)} = \frac{e}{n\cdot[\mathcal{O}_k^{\times +} : N\mathcal{O}_K^{\times +}]}\,.$$

Q.E.D.

Let $a(K/k)$ and $a^+(K/k)$ be the numbers of ambigous classes, i.e., the numbers of ideal classes in H_K and H_K^+ invariant under the action of $\mathfrak{g}$.

COROLLARY. *Notation being as above, we have*

$$a(K/k) = \frac{2^{\rho}\cdot e\cdot h(k)}{n\cdot[\mathcal{O}_k^\times : \mathcal{O}_k^\times \cap NK^\times]}$$

and

$$a^+(K/k) = \frac{e\cdot h^+(k)}{n\cdot[\mathcal{O}_k^{\times +} : \mathcal{O}_k^{\times +} \cap NK^\times]}\,.$$

Proof. By Proposition 2, Lemma 4 and Theorem 2, we have the above two formulas. Q.E.D.

Remark. The first formula in the Corollary is classical (e.g. cf. [10]). The second one is proved by G. Gras [2] in the case where K/k is a cyclic extension of a prime degree.

§ 5. Cyclotomic fields*⁾

Let K be a CM field, i.e., a totally imaginary number field containing a totally real subfield K^+ with $[K: K^+] = 2$. We denote by W_K the group of roots of unity in K. Define a homomorphism

$$g: \mathcal{O}_K^\times \longrightarrow \mathcal{O}_K^\times$$

by $g(u) = u/u^J$, where J is the complex conjugation. Then g induces an isomorphism

$$\mathcal{O}_K^\times/\mathcal{O}_{K^+}^\times \cdot W_K \longrightarrow g(\mathcal{O}_K^\times)/g(W_K)$$

and we have $g(\mathcal{O}_K^\times) \subset W_K$, $\operatorname{Ker} g = \mathcal{O}_{K^+}^\times$ and $g(W_K) = W_K^2$. We denote by Q the index $[\mathcal{O}_K^\times: \mathcal{O}_{K^+}^\times \cdot W_K]$; then it is equal to 1 or 2. For details we refer to [11].

THEOREM 3. *Let K and K^+ be a CM field and its maximal real subfield; then we have*

$$E(K/K^+) = 2^{t-1} \cdot Q$$

where t is the number of finite places ramified in K/K^+.

Proof. If a unit u in $\mathcal{O}_K$ satisfies $u^{1+J} = u \cdot \bar{u} = 1$, then any conjugation u^σ ($\sigma \in \operatorname{Gal}(K/\boldsymbol{Q})$) of u satisfies $|u^\sigma|^2 = u^\sigma \cdot \overline{u^\sigma} = u^\sigma \cdot (\bar{u})^\sigma = (u \cdot \bar{u})^\sigma = 1$; hence, by Kronecker's Theorem, we have $u \in W_K$. Since $W_K \subset \mathcal{O}_K^{(1)} = \{u \in \mathcal{O}_K^\times | u^{1+J} = 1\}$, we have $W_K = \mathcal{O}_K^{(1)}$; hence we have

$$\mathcal{O}_K^{(1)}/(\mathcal{O}_K^\times)^{1-J} \cong W_K/g(\mathcal{O}_K^\times) .$$

Therefore we have, by Theorem 2,

$$E(K/K^+) = 2^t/|W_K/g(\mathcal{O}_K^\times)| = 2^t[g(\mathcal{O}_K^\times): W_K^2]/[W_K: W_K^2] = 2^{t-1} \cdot Q .$$

Q.E.D.

*⁾ T. Ono obtained the results in this section.

COROLLARY. *Let K_n and K_n^+ be the n-th cyclotomic field and its maximal real subfield; then we have*

$$E(K_n/K_n^+) = \frac{h(K_n)}{h(K_n^+)\cdot h(K_n/K_n^+)} = 1\,.$$

Proof. If n is odd, we have $K_n = K_{2n}$; hence we may assume that $n \equiv 1 \pmod 2$ or $n \equiv 0 \pmod 4$. If n is equal to a power p^m of a prime p, then $Q = 1$ and $K_{p^m}/K_{p^m}^+$ is ramified at the only prime lying above p. If n is not a power of a prime, then $Q = 2$ and K_n/K_n^+ is unramified except at the infinite places (cf. [11]). Q.E.D.

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