ON SELBERG'S LEMMA FOR ALGEBRAIC FIELDS

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1. Introduction. Recently two Japanese authors **(1)** gave a beautifully simple proof of Selberg's fundamental lemma in the theory of distribution of primes.[1] The proof is based on a curious twist in the Möbius inversion formula. The object of this note is to show that their proof may be extended to a proof of the result for algebraic fields corresponding to Selberg's lemma. Shapiro **(2)** has already derived this result using Selberg's methods and deduced as a consequence the prime ideal theorem.

Let K be an algebraic extension of the rationals of degree k, and denote by $N(\mathfrak{a})$ the norm of the ideal $\mathfrak{a}$ and by $\mathfrak{p}$, $\mathfrak{p}_i$ etc., prime ideals of K.

We define $\mu(\mathfrak{a})$ and $\Lambda(\mathfrak{a})$ as in the case of the rational field, viz.,

$$\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = 1 \\ (-1)^r & \text{if } \mathfrak{a} = \mathfrak{p}_1 \ldots \mathfrak{p}_r, \text{ the } \mathfrak{p}_i \text{ all different,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\Lambda(\mathfrak{a}) = \begin{cases} \log N(\mathfrak{a}) & \text{if } \mathfrak{a} \text{ is a power of a prime ideal } \mathfrak{p}. \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to deduce that

$$\sum_{\mathfrak{d} | \mathfrak{a}} \mu(\mathfrak{d}) = \begin{cases} 0 & \text{if } \mathfrak{a} \neq 1 \\ 1 & \text{if } \mathfrak{a} = 1 \end{cases} \tag{1}$$

and

$$\sum_{\mathfrak{d} | \mathfrak{a}} \Lambda(\mathfrak{d}) = \log N(\mathfrak{a}). \tag{2}$$

The Möbius inversion formula is valid, i.e. if

$$f(\mathfrak{a}) = \sum_{\mathfrak{d} | \mathfrak{a}} g(\mathfrak{d})$$

then

$$g(\mathfrak{a}) = \sum_{\mathfrak{d} | \mathfrak{a}} \mu(\mathfrak{d}) f\left(\frac{\mathfrak{a}}{\mathfrak{d}}\right).$$

It follows that

$$\begin{aligned} \Lambda(\mathfrak{a}) &= \sum_{\mathfrak{d} | \mathfrak{a}} \mu(\mathfrak{d}) \log\left(\frac{N(\mathfrak{a})}{N(\mathfrak{d})}\right) \\ &= -\sum_{\mathfrak{d} | \mathfrak{a}} \mu(\mathfrak{d}) \log N(\mathfrak{d}). \end{aligned} \tag{3}$$

Define

$$\psi(x) = \sum_{N(\mathfrak{a}) \leqslant x} \Lambda(\mathfrak{a}).$$

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[1]This proof was brought to my attention by Dr. Leo Moser of the University of Alberta.

It is our object to give a new proof of

SELBERG'S LEMMA:

$$\psi(x)\log x + \sum_{N(\mathfrak{a})\leqslant x} \Lambda(\mathfrak{a})\,\psi\left(\frac{x}{N(\mathfrak{a})}\right) = 2x\log x + O(x).$$

The proof is based on the next theorem which is the essence of the Japanese method; a factor $\log x$ is introduced in the Möbius transform with interesting consequences.

THEOREM 1.1. *If*

$$f(x) = \sum_{N(\mathfrak{a})\leqslant x} h\left(\frac{x}{N(\mathfrak{a})}\right)\log x$$

then

(4) $$\sum_{N(\mathfrak{a})\leqslant x} \mu(\mathfrak{a})\,f\left(\frac{x}{N(\mathfrak{a})}\right) = h(x)\log x + \sum_{N(\mathfrak{a})\leqslant x} \Lambda(\mathfrak{a})\,h\left(\frac{x}{N(\mathfrak{a})}\right).$$

Proof.

$$\begin{aligned}\sum_{N(\mathfrak{a})\leqslant x} \mu(\mathfrak{a})\,f\left(\frac{x}{N(\mathfrak{a})}\right) &= \sum_{N(\mathfrak{a})\leqslant x} \mu(\mathfrak{a}) \sum_{N(\mathfrak{b})\leqslant x/N(\mathfrak{a})} h\left(\frac{x}{N(\mathfrak{a})\,N(\mathfrak{b})}\right)\log\frac{x}{N(\mathfrak{b})}\\ &= \sum_{N(\mathfrak{c})\leqslant x} h\left(\frac{x}{N(\mathfrak{c})}\right)\sum_{\mathfrak{b}\mid\mathfrak{c}} \mu(\mathfrak{b})\log\left(\frac{x}{N(\mathfrak{b})}\right)\\ &= h(x)\log x + \sum_{N(\mathfrak{c})\leqslant x} h\left(\frac{x}{N(\mathfrak{c})}\right)\Lambda(\mathfrak{c}),\end{aligned}$$

by (1) and (3).

