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# NEW INEQUALITIES FOR PLANAR CONVEX SETS WITH LATTICE POINT CONSTRAINTS

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We obtain new inequalities relating the inradius of a planar convex set with interior containing no point of the integral lattice, with the area, perimeter and diameter of the set. By considering a special sublattice of the integral lattice, we also obtain an inequality concerning the inradius and area of a planar convex set with interior containing exactly one point of the integral lattice.

## 1. INTRODUCTION

Let $K$ be a compact, planar convex set with interior $K^o$, and having area $A = A(K)$, perimeter $p = p(K)$, diameter $d = d(K)$ and inradius $r = r(K)$. Let $\Gamma$ denote the integral lattice and let $G(K^o, \Gamma)$ denote the number of points of $\Gamma$ in $K^o$. We prove new inequalities relating $A$, $p$, $d$ and $r$.

**THEOREM 1.** *Let $K$ be a compact, planar, convex set with $G(K^o, \Gamma) = 0$. Then*

$$(1) \qquad (2r-1)A \leqslant 2\left(\sqrt{2}-1\right) \approx 0.828,$$

*with equality when and only when $K$ is congruent to the diagonal square shown in Figure 1.*

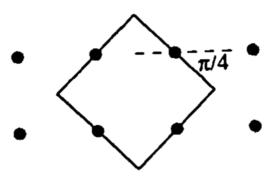


Figure 1.

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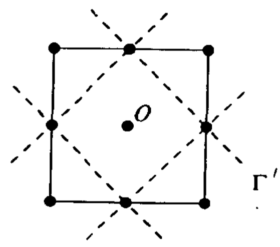


Figure 2.

**Corollary 1.** *Let $K$ be a compact, planar, convex set with $G(K^o,\Gamma) = 1$. Then*

$$(2) \qquad \left(2r - \sqrt{2}\right)A \leqslant 4\left(2 - \sqrt{2}\right) \approx 2.343,$$

*with equality when and only when $K$ is the square shown in Figure 2.*

**Theorem 2.** *Let $K$ be a compact, planar, convex set with $G(K^o,\Gamma) = 0$. Then*

$$(3) \qquad (2r - 1)\,|A - 1| < \frac{1}{2},$$

$$(4) \qquad (2r - 1)\,|p - 4| < 2,$$

$$(5) \qquad (2r - 1)(d - 1) < 1.$$

*The limiting infinite strip shows that the stated bounds are best possible.*

## 2. Proofs of Theorem 1 and Corollary 1

We first prove two useful lemmas.

**Lemma 1.** *Let $X_l$ be the Steiner symmetral of $X$ with respect to the line $l$. Then $r(X_l) \geqslant r(X)$.*

Proof: We first show that if $K \subseteq X$, then $K_l \subseteq X_l$. Let $PQ$ be any chord of $K$ perpendicular to $l$. Since $K \subseteq X$, the line $PQ$ intersects $X$ in a chord $AB$ with $|PQ| \leqslant |AB|$. Now Steiner symmetrisation maps chord $PQ$ to a chord $P'Q'$ on the line $PQ$, and having its midpoint on $l$ (see for example [1, p.90]). Similarly, the chord $AB$ is mapped to the chord $A'B'$ on the line $PQ$ and having midpoint on $l$. Since $|PQ| \leqslant |AB|$, the chord $P'Q'$ is a subset of the chord $A'B'$. Hence $K_l \subseteq X_l$.

Now let $C$ be an incircle of $X$. Then $C \subseteq X$ and $C_l \subseteq X_l$. But $C_l$ is congruent to $C$. It follows that $X_l$ contains a circle of radius $r(X)$. Therefore $r(X_l) \geqslant r(X)$. ∎

**Lemma 2.** *Let $K$ be a compact, planar, convex set with $G(K^o,\Gamma) = 0$. Then there is a compact convex set $K_*$ with $G(K_*^o,\Gamma) = 0$ satisfying the following conditions:*

(a) $A(K_*) = A(K)$, $r(K_*) \geqslant r(K)$,

(b) *$K_*$ is symmetric about the lines $x = 1/2$, $y = 1/2$.*

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PROOF: We use Steiner symmetrisation to obtain the set $K_*$. We first symmetrise $K$ with respect to the line $x = 1/2$ to obtain the set $K_1$. We recall that Steiner symmetrisation preserves convexity and areas so that $K_1$ is a convex set with $A(K_1) = A(K)$. Furthermore, by Lemma 1, $r(K_1) \geqslant r(K)$.

