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A perfect Morse function for the moduli space of flat connections

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Abstract. We show that the cohomology of the moduli space of flat $\mathrm{SU}(2)$ connections on a two-manifold may be computed using a perfect Morse function.

Key words: Moduli space, flat connections, Morse function.

Let Σ^g be a Riemann surface of genus $g > 1$. The moduli space $S_g(-1)$ of semistable holomorphic vector bundles of rank 2, degree 1, and fixed determinant on Σ^g may be described as follows. Let $R_g \subset \mathrm{SU}(2)^{2g}$ be defined by

$$R_g = \Big\{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in \mathrm{SU}(2)^{2g}: \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = -1\Big\}. \tag{1}$$

The group $\mathrm{SU}(2)$ acts freely on R_g by simultaneous conjugation: $A_i \to g^{-1}A_i g$, $B_i \to g^{-1}B_i g$, for $g \in \mathrm{SU}(2)$; and the quotient $R_g/\mathrm{SU}(2)$ may be identified by the Narasimhan-Seshadri theorem with $S_g(-1)$. This moduli space is therefore a smooth manifold of real dimension $6g - 6$; it possesses a symplectic structure which may be defined using only the structure of Σ^g as a smooth manifold and is independent of its Riemann surface structure, and a Kähler structure which does depend on the Riemann surface structure of Σ^g. The space $S_g(-1)$ may be viewed therefore as a moduli space of representations $\rho \in \mathrm{Hom}(\pi_1(\Sigma^g \backslash \{p\}), \mathrm{SU}(2))$, where $p \in \Sigma^g$, and where $\rho(c) = -1$, where c is the element of $\pi_1(\Sigma^g\backslash\{p\})$ which may be represented by an oriented curve traversing the boundary of a disc containing p.

The cohomology of $S_g(-1)$ has been extensively studied in the literature. The Betti numbers of $S_g(-1)$ were computed by Newstead in [N]; the Poincaré

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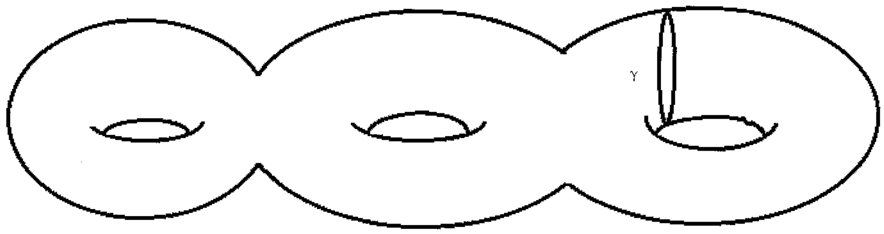

Figure 1. The surface Σ and a fixed nonseparating curve γ.

polynomials of this space, as well as of moduli spaces associated to higher-rank vector bundles, were computed by Harder [H], Harder–Narasimhan [HN], Desale–Ramanan [DR], and Atiyah–Bott [AB]. In this paper we show how the Poincaré polynomial of $S_g(-1)$ may be obtained from a perfect Morse function $f : S_g(-1) \to \mathbf{R}$. Our proof is *a posteriori*: we compute the Morse polynomial of f and recognize that it is identical with the (known) Poincaré polynomial of $S_g(-1)$. It would be interesting to construct an *a priori* argument for this function being perfect; this might enable one to understand whether our methods might extend to moduli spaces associated to higher-rank vector bundles.* Now to define our function $f : S_g(-1) \to \mathbf{R}$.

Let $\tilde{f} : R_g \to \mathbf{R}$ be given by

$$\tilde{f}((A_1, \ldots, A_g, B_1, \ldots, B_g)) = \text{trace}(A_g). \tag{2}$$

Then $\tilde{f}$ is conjugation-invariant and hence descends to a function $f : S_g(-1) \to \mathbf{R}$. If we view $S_g(-1)$ as a moduli space of representations of $\pi_1(\Sigma^g \backslash \{p\})$, the function f assigns to each equivalence class $[\rho]$ of such representations the trace tr $\rho(\gamma)$ of the value of ρ on the homotopy class of a fixed *nonseparating* simple closed curve $\gamma \in \Sigma^g$ (See figure 1).

Our main result is as follows:

THEOREM. *The function f is a perfect Morse function on $S_g(-1)$.*

Proof. We study the critical values of f. There are two obvious critical values, corresponding to the minimum of f, attained where $f = -2$, and the maximum of f, attained where $f = 2$. These are easily seen to be nondegenerate. Any other critical values of f occur where $-2 < f < 2$, and are also critical values for the function $\mu = 1/\pi \cos^{-1} \frac{1}{2} f$. But by the results of [D, JW], the function $\mu|_{f^{-1}((-2,2))}$ is the moment map for a circle action on $f^{-1}((-2,2))$, and hence its critical manifolds correspond to the fixed manifolds of this circle action. These were computed by Donaldson in [D]; they are given by the image C_g in $S_g(-1)$ of

* While this paper was being revised for publication we learned of recent work of Thaddeus [T] which gives such an *a priori* proof.