2. Some estimates. We make the following abbreviation: we denote simply by the index a summation over the range 0 to x, for example,

$$\sum_{\mathfrak{a}} f(\mathfrak{a}) \text{ means } \sum_{N(\mathfrak{a})\leqslant x} f(\mathfrak{a}), \text{ while } \sum_{n} f(n) \text{ means } \sum_{n\leqslant x} f(n).$$

In other cases the range of summation will be specified. We sometimes use the notation $A \ll B$ to mean $A = O(B)$. We assume known the following classical result of Weber **(3)**:

(5) $$[x] = \sum_{\mathfrak{a}} 1 = g\,x + a(x),$$

where $a(x) = O(x^{1-m})$ with $m = 1/k$, g is the residue of $\zeta_k(s)$ at $s = 1$, i.e.,

$$g = \frac{2^{r_1+r_2}\pi^{r_2}R}{w\sqrt{|d|}}\,h.$$

Here r_1 and r_2 are the numbers of real and pairs of complex conjugate fields, w is the order of the group of roots of unity, d is the discriminant, R the regulator and h is the class number.

THEOREM 2.1.

$$\sum_{\mathfrak{a}} N(\mathfrak{a})^{-1} = g\log x + c + O(x^{-m}).$$

Proof. Using (5), we get

$$\begin{aligned}\sum_{\mathfrak{a}} N(\mathfrak{a})^{-1} &= \sum_n \frac{[n]-[n-1]}{n} \\ &= \sum_n \frac{gn - g(n-1)}{n} + \sum_n \frac{a(n)-a(n-1)}{n} \\ &= g\sum_n \frac{1}{n} + \sum_n a(n)\left(\frac{1}{n} - \frac{1}{n+1}\right) + O(x^{-m}) \\ &= g\log x + g\gamma + O(x^{-1}) + O\left(\sum_{n=1}^{\infty} n^{-1-m}\right) + O(x^{-m}) \\ &= g\log x + g\gamma + O(1) + O(x^{-m}) \\ &= g\log x + c + O(x^{-m}),\end{aligned}$$

where γ is Euler's constant.

THEOREM 2.2.

$$\sum_{\mathfrak{a}} N(\mathfrak{a})^{v-1} = O(x^v) \qquad \text{if } 0 < v \leqslant 1.$$

Proof. Using (5) again,

$$\begin{aligned}\sum_{\mathfrak{a}} N(\mathfrak{a})^{v-1} &= \sum_n ([n]-[n-1])\, n^{v-1} \\ &\ll \sum_n n^{v-1} + \sum_n \{a(n) - a(n-1)\}\, n^{v-1} \\ &\ll x^v + \sum_n n^{v-m}\left\{1 - \left(1+\frac{1}{n}\right)^{v-1}\right\} \\ &\ll x^v + \sum_n n^{v-m-1} \\ &\ll x^v + x^{v-m}\log x \\ &\ll x^v.\end{aligned}$$

THEOREM 2.3.

$$\sum_{\mathfrak{a}} \log N(\mathfrak{a}) = g\,x\log x - g\,x + O(x^{1-m}\log x).$$

Proof. By (5),

$$\begin{aligned}\sum_{\mathfrak{a}} \log N(\mathfrak{a}) &= \sum_n ([n]-[n-1])\log n \\ &= g\sum_n \log n + \sum_n \{a(n)-a(n-1)\}\log n.\end{aligned}$$

The second sum, however, is

$$\begin{aligned}&\ll \sum_n a(n)\log\left(1+\frac{1}{n}\right) + x^{1-m}\log x \\ &\ll \sum_n n^{-m} + x^{1-m}\log x \\ &\ll x^{1-m}\log x.\end{aligned}$$

Consequently,

$$\sum_{\mathfrak{a}} \log N(\mathfrak{a}) = g\,x\log x - g\,x + O(x^{1-m}\log x).$$

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The object of the next paragraph is to prove

THEOREM 2.4.

$$\sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})} = \log x + O(1).$$

Shapiro's proof is based on several auxiliary results which are needed for the proof of Selberg's lemma. We prove the theorem here directly, using Chebychev's ideas. We first notice that

$$\sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})} = \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} + \sum_{\mathfrak{p}^2} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^2} + \dots$$
$$= \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} + O(1).$$

