ON EXTENSIONS OF INEQUALITIES OF KOLMOGOROFF AND OTHERS AND SOME APPLICATIONS TO ALMOST PERIODIC FUNCTIONS

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Introduction. Let $f(x)$ be a complex function of a real variable, defined over the whole real line, which possesses n derivatives (the nth at least almost everywhere) and is such that $f^{(n-1)}(x) = \int^x f^{(n)}(t)\,dt$. Then, if k is any integer for which $0 < k < n$, Kolmogoroff's inequality may be written as

$$\sup_x |f^{(k)}(x)| \leqq K\{\sup_x |f^{(n)}(x)|\}^{k/n}\{\sup_x |f(x)|\}^{1-k/n}, \tag{0.1}$$

or, by putting $\| f \|_U = \sup_x |f(x)|$ and $\sigma = k/n$,

$$\| f^{(k)} \|_U \leqq K \| f^{(n)} \|_U^{\sigma} \| f \|_U^{1-\sigma}. \tag{0.2}$$

The constant $K = K(k, n)$ is known explicitly and is the best possible, i.e., there is a (real) function for which equality holds (see Bang **[1]**).

In the first section of this paper we show that in (0.2) the norm $\|\cdot\|_U$ can be replaced by any one of the norms used in defining classes of almost periodic functions by Stepanoff, Weyl, Besicovitch, Love and the present author **[2, 9, 11]**, but no assertion is made as to whether K is still the best possible constant. Part of the proof utilises a convolution introduced by Ogiewetski **[10]**, who established a special case of Theorem 4 of this paper.

In the second section similar extensions involving fractional integrals are made to inequalities between trigonometric polynomials, originated by Bohr, Favard and Bernstein and generalized by Bang.

In the third section various classes of almost periodic functions are introduced, and the theorems of the previous section are applied to them. Finally, a theorem proved in the first section is used to establish alternative characterizations of the almost periodic functions defined by the present author.

Definitions. For completeness we state here the definitions of the various norms which will be considered. Let $p \geqq 1$ and $L > 0$; then

$$\left.\begin{aligned}
&\| f \|_U = \sup_x |f(x)|,\\
&\| f \|_{S_L^p} = \sup_x \{L^{-1}\int_x^{x+L} |f(t)|^p\,dt\}^{1/p},\\
&\| f \|_{W^p} = \lim_{L\to\infty} \| f \|_{S_L^p},\\
&\| f \|_{B^p} = [\overline{M}_x\{|f(x)|^p\}]^{1/p} = \limsup_{T\to\infty}\{(2T)^{-1}\int_{-T}^{T}|f(t)|^p\,dt\}^{1/p},\\
&\| f \|_{V^p} = \| f \|_U + \sup_x V_p(f;\, x,\, x+1),
\end{aligned}\right\} \tag{0.3}$$

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where the Wiener pth variation

$$V_p(f;\, a,\, b) = \sup_{\pi} \sum_{i=1}^{m} |f(x_i)-f(x_{i-1})|^p, \tag{0.4}$$

the supremum on the right hand side being taken over all partitions

$$\pi:\ a = x_0 < x_1 < \ldots < x_m = b$$

of the interval (a, b).

If either L or p is equal to 1, it is omitted.

We also consider the norms

$$\| f \|_{G_{(\kappa)}} = \sum_{r=0}^{\kappa-1} \| f^{(r)} \|_U + \| f^{(\kappa)} \|_{G_{(0)}} \qquad (\kappa \geqq 1), \tag{0.5}$$

where $f(x)$ is assumed to be differentiable κ times and $G_{(0)}$ stands for any one of U, S^p, V^p, W^p and B^p. However, for reasons that will appear later, the use of $\| \cdot \|_{G_{(\kappa)}}$ when $G_{(0)}$ stands for any one of W^p and B^p will be restricted to the first two sections of the paper.

If $p > 1$, we define p' by $1/p + 1/p' = 1$.

1. Extensions of Kolmogoroff's inequality.

THEOREM 1. *Let $f(x)$ be any complex function of a real variable such that $f^{(r)}(x)$ exists for $r = 1, 2, \ldots, n-1$, $f^{(n)}(x)$ exists almost everywhere and $f^{(n-1)}(x) = \int^x f^{(n)}(t)\,dt$; and let k be any integer for which $0 < k < n$. Then, if G is any one of S^p, W^p or B^p $(p \geqq 1)$,*

$$\| f^{(k)} \|_G \leqq K \, \| f^{(n)} \|_G^{\sigma} \, \| f \|_G^{1-\sigma},$$

where K is a constant independent of $f(x)$ and $\sigma = k/n$.

Proof. For any bounded measurable set E, we define the convolution

$$H(x) = \int_E f(x+y) g(y)\, dy,$$

where $g(y)$ is a bounded measurable function whose specific values will be determined later. Then, if m is any non-negative integer $(0 \leqq m \leqq n)$,

$$H^{(m)}(x) = \int_E f^{(m)}(x+y) g(y)\, dy. \tag{1.1}$$

We first suppose that $p > 1$ and that $g(y)$ satisfies the condition

$$\int_E |g(y)|^{p'}\, dy = 1. \tag{1.2}$$

Then, by (0.1) and Hölder's inequality,

$$\begin{aligned} |H^{(k)}(0)| &\leqq \sup_x |H^{(k)}(x)| \leqq K\{\sup_x |H^{(n)}(x)|\}^{\sigma}\{\sup_x |H(x)|\}^{1-\sigma} \\ &\leqq K\left\{\sup_x \left[\int_E |f^{(n)}(x+y)|^p\, dy\right]^{1/p}\right\}^{\sigma}\left\{\sup_x \left[\int_E |f(x+y)|^p\, dy\right]^{1/p}\right\}^{1-\sigma}. \end{aligned} \tag{1.3}$$

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If we now put

$$g(y) = |f^{(k)}(y)|^{p-1} \operatorname{sgn}[f^{(k)}(y)] \Big/ \left\{\int_E |f^{(k)}(y)|^p\, dy\right\}^{1/p'},$$

then $g(y)$ satisfies (1.2) and, from (1.1),

$$H^{(k)}(0) = \left\{\int_E |f^{(k)}(y)|^p\, dy\right\}^{1/p}.$$

It therefore follows from (1.3) that

$$\left\{\int_E |f^{(k)}(y)|^p\, dy\right\}^{1/p} \leqq K\left\{\sup_x \int_E |f^{(n)}(x+y)|^p\, dy\right\}^{\sigma/p} \left\{\sup_x \int_E |f(x+y)|^p\, dy\right\}^{(1-\sigma)/p}. \tag{1.4}$$

We next suppose that $p = 1$ and that $g(y)$ satisfies the condition

$$\sup_y |g(y)| = 1. \tag{1.5}$$

Then, by a similar argument,

$$|H^{(k)}(0)| \leqq K\left\{\sup_x \int_E |f^{(n)}(x+y)|\, dy\right\}^{\sigma} \left\{\sup_x \int_E |f(x+y)|\, dy\right\}^{1-\sigma}.$$

If we now put $g(y) = \operatorname{sgn}[f^{(k)}(y)]$, then (1.5) is satisfied and, from (1.1), $H^{(k)}(0) = \int_E |f^{(k)}(y)|\, dy$. It follows that (1.4) is true for $p = 1$ and hence for all $p \geqq 1$.