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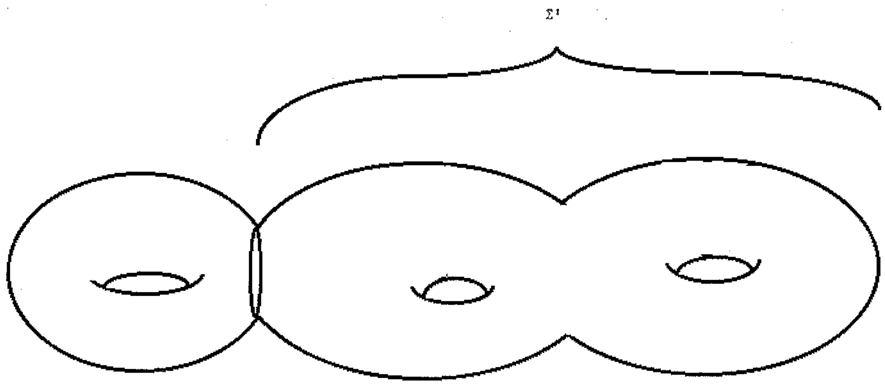

Figure 2. The subsurface Σ' on which representations corresponding to points in V_g are reducible.

the subvariety $V_g \subset R_g$ defined by

$$V_g = \{(A_1, \dots, A_g, B_1, \dots, B_g) \in T^{2g-1} \times \mathrm{SU}(2) \subset \mathrm{SU}(2)^{2g} : \mathrm{tr}\, A_g = 0\} \cap R_g, \tag{3}$$

where $T \subset \mathrm{SU}(2)$ denotes a fixed maximal torus of $\mathrm{SU}(2)$. Geometrically, consider the two-manifold Σ' of genus $g-1$ obtained by removing one handle from Σ^g (see figure 2). Then V_g corresponds to representations of $\pi_1(\Sigma^g \backslash \{p\})$ which send the homotopy classes in $\pi_1(\Sigma^g - \{p\})$ represented by loops lying entirely in the two-manifold Σ' to elements of T.

In any event, the corresponding critical manifold C_g is immediately non-degenerate (as it is a fixed point set of a Hamiltonian circle action).

Let us now compute the Poincaré polynomials and indices of these critical manifolds. The maximum $f^{-1}(2)$ is given by the image in $S_g(-1)$ of the subvariety $M_g \subset R_g$ given by

$$M_g = \left\{(A_1, \dots, A_{g-1}, 1, B_1, \dots, B_g) \in \mathrm{SU}(2)^{2g} : \prod_{i=1}^{g-1} A_i B_i A_i^{-1} B_i^{-1} = -1\right\}. \tag{4}$$

We see that $M_g = R_{g-1} \times \mathrm{SU}(2)$; furthermore the $\mathrm{SO}(3)$-bundle $M_g \to M_g/\mathrm{SU}(2) = f^{-1}(2)$ has a section, so that $H^*(f^{-1}(2), \mathbf{Q}) = H^*(\mathrm{SU}(2), \mathbf{Q}) \times H^*(S_{g-1}(-1), \mathbf{Q})$. Thus the Poincaré polynomial $P_t(f^{-1}(2))$ is given by $(1 + t^3)P_t(S_{g-1})$, while the index of $f^{-1}(2)$ is given by its codimension, which is 3. Hence the contribution of $f^{-1}(2)$ to the Morse polynomial of f is

$$S_t(f^{-1}(2)) = t^3(1 + t^3)P_t(S_{g-1}(-1)). \tag{5}$$

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Similarly the minimum $f^{-1}(-2)$ is the image in $S_g(-1)$ of the subvariety $N_g \subset R_g$ given by

$$N_g = \Big\{(A_1, \dots, A_{g-1}, -1, B_1, \dots, B_g) \in \mathrm{SU}(2)^{2g}: \prod_{i=1}^{g-1} A_i B_i A_i^{-1} B_i^{-1} = -1\Big\}. \tag{6}$$

Thus again, $H^*(f^{-1}(-2), \mathbf{Q}) = H^*(\mathrm{SU}(2), \mathbf{Q}) \times H^*(S_{g-1}(-1), \mathbf{Q})$, while the index of the minimum $f^{-1}(-2)$ is 0; so that the contribution of $f^{-1}(-2)$ to the Morse polynomial of f is

$$S_t(f^{-1}(2)) = (1+t^3)P_t(S_{g-1}(-1)). \tag{7}$$

Finally we must compute the contribution of C_g to the Morse polynomial of f. By Equation (3) we see that $C_g = (S^1)^{2g-2}$. To compute the index of C_g, we note that the involution $a\colon S_g(-1) \to S_g(-1)$ arising from $\tilde{a}\colon R_g \to R_g$ defined by

$$\tilde{a}((A_1, \dots, A_g, B_1, \dots, B_g)) = (A_1, \dots, -A_g, B_1, \dots, B_g)$$

interchanges the ascending and descending flows of f at C_g. Hence $\mathrm{index}(C_g) = \frac{1}{2}\mathrm{codim}(C_g) = 2g-2$, so that the contribution of C_g to the Morse polynomial of f is given by

$$S_t(C_g) = t^{2g-2}(1+t)^{2g-2}. \tag{8}$$

Combining (5), (7), and (8) we see that the Morse polynomial $M_t(f)$ is given by

$$M_t(f) = (1+t^3)^2 P_t(S_{g-1}(-1)) + t^{2g-2}(1+t)^{2g-2}. \tag{9}$$

On the other hand the Poincaré polynomial of $S_g(-1)$ is given by

$$P_t(S_g(-1)) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Given that $P_t(S_1(-1)) = 1$, it is easily seen that $M_t(f) = P_t(S_g(-1))$.

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