THE ORDER OF CERTAIN DIRICHLET SERIES

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This paper is a continuation of [1] [1]. We shall use the same notations as those in [1]. Let $F(X) \in \boldsymbol{R}[X]$, $X = (X_1, \cdots, X_n)$, be a polynomial of degree $d > 0$ and $h(x) \in SP(\boldsymbol{R}^n)$, i.e. $h(x)$ is the sum of a polynomial and a Schwartz function. We shall consider Dirichlet series of the type

$$Z(h, F, s) = \sum_{v \in \boldsymbol{Z}^n - N_F} h(v) F(v)^{-s}, \qquad s = \sigma + ti,$$

where $N_F = \{x \in \boldsymbol{R}^n : F(x) = 0\}$. We proved, in [1], that $Z(h, F, s)$ is regular for $\sigma > (n+p)/d$ and possesses the analytic continuation to the whole s-plane when $F_d(x)$ (the highest homogeneous part of $F(X)$) $\neq 0$ for $x \neq 0$. In this paper, we shall say the following.

$$Z(h, F, s) = O(|t|^{k(n+1)} e^{\pi|t|}), \qquad \text{for } \sigma_1 \geqq \sigma \geqq \sigma_2 > \frac{n+p-k}{d}.$$

Let $h(x)$ be a Schwartz function and K be a suitable positive integer. Put

$$J_2(s) = \int_{|x| \geqq K} h(x) F(x)^s dx.$$

From the proof of [1, Theorem 1], we see that

$$J_2(s) = O(e^{\pi|t|}), \qquad \text{for } |\sigma| \leqq \sigma_2,$$

where σ_2 is a positive real number and

$$J_2(s) = O(1), \qquad \text{for } |\sigma| \leqq \sigma_2,$$

when $F(x) > 0$ for $|x| \geqq K$.

Let $G(X) \in \boldsymbol{R}[X]$ be a polynomial of degree p and

$$I(s) = \int_{|x| \geqq K} G(x) F(x)^s dx.$$

[1] The results in [1] have appeared in the Bulletin of the American Mathematical Society, May, 1969.

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We can rewrite $I(s)$ as

$$I(s) = \sum_{u=0}^{p} I_u(s)$$

where

$$I_u(s) = \int_{|x| \geqq K} G_u(x) F(x)^s dx$$

and $G_u(X)$ is the homogeneous part of $G(X)$ of degree u. Following Mahler's method [2], we get

$$I_u(s) = \sum_{q=0}^{k-1} \binom{s}{q} M_q(s) + N_k(s)$$

where

$$M_q(s) = \int_{S^{n-1}} \int_K^\infty G_u(w) F_d(w)^s R(rw)^q r^{n+u+ds-1} dr\, dw,$$

$$N_k(s) = \int_{S^{n-1}} \int_K^\infty \int_0^1 k \binom{s}{q} G_u(w) F_d(w)^s R(rw)^k r^{n+u+ds-1}$$
$$\{1+\tau R(rw)\}^{s-k} (1-\tau)^{k-1}\, d\tau\, dr\, dw$$

and

$$R(x) = \frac{F_{d-1}(x) + \cdots + F_0(x)}{F_d(x)}, \qquad \text{for } x \neq 0.$$

Then, it is easy to see that, for $\beta_2 \leqq \sigma \leqq \beta_1 < -(n+u-k)/d$,

$$N_k(s) = O(|t|^k e^{\pi|t|})$$

and

$$M_q(s) = O(e^{\pi|t|}).$$

We have $(s/q) = O(|t|^k)$ for $\beta_2 \leqq \sigma \leqq \beta_1 < -(n+p-k)/d$. Hence

$$I(s) = O(|t|^k e^{\pi|t|}) \qquad \text{for } \beta_2 \leqq \sigma \leqq \beta_1 < -\frac{n+p-k}{d}.$$

Put, for suitable K as in [1, Theorem 1],

$$V_1 = \{v \in \mathbf{Z}^n : -K+1 \leqq v_i \leqq K, \text{ for all } i = 1, \cdots, n\}, \ V_2 = \mathbf{Z}^n - V_1.$$

We see that

$$Z(h, F, s) = Z_1(h, F, s) + Z_2(h, F, s)$$

where

$$Z_1(h, F, s) = \sum_{v \in V_1 - N_F} h(v) F(v)^{-s}$$

$$Z_2(h, F, s) = \sum_{v \in V_2} h(v) F(v)^{-s}.$$

It follows immediately that

$$Z_1(h, F, s) = O(e^{\pi|t|}), \qquad \text{for } \sigma_1 \geqq \sigma \geqq \sigma_2 > \frac{n+p-k}{d}.$$

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If we apply the generalized Euler's summation formula [1, Lemma 2] and use Mahler's method, we shall have the following.

$$Z_2(h, F, s) = O(|t|^{k(n+1)}e^{\pi|t|}), \qquad \text{for } \sigma_1 \geqq \sigma \geqq \sigma_2 > \frac{n+p-k}{d}.$$

Hence, we obtain

$$Z(h, F, s) = O(|t|^{k(n+1)}e^{\pi|t|}), \qquad \text{for } \sigma_1 \geqq \sigma \geqq \sigma_2 \geqq \frac{n+p-k}{d}.$$

Furthermore, we may assume that $F(X)$ is homogeneous and $n > 1$. Since the n-sphere S^{n-1} is connected, we see that either $F(x) > 0$ for all $x \neq 0$ or $F(x) < 0$ for all $x \neq 0$. Without loss of generality, we may assume $F(x) > 0$ for all $x \neq 0$. Thus

$$Z(h, F, s) = O(|t|^{k(n+1)}), \qquad \text{for } \sigma_1 \geqq \sigma \geqq \sigma_2 > \frac{n+p-k}{d}.$$

References

[1] An, Chung-ming, *On a generalization of Gamma function and its application to certain Dirichlet series* (Dissertation, University of Pennsylvania, 1969).
[2] Mahler, K., 'Über einer Satz von Mellin', *Math. Ann.* 100 (1928), 384–395.

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