We complete the proof of the theorem by replacing G in turn by S^p, W^p and B^p.

Case 1. Let G be S^p ($p \geqq 1$). In (1.4) we let E be the interval $(t, t+L)$. Then

$$\left\{L^{-1}\int_t^{t+L} |f^{(k)}(y)|^p\, dy\right\}^{1/p} \leqq K\left\{\sup_x L^{-1}\int_{t+x}^{t+x+L} |f^{(n)}(y)|^p\, dt\right\}^{\sigma/p}$$
$$\times \left\{\sup_x L^{-1}\int_{t+x}^{t+x+L} |f(y)|^p\, dy\right\}^{(1-\sigma)/p}$$
$$\leqq K\| f^{(n)}\|_{S_L^p}^{\sigma}\ \| f\|_{S_L^p}^{1-\sigma}. \tag{1.6}$$

Put $L = 1$. Then, if we take the supremum of the left hand side of (1.6) over all t, we find that

$$\| f^{(k)}\|_{S^p} \leqq K\| f^{(n)}\|_{S^p}^{\sigma}\ \| f\|_{S^p}^{1-\sigma}. \tag{1.7}$$

Case 2. Let G be W^p ($p \geqq 1$), and let $\varepsilon > 0$ be chosen arbitrarily. Then, by (0.3) and (1.6), there exists $L_0(\varepsilon) > 0$ such that, if $L > L_0$,

$$\left\{L^{-1}\int_t^{t+L} |f^{(k)}(y)|^p\, dy\right\}^{1/p} \leqq K\{\| f^{(n)}\|_{W^p}+\varepsilon\}^{\sigma}\{\| f\|_{W^p}+\varepsilon\}^{1-\sigma}.$$

If then, on the left hand side, we first take the supremum over all t and then let $L \to \infty$, we see

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that the left hand side can be replaced by $\| f^{(k)} \|_{W^p}$. As the resulting inequality is true for all $\varepsilon > 0$, it follows that

$$\| f^{(k)} \|_{W^p} \leqq K \| f^{(n)} \|_{W^p}^{\sigma} \| f \|_{W^p}^{1-\sigma}. \tag{1.8}$$

Case 3. Let G be B^p $(p \geqq 1)$. We first note that, for fixed x and $T > |x|$,

$$\frac{T-|x|}{T} \frac{1}{2(T-|x|)} \int_{-T+|x|}^{T-|x|} |f(t)|^p \, dt$$
$$\leqq \frac{1}{2T} \int_{-T+x}^{T+x} |f(t)|^p \, dt \leqq \frac{T+|x|}{T} \frac{1}{2(T+|x|)} \int_{-T-|x|}^{T+|x|} |f(t)|^p \, dt.$$

If we let $T \to \infty$, it follows easily that

$$\overline{M}_t\{|f(x+t)|^p\} = \overline{M}_t\{|f(t)|^p\} \, (= \| f \|_{B^p}^p). \tag{1.9}$$

In (1.4) we now let E be the interval $(-T, T)$, obtaining the inequality

$$\left\{\frac{1}{2T} \int_{-T}^{T} |f^{(k)}(y)|^p \, dy\right\}^{1/p}$$
$$\leqq K \left\{\sup_x \frac{1}{2T} \int_{-T}^{T} |f^{(n)}(x+y)|^p \, dy\right\}^{\sigma/p} \left\{\sup_x \frac{1}{2T} \int_{-T}^{T} |f(x+y)|^p \, dy\right\}^{(1-\sigma)/p}.$$

Let $\varepsilon > 0$ be chosen arbitrarily. Then, from (1.9), there exists $T_0(\varepsilon) > 0$ such that, if $T > T_0$,

$$\left\{\frac{1}{2T} \int_{-T}^{T} |f^{(k)}(y)|^p \, dy\right\}^{1/p} \leqq K\{\| f^{(n)} \|_{B^p} + \varepsilon\}^{\sigma} \{\| f \|_{B^p} + \varepsilon\}^{1-\sigma}.$$

If we let $T \to \infty$, it follows that the left hand side can be replaced by $\| f^{(k)} \|_{B^p}$; and the resulting inequality is true for all $\varepsilon > 0$. Therefore

$$\| f^{(k)} \|_{B^p} \leqq K \| f^{(n)} \|_{B^p}^{\sigma} \| f \|_{B^p}^{1-\sigma}. \tag{1.10}$$

The inequalities (1.7), (1.8) and (1.10) establish Theorem 1.

COROLLARY. *If in* (1.4) *we replace E by the interval* $(-T, T)$ *and then let* $T \to \infty$ *first on the right-hand side and then on the left-hand side, we obtain the inequality*

$$\int_{-\infty}^{\infty} |f^{(k)}(y)|^p \, dy \leqq K^p \left\{\int_{-\infty}^{\infty} |f^{(n)}(y)|^p \, dy\right\}^{\sigma} \left\{\int_{-\infty}^{\infty} |f(y)|^p \, dy\right\}^{1-\sigma}.$$

In the case when $n = p = 2$ and $k = 1$, this inequality is proved in [**8**, p. 193], and the best possible value of K is shown to be 1. (For the same values of n, p and k, the least value of K obtainable by the present argument is $\sqrt{2}$.)

Before we consider extensions of Kolmogoroff's inequality which involve the remaining norms defined in (0.3) and (0.5), we first obtain an inequality analogous to (1.4) for the Wiener pth variation defined in (0.4).

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LEMMA 1. *Let π be the partition $a = x_0 < x_1 < \ldots < x_m = b$ of the interval (a, b), and let $p > 1$. Then, with the definitions and notation of Theorem* 1,

$$\left\{\sum_{i=1}^{m} |f^{(k)}(x_i)-f^{(k)}(x_{i-1})|^p\right\}^{1/p} \leqq K\{\sup_x V_p(f^{(n)};\, a+x,\, b+x)\}^{\sigma}\{\sup_x V_p(f;\, a+x,\, b+x)\}^{1-\sigma},$$

where K is the constant defined in (0.1).