It is therefore enough to show that the sum on the right is $\log x + O(1)$. The number of ideals $\mathfrak{a}$ with $N(\mathfrak{a}) \leqslant x$ and divisible by a prime ideal $\mathfrak{p}$ is $[x/N(\mathfrak{p})]$ and so on for $\mathfrak{p}^2$ etc. Hence

$$\prod_{\mathfrak{a}} N(\mathfrak{a}) = \prod_{\mathfrak{p}} N(\mathfrak{p})^{\left[\frac{x}{N(\mathfrak{p})}\right] + \left[\frac{x}{N(\mathfrak{p})^2}\right] + \dots}$$

and

$$\sum_{\mathfrak{a}} \log N(\mathfrak{a}) = \sum_{\mathfrak{p}} \log N(\mathfrak{p})\left\{\left[\frac{x}{N(\mathfrak{p})}\right] + \left[\frac{x}{N(\mathfrak{p})^2}\right] + \dots\right\} \tag{6}$$
$$= \sum_{\mathfrak{p}} \log N(\mathfrak{p})\left[\frac{x}{N(\mathfrak{p})}\right]$$
$$+ O(x) \sum_{\mathfrak{p}} \log N(\mathfrak{p}) \{N(\mathfrak{p})^{-2} + N(\mathfrak{p})^{-3} + \dots\}$$
$$= g\,x \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} + O(x^{1-m}) \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{1-m}}$$
$$+ O(x) \sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^2}.$$

The third sum on the right is $O(1)$; we now evaluate the second one. For this purpose we introduce the function $\theta(x) = \sum_{\mathfrak{p}} \log N(\mathfrak{p})$. Since $N(\mathfrak{p})$ is at most p^k for some rational prime p, we conclude that $\theta(x)$ is $O(\sum_p \log p) = O(x)$. Hence

$$\sum_{\mathfrak{p}} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{1-m}} = \sum \frac{\theta(n) - \theta(n-1)}{n^{1-m}}$$
$$\ll \sum_n \theta(n)\{n^{m-1} - (n+1)^{m-1}\} + x^m$$
$$\ll \sum_n n^{m-1} + x^m$$
$$\ll x^m.$$

Using Theorem 2.3 and (6), we deduce Theorem 2.4.

THEOREM 2.5.

$$\sum_{\mathfrak{a}} \psi\left(\frac{x}{N(\mathfrak{a})}\right) = g\,x \log x - g\,x + O(x^{1-m} \log x).$$

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Proof.

$$\sum_{\mathfrak{a}} \psi\left(\frac{x}{N(\mathfrak{a})}\right) = \sum_{\mathfrak{a}} \sum_{\mathfrak{a}\mathfrak{b}} \Lambda(\mathfrak{b})$$
$$= \sum_{\mathfrak{c}} \sum_{\mathfrak{b}|\mathfrak{c}} \Lambda(\mathfrak{b})$$
$$= \sum_{\mathfrak{c}} \log N(\mathfrak{c})$$
$$= g\,x \log x - g\,x + O(x^{1-m} \log x),$$

using (2) and Theorem 2.3.

3. Proof of Selberg's Lemma. In (4), we put $h(x) = \psi(x) - x + c/g + 1$, where c is the constant of Theorem 2.1. Then

$$h(x) \log x + \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\, h\left(\frac{x}{N(\mathfrak{a})}\right)$$
$$= \log x \left\{\psi(x) - x + \frac{c}{g} + 1\right\} + \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\, \psi\left(\frac{x}{N(\mathfrak{a})}\right) - x \sum_{\mathfrak{a}} \frac{\Lambda(\mathfrak{a})}{N(\mathfrak{a})} + O(\psi(x))$$
$$= \log x\, \psi(x) + \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\, \psi\left(\frac{x}{N(\mathfrak{a})}\right) - 2x \log x + O(x) + O(\psi(x)),$$

by Theorem 2.4. On the other hand,

$$f(x) = \log x \left[\sum_{\mathfrak{a}} \psi\left(\frac{x}{N(\mathfrak{a})}\right) - x \sum_{\mathfrak{a}} N(\mathfrak{a})^{-1} + \left(\frac{c}{g} + 1\right) \sum_{\mathfrak{a}} 1\right]$$
$$= \log x \{g\,x \log x - g\,x + O(x^{1-m} \log x) - g\,x \log x - cx - O(x^{-m+1}) + \left(\frac{c}{g} + 1\right)(g\,x - O(x^{1-m}))\}$$
$$= O(x^{1-m} \log^2 x) = O(x^{1-\frac{1}{2}m}),$$

by (5) and Theorems 2.1 and 2.5. Consequently

$$\sum_{\mathfrak{a}} \mu(\mathfrak{a}) f\left(\frac{x}{N(\mathfrak{a})}\right) = O(x^{1-\frac{1}{2}m}) \sum_{\mathfrak{a}} N(\mathfrak{a})^{-1+\frac{1}{2}m}$$
$$= O(x),$$

by Theorem 2.2.

Combining these results, we conclude that

$$\psi(x) \log x + \sum_{\mathfrak{a}} \Lambda(\mathfrak{a})\, \psi\left(\frac{x}{N(\mathfrak{a})}\right) = 2x \log x + O(x) + O(\psi(x)).$$

Since $\theta(x) = O(x)$, then $\psi(x) = O(x)$, but it will be noticed that this fact is a consequence of the above inequality. The proof of the theorem is therefore complete.

REFERENCES

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