Proof. As in the proof leading to (1.4), we define a new function

$$H_\pi(x) = \sum_{i=1}^{m} \{f(x+x_i)-f(x+x_{i-1})\}\{g(x_i)-g(x_{i-1})\}$$
$$= \sum_{i=1}^{m} \{f(x+x_i)-f(x+x_{i-1})\}\,\Delta g_i, \text{ say,}$$

where $g(x)$ is an auxiliary function to be determined later. Then

$$H_\pi^{(j)}(x) = \sum_{i=1}^{m} \{f^{(j)}(x+x_i)-f^{(j)}(x+x_{i-1})\}\,\Delta g_i \qquad (j = 0, 1, \ldots, n). \tag{1.11}$$

If $g(x)$ satisfies the condition

$$\sum_{i=1}^{m} |\Delta g_i|^{p'} = 1, \tag{1.12}$$

it follows, by (0.1) and Hölder's inequality, that

$$|H_\pi^{(k)}(0)| \leqq \sup_x |H_\pi^{(k)}(x)|$$
$$\leqq K\{\sup_x |H_\pi^{(n)}(x)|\}^{\sigma}\{\sup_x |H_\pi(x)|\}^{1-\sigma}$$
$$\leqq K\{\sup_x [\sum_i |f^{(n)}(x+x_i)-f^{(n)}(x+x_{i-1})|^p]^{1/p}\}^{\sigma}$$
$$\times\{\sup_x [\sum_i |f(x+x_i)-f(x+x_{i-1})|^p]^{1/p}\}^{1-\sigma}$$
$$\leqq K\{\sup_x V_p(f^{(n)};\, a+x,\, b+x)\}^{\sigma}\{\sup_x V_p(f;\, a+x,\, b+x)\}^{1-\sigma}. \tag{1.13}$$

We now define $g(x)$ as a function whose values at the points of the partition π are as follows. We fix $g(x_0)$ arbitrarily and set, for $i = 1, 2, \ldots, m$,

$$g(x_i) = g(x_{i-1}) + |\Delta f_i^{(k)}|^{p-1}\,\mathrm{sgn}\,[\Delta f_i^{(k)}]/\{\sum_{i=1}^{m} |\Delta f_i^{(k)}|^p\}^{1/p'},$$

where $\Delta f_i^{(k)} = f^{(k)}(x_i)-f^{(k)}(x_{i-1})$. No restriction is placed on the values that $g(x)$ takes elsewhere. Then $g(x)$ satisfies (1.12) and, from (1.11),

$$H_\pi^{(k)}(0) = \{\sum_{i=1}^{m} |\Delta f_i^{(k)}|^p\}^{1/p}.$$

This and (1.13) complete the proof of the lemma.

We also state as a second lemma the following inequality.

LEMMA 2. *If u_i and v_i and α are real, $u_i \geqq 0$, $v_i \geqq 0$ $(i = 1, 2, \ldots, m)$, and $0 < \alpha < 1$, then*

$$\sum_{i=1}^{m} u_i^{\alpha} v_i^{1-\alpha} \leqq \left(\sum_{i=1}^{m} u_i\right)^{\alpha} \left(\sum_{i=1}^{m} v_i\right)^{1-\alpha}.$$

For proof see [8; 2.9.2].

We now extend the result of Theorem 1 to other norms.

THEOREM 2. *Under the hypotheses and notation of Theorem* 1,

$$\| f^{(k)} \|_{VP} \leqq K \| f^{(n)} \|_{VP}^{\sigma} \, \| f \|_{VP}^{1-\sigma} \qquad (p > 1).$$

Proof. In Lemma 1 put $a = y$, $b = y+1$. Then

$$\begin{aligned}\{\sum_{i=1}^{m} |\Delta f_i^{(k)}|^p\}^{1/p} &\leqq K\{\sup_x V_p(f^{(n)}; y+x, y+x+1)\}^{\sigma}\{\sup_x V_p(f; y+x, y+x+1)\}^{1-\sigma} \\ &= K\{\sup_y V_p(f^{(n)}; y, y+1)\}^{\sigma}\{\sup_y V_p(f; y, y+1)\}^{1-\sigma}.\end{aligned}$$

If, on the left-hand side, we take the supremum firstly over all partitions of the interval $(y, y+1)$ and then over all values of y, it follows that

$$\sup_y V_p(f^{(k)}; y, y+1) \leqq K\{\sup_y V_p(f^{(n)}; y, y+1)\}^{\sigma}\{\sup_y V_p(f; y, y+1)\}^{1-\sigma}.$$

Hence, by (0.2) and Lemma 2,

$$\begin{aligned}\| f^{(k)} \|_{VP} &= \| f^{(k)} \|_U + \sup_y V_p(f^{(k)}; y, y+1) \\ &\leqq K\{\| f^{(n)} \|_U^{\sigma} \| f \|_U^{1-\sigma} + [\sup_y V_p(f^{(n)}; y, y+1)]^{\sigma}[\sup_y V_p(f; y, y+1)]^{1-\sigma}\} \\ &\leqq K\{\| f^{(n)} \|_U + \sup_y V_p(f^{(n)}; y, y+1)\}^{\sigma}\{\| f \|_U + \sup_y V_p(f; y, y+1)\}^{1-\sigma} \\ &= K \| f^{(n)} \|_{VP}^{\sigma} \| f \|_{VP}^{1-\sigma}.\end{aligned}$$

When $p = 1$ we note that $\| f \|_V = \| f \|_U + \| f' \|_S = \| f \|_{S_{(1)}}$. This norm is included in the next theorem.

THEOREM 3. *Let $f(x)$ be an $(n+\kappa)$-times differentiable complex function of a real variable (the $(n+\kappa)$th derivative existing at least almost everywhere) such that $f^{(n+\kappa-1)}(x) = \int^x f^{(n+\kappa)}(t)\,dt$, where n and κ are positive integers. Then, if k is any integer such that $0 < k < n$ and $\sigma = k/n$,*

$$\| f^{(k)} \|_{G(\kappa)} \leqq K \| f^{(n)} \|_{G(\kappa)}^{\sigma} \, \| f \|_{G(\kappa)}^{1-\sigma},$$

where $\| \cdot \|_{G(\kappa)}$ is any one of the norms defined in (0.5) *and K is a constant independent of $f(x)$ and σ.*

Proof. If, in (0.2), f is replaced in turn by $f', f'', \ldots, f^{\kappa-1}$, it follows from Lemma 2 that

$$\sum_{r=0}^{\kappa-1} \| f^{(k+r)} \|_U \leqq K\{\sum_{r=0}^{\kappa-1} \| f^{(n+r)} \|_U\}^{\sigma}\{\sum_{r=0}^{\kappa-1} \| f^{(r)} \|_U\}^{1-\sigma}. \qquad (1.14)$$

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From the definitions of the norms in (0.5), the proof of the theorem then follows from (1.14), Theorems 1 and 2 and a further application of Lemma 2.

2. Extensions of inequalities of Bernstein and others. Let $t(x)$ be a trigonometric polynomial

$$t(x) = \sum_{h=1}^{m} a_h e^{i\lambda_h x} \qquad (a_h \text{ complex, } \lambda_h \text{ real})$$

and let α be any complex number. Then the derivative of $t(x)$ of order α is

$$t^{(\alpha)}(x) = \sum_{h=1}^{m} a_h (i\lambda_h)^{\alpha} e^{i\lambda_h x},$$

where $(i\lambda)^{\alpha} = \exp[(\tfrac{1}{2}i\pi \operatorname{sgn}\lambda + \log|\lambda|)\alpha]$. Similarly the integral of $t(x)$ of order α is

$$t_{\alpha}(x) = \sum_{h=1}^{m} a_h (i\lambda_h)^{-\alpha} e^{i\lambda_h x} \qquad (= t^{(-\alpha)}(x)),$$

provided that none of the λ_h is zero.

We denote by $\bar{\alpha}$ the real part of α. Then the following inequalities hold, where $\|\cdot\|_G$ is any one of the norms defined in (0.3) and (0.5).

THEOREM 4. $\| t^{(\alpha)} \|_G \leqq A(\alpha) \| t \|_G \max_h |\lambda_h|^{\bar{\alpha}}$, *where* $A(\alpha)$ *depends only upon* α *and* $\bar{\alpha} > 0$.

THEOREM 5. $\| t_{\alpha} \|_G \leqq A(\alpha) \| t \|_G / \min_h |\lambda_h|^{\bar{\alpha}}$, *where* $A(\alpha)$ *depends only upon* α *and* $\bar{\alpha} > 0$.

THEOREM 6. *If* $\min_h |\lambda_h| > 0$ *and* α, β, γ *are complex numbers such that* $\bar{\gamma} < \bar{\alpha} < \bar{\beta}$, *and* $t_{\alpha}(x)$, $t_{\beta}(x)$, $t_{\gamma}(x)$ *are, respectively, the* α*th,* β*th and* γ*th integrals of* $t(x)$, *then*

$$\| t_{\alpha} \|_G \leqq K \| t_{\gamma} \|_G^{\tau} \| t_{\beta} \|_G^{1-\tau},$$

where $\tau = (\bar{\beta} - \bar{\alpha})/(\bar{\beta} - \bar{\gamma})$ *and* K *is a constant depending only upon* $\alpha - \gamma$ *and* $\beta - \gamma$.

When G is U, all three theorems have been established by Bang. When, in addition, α is real, the inequalities in Theorems 4 and 5 are proved, respectively, by Civin [6, a special case of Theorem 6], and Sz. Nagy; when, further, $\lambda_h \equiv h$, they are due, respectively, to Bernstein and to Bohr and Favard. For references see [1].

If α, β and γ are real, and $\|\cdot\|_G$ is replaced by

$$\int_0^{\infty} |\cdot|^p \, dt \qquad (p \geqq 1),$$

Theorem 6 becomes a special case of [7, pp. 688 and 695]. This special case is included in Corollary (ii), below.

As the proofs of all three theorems follow closely those of Theorems 1, 2 and 3, making use of Bang's results when G is U, it will suffice to prove one of them, Theorem 4, in the cases when G is S^p, W^p or B^p. The proofs in the other cases and the proofs of Theorems 5 and 6 will follow, *mutatis mutandis.*

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Suppose then that $p \geqq 1$ and E is any bounded measurable set. We define an auxiliary function $g(y)$ as follows:

$$g(y) = \begin{cases} \operatorname{sgn}[t^{(\alpha)}(y)] & (p=1), \\ |t^{(\alpha)}(y)|^{p-1}\operatorname{sgn}[t^{(\alpha)}(y)] \Big/ \left\{\int_E |t^{(\alpha)}(y)|^p\,dt\right\}^{1/p'} & (p>1), \end{cases} \tag{2.1}$$

so that

$$\sup_y |g(y)| = 1 \qquad (p=1), \qquad \text{and} \qquad \int_E |g(y)|^{p'}\,dy = 1 \qquad (p>1). \tag{2.2}$$

Let

$$T(x) = \int_E t(x+y)g(y)\,dy = \sum_{h=1}^{m} e^{i\lambda_h x}a_h \int_E e^{i\lambda_h y}g(y)\,dy.$$

Then $T(x)$ is a trigonometric polynomial and, for any complex α,

$$T^{(\alpha)}(x) = \int_E t^{(\alpha)}(x+y)g(y)\,dy.$$

Hence, using the definition (2.1) of $g(y)$, we have

$$T^{(\alpha)}(0) = \left\{\int_E |t^{(\alpha)}(y)|^p\,dy\right\}^{1/p} \qquad (p \geqq 1).$$

Let $C_\alpha = A(\alpha)\max_h |\lambda_h|^{\bar{\alpha}}$, where $A(\alpha)$ is the constant determined in [1] for the case when G is U. Then, by (2.1) and (2.2) and by using Hölder's inequality when $p > 1$, it follows that

$$\begin{aligned} \left\{\int_E |t^{(\alpha)}(y)|^p\,dy\right\}^{1/p} &= T^{(\alpha)}(0) \leqq \sup_x |T^{(\alpha)}(x)| \\ &\leqq C_\alpha \sup_x |T(x)| \\ &= C_\alpha \sup_x \left|\int_E t(x+y)g(y)\,dy\right| \\ &\leqq C_\alpha \sup_x \left\{\int_E |t(x+y)|^p\,dy\right\}^{1/p}. \end{aligned} \tag{2.3}$$

If E is replaced by the interval $(s, s+L)$, then it quickly follows, as in the argument leading to (1.6) and (1.7), that

$$\left\{L^{-1}\int_s^{s+L} |t^{(\alpha)}(y)|^p\,dy\right\}^{1/p} \leqq C_\alpha \| t \|_{S_L^p}.$$

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Therefore, if $L = 1$,

$$\| t^{(\alpha)} \|_{S^p} \leqq C_\alpha \| t \|_{S^p}.$$

Similarly, arguments analogous to those leading to (1.8) and (1.10) show that

$$\| t^{(\alpha)} \|_{W^p} \leqq C_\alpha \| t \|_{W^p}$$

and

$$\| t^{(\alpha)} \|_{B^p} \leqq C_\alpha \| t \|_{B^p}.$$

COROLLARY. *Theorems* 4, 5 *and* 6 *are also true when* $\| \cdot \|_G$ *is replaced by either*

$$\text{(i)} \left\{ \int_{-\infty}^{\infty} | \cdot |^p \, dy \right\}^{1/p} \quad \textit{or} \quad \text{(ii)} \left\{ \int_0^{\infty} | \cdot |^p \, dy \right\}^{1/p}.$$

We again consider Theorem 4. Corollary (i) follows from (2.3) if we take E as the interval $(-T, T)$ and let $T \to \infty$ first on the right-hand side and then on the left-hand side.

To prove Corollary (ii) we first note that, for any trigonometric polynomial $\phi(x)$, $\sup_x |\phi(x)| = \sup_{x \geqq 0} |\phi(x)|$. For, if $\sup_x |\phi(x)| = M$ and $\sup_{x \geqq 0} |\phi(x)| = M_1$, then $M_1 \leqq M$. To obtain the reverse inequality, let $\varepsilon > 0$ be chosen arbitrarily and let x_0 be such that $|\phi(x_0)| > M - \varepsilon/2$. Then there exists a number τ such that $x_0 + \tau > 0$ and $|\phi(x_0+\tau) - \phi(x_0)| < \varepsilon/2$. (Its existence can be seen, for example, from the fact that $\phi(x)$ is uniformly almost periodic.) Hence

$$M_1 \geqq |\phi(x_0+\tau)| > |\phi(x_0)| - \varepsilon/2 > M - \varepsilon.$$

Since ε is arbitrary, $M_1 \geqq M$ and thus $M_1 = M$.

It follows that the argument leading to (2.3) is equally valid when $\sup_x$ is replaced by $\sup_{x \geqq 0}$. If we then take E as the interval $[0, T]$, and let $T \to \infty$ as before, first on the right-hand side and then on the left-hand side, we obtain

$$\begin{aligned} \left\{ \int_0^{\infty} | t^{(\alpha)}(y) |^p \, dy \right\}^{1/p} &\leqq C_\alpha \sup_{x \geqq 0} \left\{ \int_0^{\infty} | t(x+y) |^p \, dy \right\}^{1/p} \\ &= C_\alpha \sup_{x \geqq 0} \left\{ \int_x^{\infty} | t(y) |^p \, dy \right\}^{1/p} \\ &\leqq C_\alpha \left\{ \int_0^{\infty} | t(y) |^p \, dy \right\}^{1/p}. \end{aligned}$$

Ogiewetski's generalization **[10]** follows directly from (2.3) by letting α be real and by replacing λ_h by h and E by the interval $(-\pi, \pi)$. The trigonometric polynomial $t(x)$ is then periodic with period 2π and

$$\left\{ \int_{-\pi}^{\pi} | t^{(\alpha)}(y) |^p \, dy \right\}^{1/p} \leqq m^\alpha A(\alpha) \left\{ \int_{-\pi}^{\pi} | t(y) |^p \, dy \right\}^{1/p}.$$

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3. Applications to almost periodic functions. For many of the norms that have been used hitherto, classes of almost periodic functions have been defined. More specifically, a G-almost periodic function (Gap) has been defined when $\|\cdot\|_G$ denotes any one of the norms given in (0.3) or any one of the norms given in (0.5) when $G_{(0)}$ is U, S^p or V^p [**2**, **9**, **11**]. For our present purposes, however, we need to consider spaces of Gap functions which are complete, and we must therefore exclude W^pap functions (see [**4**, p. 58]). In this section we therefore restrict G to be any one of U, S^p, V^p, B^p or $G_{(\kappa)}$, where $G_{(0)}$ is U, S^p or V^p.

We first state as lemmas two properties of Gap functions that will be needed.

Lemma 3. *Let $f(x)$ be* Gap *and let $\{s_m(x)\}$ be a corresponding Bochner–Fejér sequence of trigonometric polynomials. Then $\{s_m(x)\}$ G-converges to $f(x)$, i.e., $\| s_m - f \|_G \to 0$ as $m \to \infty$.*

Lemma 4. *The space of* Gap *functions is complete.*

These are well-known properties of B^pap, S^pap, Uap and V^pap functions. To establish them for $G_{(\kappa)}$ap functions ($\kappa \geqq 1$) is not difficult.

To prove Lemma 3 we suppose, then, that $f(x)$ is $G_{(\kappa)}$ap ($\kappa \geqq 1$). It follows from the definitions [**11**] that $f^{(\kappa)}(x)$ is $G_{(0)}$ap and that each $f^{(r)}(x)$ ($r = 0, 1, \ldots, \kappa-1$) is Uap.

Let $f(x)$ have the Fourier series $\sum_n a_n e^{i\lambda_n x}$. Then $s_m(x)$ can be written as $\sum_n d_m^n a_n e^{i\lambda_n x}$, where the d_m^n are constants (only a finite number of which are nonzero) which are independent of the coefficients a_n. Since, for each $r = 0, 1, \ldots, \kappa$, $f^{(r)}(x)$ has the Fourier series $\sum_n a_n (i\lambda_n)^r e^{i\lambda_n x}$, $\{s_m^{(r)}(x)\}$ is a corresponding Bochner–Fejér sequence, as $m \to \infty$, and this sequence converges to $f^{(r)}(x)$ in the appropriate norm. That is, as $m \to \infty$, $\{s_m^{(r)}(x)\}$ U-converges to $f^{(r)}(x)$ for $r = 0, 1, \ldots, \kappa-1$ and $\{s_m^{(\kappa)}(x)\}$ $G_{(0)}$-converges to $f^{(k)}(x)$ [**2**, pp. 50 and 105; **9**, p. 23]. Hence the sequence $\{s_m(x)\}$ $G_{(\kappa)}$-converges to $f(x)$ as $m \to \infty$, and Lemma 3 is established.

To prove Lemma 4 we make use of another lemma.

Lemma 5. *If $\kappa \geqq 1$ and $G_{(\kappa)}$ is fixed, then $f(x)$ is $G_{(\kappa)}$*ap *if and only if it is bounded and $f'(x)$ is $G_{(\kappa-1)}$*ap [**11**, Theorem 9].

Suppose, then, that $\kappa \geqq 1$ and that $\{f_n(x)\}$ is a sequence of $G_{(\kappa)}$ap functions such that $\| f_m - f_n \|_{G_{(\kappa)}} \to 0$ as $m > n \to \infty$. Then $\| f_n^{(r)} - f_m^{(r)} \|_U \to 0$ ($r = 0, 1, \ldots, \kappa-1$) and $\|f_n^{(\kappa)} - f_m^{(\kappa)}\|_{G_{(0)}} \to 0$ as $m > n \to \infty$. It follows from the completeness of the $G_{(0)}$ap spaces that there exist unique functions $\theta_r(x)$ such that $\theta_0(x), \theta_1(x), \ldots, \theta_{\kappa-1}(x)$ are all Uap, $\theta_\kappa(x)$ is $G_{(0)}$ap,

$$\|f_n^{(r)} - \theta_r\|_U \to 0 \quad \text{as} \quad n \to \infty \quad (r = 0, \ldots, \kappa-1), \tag{3.1}$$

and $\|f_n^{(\kappa)} - \theta_\kappa\|_{G_{(0)}} \to 0$ as $n \to \infty$.

Further, if $r = 1, 2, \ldots, \kappa$ and $a < x$, then

$$\int_a^x \theta_r(t)\,dt = \lim_{n\to\infty} \int_a^x f_n^{(r)}(t)\,dt = \lim_{n\to\infty} [f_n^{(r-1)}(x) - f_n^{(r-1)}(a)]$$
$$= \theta_{r-1}(x) - \theta_{r-1}(a).$$

Therefore $\theta_r(x) = \dfrac{d}{dx}\theta_{r-1}(x)$. If, then, we put $f(x) = \theta_0(x)$, it follows from κ applications

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of Lemma 5, since each of the Uap functions $\theta_0(x), \theta_1(x), \ldots, \theta_{\kappa-1}(x)$ is bounded, that $f(x)$ is $G_{(\kappa)}$ap. Finally, from (3.1),

$$\|f_n - f\|_{G(\kappa)} \to 0 \quad \text{as} \quad n \to \infty,$$

and the space of $G_{(\kappa)}$ap functions is therefore complete.

NOTE. We may note in passing, that, when $\kappa \geqq 1$, the space of $G_{(\kappa)}$ap functions shares with the other spaces of almost periodic functions the properties of being closed and of being identical to the closure, with respect to $\|\cdot\|_{G(\kappa)}$, of the space of trigonometric polynomials. The proofs of these properties follow readily from Lemmas 3 and 4 and from the corresponding properties of $G_{(0)}$ap functions.

We now show that Theorems 4, 5 and 6, which concern finite trigonometric sums, can be extended as follows to the classes of Gap functions that are being considered in this section.

THEOREM 7. *Let G be fixed and let $f(x)$ be a* Gap *function whose Fourier series is*

$$\sum a_h e^{i\lambda_h x}. \tag{3.2}$$

Then, if $\sup_h |\lambda_h| = M < \infty$ *and if* α *is any complex number such that* $\bar{\alpha} > 0$,

$$\sum a_h (i\lambda_h)^\alpha e^{i\lambda_h x} \tag{3.3}$$

is the Fourier series of another Gap *function* $f_{-\alpha}(x)$ *and*

$$\|f_{-\alpha}\|_G \leqq A(\alpha) M^{\bar{\alpha}} \|f\|_G,$$

where $A(\alpha)$ *depends only upon* α.

THEOREM 8. *Let G be fixed and let $f(x)$ be a* Gap *function whose Fourier series is* (3.2). *Then, if* $\inf_h |\lambda_h| = \Lambda > 0$ *and* α *is any complex number such that* $\bar{\alpha} > 0$,

$$\sum a_h (i\lambda_h)^{-\alpha} e^{i\lambda_h x}$$

is the Fourier series of another Gap *function* $f_\alpha(x)$ *and*

$$\| f_\alpha \|_G \leqq A(\alpha) \Lambda^{-\bar{\alpha}} \| f \|_G,$$

where $A(\alpha)$ *depends only upon* α.

THEOREM 9. *Let G be fixed and let* $\sum a_h (i\lambda_h)^{-\gamma} e^{i\lambda_h x}$ *and* $\sum a_h (i\lambda_h)^{-\beta} e^{i\lambda_h x}$ *be the Fourier series of two* Gap *functions* $f_\gamma(x)$ *and* $f_\beta(x)$, *respectively, where* γ *and* β *are complex and* $\bar{\gamma} < \bar{\beta}$. *Then, if* α *is any complex number such that* $\bar{\gamma} < \bar{\alpha} < \bar{\beta}$, *the series*

$$\sum a_h (i\lambda_h)^{-\alpha} e^{i\lambda_h x}$$

is the Fourier series of another Gap *function* $f_\alpha(x)$, *and*

$$\| f_\alpha \|_G \leqq K \| f_\gamma \|_G^\tau \| f_\beta \|_G^{1-\tau},$$

where $\tau = (\bar{\beta} - \bar{\alpha})/(\bar{\beta} - \bar{\gamma})$ *and* K *is a constant depending only upon* $\alpha - \gamma$ *and* $\beta - \gamma$.

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Since, in the case when G is U, Bang proves Theorem 9 and asserts (without proof) that Theorems 4 and 5 can be extended to include Uap functions, these last three theorems are not unexpected in the light of earlier results given in this paper.

To prove Theorem 7 let

$$s_m(x) = \sum a_h d_m^h \, e^{i\lambda_h x},$$

where $0 \leqq d_m^h \leqq 1$ and only a finite number of the set $\{d_m^h\}$ are nonzero, be a Bochner–Fejér polynomial of (3.2). Then

$$s_m^{(\alpha)}(x) = \sum a_h (i\lambda_h)^\alpha d_m^h \, e^{i\lambda_h x}$$

is a Bochner–Fejér polynomial of (3.3). Further, by Theorem 4 and Lemma 3,

$$\| s_m^{(\alpha)} - s_n^{(\alpha)} \|_G \leqq A(\alpha) M^{\bar{\alpha}} \| s_m - s_n \|_G$$

and

$$\| s_m - s_n \|_G \to 0 \quad \text{as} \quad m > n \to \infty.$$

Hence

$$\| s_m^{(\alpha)} - s_n^{(\alpha)} \|_G \to 0 \quad \text{as} \quad m > n \to \infty.$$

Now, by Lemma 4, the space of Gap functions is complete. Therefore there exists a Gap function $f_{-\alpha}(x)$ such that

$$\| s_m^{(\alpha)} - f_{-\alpha} \|_G \to 0 \quad \text{as} \quad m \to \infty,$$

and it follows that the Fourier series of $f_{-\alpha}(x)$ is (3.3). Finally, for arbitrary $\varepsilon > 0$ we choose $m(\varepsilon)$ such that

$$\| s_m - f \|_G \leqq \varepsilon \quad \text{and} \quad \| s_m^{(\alpha)} - f_{-\alpha} \|_G \leqq \varepsilon.$$

Then, by Theorem 4 again,

$$\begin{aligned} \| f_{-\alpha} \|_G &\leqq \varepsilon + \| s_m^{(\alpha)} \|_G \\ &\leqq \varepsilon + A(\alpha) M^{\bar{\alpha}} \| s_m \|_G \\ &\leqq \varepsilon + A(\alpha) M^{\bar{\alpha}} \{ \| f \|_G + \varepsilon \}. \end{aligned}$$

Since ε may be arbitrarily small, it follows that

$$\| f_{-\alpha} \|_G \leqq A(\alpha) M^{\bar{\alpha}} \| f \|_G.$$

The proof of Theorem 8 follows similarly from Theorem 5.

To prove Theorem 9 we consider the polynomials

$$s_{m,j}(x) = \sum a_h (i\lambda_h)^{-j} d_m^h \, e^{i\lambda_h x} \qquad (m = 1, 2, 3, \ldots;\ j = \alpha, \beta, \gamma),$$

where $\{s_{m,\beta}(x)\}$ and $\{s_{m,\gamma}(x)\}$ are sequences of Bochner–Fejér polynomials which G-converge, respectively, to $f_\beta(x)$ and $f_\gamma(x)$ as $m \to \infty$.

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As $s_{m,\alpha}(x)$ and $s_{m,\beta}(x)$ are integrals of $s_{m,\gamma}(x)$ of orders $\alpha-\gamma$ and $\beta-\gamma$, respectively, it follows from Theorem 6 that

$$\| s_{m,\alpha}-s_{n,\alpha} \|_G \leqq K \| s_{m,\gamma}-s_{n,\gamma} \|_G^{\tau} \| s_{m,\beta}-s_{n,\beta} \|_G^{1-\tau},$$

where $\tau=(\bar{\beta}-\bar{\alpha})/(\bar{\beta}-\bar{\gamma})$.

The proof is completed by an argument similar to that for Theorem 7.

4. Characterization of $G_{(\kappa)}$ap functions ($\kappa \geqq 0$). In this section we are only concerned with Gap functions when G is any one of U, S^p, V^p or $G_{(\kappa)}$, where $G_{(0)}$ is U, S^p or V^p, i.e., we do not consider W^pap or B^pap functions. We can conveniently denote any such function as $G_{(\kappa)}$ap where, now, $\kappa \geqq 0$. We use some of the results of § 1 to establish alternative definitions for these almost periodic functions.

We begin with two lemmas.

LEMMA 6. *If $f(x)$ is $G_{(\kappa)}$ap ($\kappa \geqq 0$), then*
(i) *it is $G_{(\kappa)}$-bounded, i.e. $\| f \|_{G_{(\kappa)}} < \infty$, and*
(ii) *it is $G_{(\kappa)}$-continuous, i.e. $\| f(x+h)-f(x) \|_{G_{(\kappa)}} \to 0$ as $h \to 0$.*

These are well-known properties for $G_{(0)}$ap functions. When $\kappa \geqq 1$, then, as was pointed out in the proof of Lemma 3, each of $f(x), f'(x), \ldots, f^{(\kappa-1)}(x)$ is Uap (and hence is U-bounded and U-continuous) and $f^{(\kappa)}(x)$ is $G_{(0)}$ap (and hence is $G_{(0)}$-bounded and $G_{(0)}$-continuous).

The rest of the proof follows at once from the definitions.

LEMMA 7. *If $f(x)$ is differentiable almost everywhere and $f(x)=\int^x f'(t)\,dt$, then*

$$\| f \|_U \leqq \| f \|_S + \| f' \|_S.$$

Proof.
$$\| f \|_U = \sup_x | f(x) | \leqq \sup_x \left\{ \left| \int_0^1 f(x+t)\,dt \right| + \left| \int_0^1 \{f(x)-f(x+t)\}\,dt \right| \right\}$$
$$\leqq \sup_x \int_x^{x+1} | f(t) |\,dt + \sup_x \int_0^1 dt \int_x^{x+t} | f'(u) |\,du$$
$$\leqq \| f \|_S + \| f' \|_S.$$

For any fixed $G_{(0)}$ we can now identify the class of $G_{(\kappa)}$ap functions ($\kappa \geqq 1$) with the class of bounded κth integrals of $G_{(0)}$ap functions.

THEOREM 10. *If $\kappa \geqq 1$ and $G_{(0)}$ is fixed, $f(x)$ is $G_{(\kappa)}$ap if and only if it is bounded and $f^{(\kappa)}(x)$ exists and is $G_{(0)}$ap.*

Proof. The necessity of the conditions follows directly from the definitions.

Sufficiency. Suppose that $f(x)$ is bounded and that $f^{(\kappa)}(x)$ is $G_{(0)}$ap. Then $f(x)$ is S-bounded and $f^{(\kappa)}(x)$, being certainly Sap, is S-bounded also. It follows then, from Theorem 1, that $f^{(r)}(x)$ is S-bounded for $r=0, 1, \ldots, \kappa$, and, from Lemma 7, that $f^{(r)}(x)$ is U-bounded for $r=0, 1, \ldots, \kappa-1$. Hence, by κ applications of Lemma 5, $f(x)$ is $G_{(\kappa)}$ap.

COROLLARY. *If* $\kappa \geqq 1$ *and* $G_{(0)}$ *is fixed,* $f(x)$ *is* $G_{(\kappa)}$ap *if and only if it is S-bounded and* $f^{(\kappa)}(x)$ *exists and is* $G_{(0)}$ap.

The proof is immediate.

Since each $G_{(\kappa)}$ap function is Sap it can also be characterized as an Sap function which is $G_{(\kappa)}$-continuous. We shall need the following lemma.

LEMMA 8. *If, for* $\kappa \geqq 0$, $f(x)$ *is* $G_{(\kappa)}$*-continuous and*

$$f_h(x) = h^{-1}\int_0^h f(x+t)\,dt$$

for $h > 0$, *then*

$$\| f_h - f \|_{G_{(\kappa)}} \to 0 \quad as \quad h \to 0.$$

Proof. If $\kappa = 0$ and $G_{(0)}$ is U or S^p ($p \geqq 1$), the result is well known (see e.g. [5], where Burkill also proves the lemma when $G_{(0)}$ is W^p or B^p). It will therefore suffice to prove it true when $G_{(0)}$ is V^p ($p > 1$). For then the lemma will be true when $\kappa = 0$ and, hence, for all $\kappa \geqq 0$. For, if $\kappa \geqq 1$, $f(x), f'(x), \ldots, f^{(\kappa-1)}(x)$ will all be U-continuous and $f^{(\kappa)}(x)$ will be $G_{(0)}$-continuous. Since

$$f_h^{(r)}(x) = h^{-1}\int_0^h f^{(r)}(x+t)\,dt$$

for $r = 0, 1, \ldots, \kappa$, the proof follows at once from the definition of $\|\cdot\|_{G_{(\kappa)}}$.

Suppose, therefore, that $G_{(0)}$ is V^p ($p > 1$) and, for any fixed y, π is the partition $y = x_0 < x_1 < \ldots < x_m = y+1$ of the interval $(y, y+1)$. Then (by use of Hölder's inequality)

$$\begin{aligned}
&\sum_{i=1}^{m} \left| [f_h(x_i) - f(x_i)] - [f_h(x_{i-1}) - f(x_{i-1})] \right|^p \\
&= \sum_{i=1}^{m} \left| h^{-1}\int_0^h \{[f(x_i+t) - f(x_i)] - [f(x_{i-1}+t) - f(x_{i-1})]\}\,dt \right|^p \\
&\leqq \sum_{i=1}^{m} h^{-p}\int_0^h \left| [f(x_i+t) - f(x_i)] - [f(x_{i-1}+t) - f(x_{i-1})] \right|^p dt \left\{\int_0^h dt\right\}^{p-1} \\
&\leqq h^{-1}\int_0^h V_p\{f(x+t) - f(x);\ y \leqq x \leqq y+1\}^p\,dt \\
&\leqq \sup_{0 \leqq t \leqq h} \sup_{y} V_p\{f(x+t) - f(x);\ y \leqq x \leqq y+1\}^p.
\end{aligned}$$

This last term tends to 0 as $h \to 0$ since, by hypothesis, $f(x)$ is V^p-continuous. If we take on the left-hand side the supremum first over all possible partitions π of the interval $(y, y+1)$ and then over all y, we see that

$$\sup_{y} V_p\{f_h(x) - f(x);\ y \leqq x \leqq y+1\} \to 0 \quad \text{as} \quad h \to 0.$$

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From this, the definition of $\|\cdot\|_{VP}$ and the present lemma when $G_{(0)}$ is U, it follows that

$$\| f_h - f \|_{VP} \to 0 \quad \text{as} \quad h \to 0.$$

Hence the lemma is established.

The second characterization of $G_{(\kappa)}$ap functions can now be given.

THEOREM 11. *If $\kappa \geqq 0$ and $G_{(0)}$ is fixed, $f(x)$ is $G_{(\kappa)}$*ap *if and only if it is S*ap *and $G_{(\kappa)}$-continuous.*

Proof. That these conditions are necessary for $f(x)$ to be $G_{(\kappa)}$ap follows at once from the definitions of $G_{(\kappa)}$ap functions and Lemma 6(ii).

To prove the converse, first let $\kappa = 0$ and suppose that $f(x)$ is Sap and $G_{(0)}$-continuous. For $0 < h \leqq 1$ we define

$$f_h(x) = h^{-1} \int_0^h f(x+t)\,dt = h^{-1} \int_x^{x+h} f(u)\,du.$$

Then, for almost all x, $f_h'(x) = h^{-1}\{f(x+h) - f(x)\}$ and is therefore Sap, being the difference between two Sap functions. Also

$$\| f_h \|_U \leqq \sup_x h^{-1} \int_x^{x+1} |f(u)|\,du \leqq h^{-1} \| f \|_S,$$

which is finite by Lemma 6(i). Hence, by Theorem 10, $f_h(x)$ is $S_{(1)}$ap and thus is $G_{(0)}$ap for all possible $G_{(0)}$ (see [**11**, pp. 423–424]). Furthermore, it follows from the hypothesis and Lemma 8 that $\| f_h - f \|_{G_{(0)}} \to 0$ as $h \to 0$. Hence $f(x)$, being the $G_{(0)}$-limit of a sequence of $G_{(0)}$ap functions, is itself $G_{(0)}$ap.

If $\kappa \geqq 1$, it follows from the hypothesis that $f(x), f'(x), \ldots, f^{(\kappa-1)}(x)$ are all U-continuous and that $f^{(\kappa)}(x)$ is S-continuous. Now it is known that, if an Sap function has an S-continuous derivative, then that derivative is also Sap. (See Bochner [**3**]. Alternatively, if

$$\Phi(x) = \int_0^x \phi(t)\,dt$$

is Sap, then so is $\phi_h(x) = h^{-1}\{\Phi(x+h) - \Phi(x)\}$ for $h > 0$. If $\phi(x)$ is S-continuous, then Lemma 8 shows that, as $h \to 0$, $\phi_h(x)$ S-converges to $\phi(x)$, which is therefore Sap.) Successive applications of this result show that $f(x), f'(x), \ldots, f^{(\kappa)}(x)$ are all Sap. Since, by hypothesis, $f^{(\kappa)}(x)$ is $G_{(0)}$-continuous, the argument in the earlier part of this proof shows that it is also $G_{(0)}$ap. Hence $f(x)$, which is S-bounded, being Sap, is $G_{(\kappa)}$ap by Theorem 10, Corollary.

COROLLARY. *For any fixed $G_{(0)}$, $f(x)$ is $G_{(\kappa)}$*ap *if and only if it is S*ap *and $f^{(\kappa)}(x)$ is $G_{(0)}$-continuous.*

Proof. As in Theorem 11, the conditions are clearly necessary. If, on the other hand, $f(x)$ satisfies them, then $f(x)$ and $f^{(\kappa)}(x)$ are both S-continuous, so that, by Theorem 1, with $G = S$ and $f(x)$ replaced by $f(x+h) - f(x)$, it follows that $f^{(r)}(x)$ is S-continuous for $r = 0, 1, \ldots, \kappa$. Hence. by Lemma 7, with $f(x)$ replaced, in turn, by $f^{(r)}(x+h) - f^{(r)}(x)$, for

$r = 0, 1, 2, \ldots, \kappa - 1$, we see that $f(x), f'(x), \ldots, f^{(\kappa-1)}(x)$ are all U-continuous. Hence $f(x)$ is $G_{(\kappa)}$-continuous. The result then follows from Theorem 11.

In [5], Burkill established a theorem similar to Theorem 11 for classes of Denjoy and Cesàro–Perron almost periodic functions (Dap and $C_rP$ap respectively) each of which contains the Uap and Sap classes. If the two theorems are combined we have the following theorem.

THEOREM 12. *If* $f(x)$ *is* $C_rP$ap *for some* $r \geqq 0$ *and is G-continuous, where G is any one of* $G_{(\kappa)}(\kappa \geqq 0)$, W^p, B^p, D, *and* C_sP $(s \geqq 0)$, *then* $f(x)$ *is* Gap.

Since W^pap functions and B^pap functions need not be $C_sP$ap, for they can be uniformly continuous without even being Uap, the converse of Theorem 12 will only be true if W^pap and B^pap functions are excluded.

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