We now show that $G(K_1^o, \Gamma) = 0$. Since $G(K^o, \Gamma) = 0$, $K^o$ intersects the line $y = k$, where $k$ is an integer, either in the empty set or in a line segment of length at most 1. Hence the symmetric set $K_1^o$ intersects the line $y = k$ either in the empty set or between the points $(0, k)$ and $(1, k)$. Clearly, $G(K_1^o, \Gamma) = 0$.

We now symmetrise $K_1$ with respect to the line $y = 1/2$ to obtain $K_*$. Using the same arguments as above, we have $A(K_*) = A(K_1)$, $r(K_*) \geqslant r(K_1)$ and $G(K_*^o, \Gamma) = 0$. Hence $A(K_*) = A(K)$ and $r(K_*) \geqslant r(K)$. By construction, $K_*$ is symmetric about the lines $x = 1/2$ and $y = 1/2$ and the lemma is proved. □

Let $f(K) = (2r(K) - 1)A(K)$. By Lemma 2 we have $f(K) \leqslant f(K_*)$. It therefore suffices to prove Theorem 1 for sets $K$ which are symmetric about the lines $x = 1/2$ and $y = 1/2$.

To fully utilise the symmetry of $K$ about the lines $x = 1/2$ and $y = 1/2$, we move the origin to the point $(1/2, 1/2)$. If $r \leqslant 1/2$, (1) is trivially true. Hence we may assume that $r > 1/2$. Since $K^o$ does not contain the points $P_1(1/2, 1/2)$, $P_2(-1/2, 1/2)$, $P_3(-1/2, -1/2)$ and $P_4(1/2, -1/2)$, it follows by the convexity of $K$ that for each $i = 1, \dots, 4$, $K$ is bounded by a line $l_i$ through the point $P_i$, with $l_1$ and $l_3$ having negative slope and $l_2$ and $l_4$ having positive slope. Furthermore since $K$ is symmetric about the coordinate axes, $K$ is contained in a rhombus $Q$ determined by the lines $l_i$, $i = 1, \dots, 4$. Since $K \subseteq Q$, $A(K) \leqslant A(Q)$ and $r(K) \leqslant r(Q)$ we have $f(K) \leqslant f(Q)$. It is therefore sufficient to maximise $f(K)$ over the set of all rhombi, $K = Q$, determined by the lines $l_i$, $i = 1, \dots, 4$ (see Figure 3).

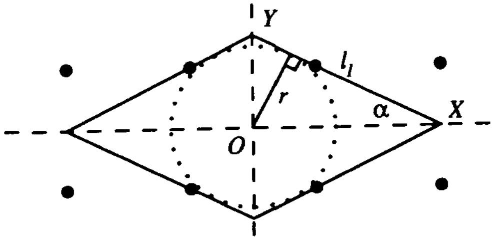


Figure 3.

Let side $l_1$ make an acute angle of $\alpha$ with the $x$-axis and let it intercept the $x$ and $y$ axes in the points $X(x, 0)$ and $Y(0, y)$ respectively. Since $l_1$ passes through

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$(1/2, 1/2)$, similar triangles give

$$\frac{y}{x} = \frac{\frac{1}{2}}{x - \frac{1}{2}},$$

that is,

$$\frac{1}{x} + \frac{1}{y} = 2.$$

Multiplying both sides of the equation by $r$, we get

$$2r = \frac{r}{x} + \frac{r}{y} = \sin\alpha + \cos\alpha.$$

Now

$$\begin{aligned} A &= 4.A(\triangle OXY) \\ &= 2xy \\ &= \frac{2r^2}{\sin\alpha\cos\alpha} \\ &= \frac{4r^2}{(\sin\alpha + \cos\alpha)^2 - 1} \\ &= \frac{4r^2}{4r^2 - 1} \\ &= 1 + \frac{1}{4r^2 - 1}. \end{aligned}$$

Hence

$$f(K) = 2r - 1 + \frac{1}{2r+1} = g(r). \tag{6}$$

Now $(1/2)g'(r) = 1 - 1/(2r+1)^2 > 0$. Hence $g$ is an increasing function of $r$. Noting that $1/2 < r \leqslant \sqrt{2}/2$, the maximal value of $g$ is therefore attained at $r = \sqrt{2}/2$, that is, when and only when $K$ is congruent to the diagonal square shown in Figure 1. In this case

$$f(K) \leqslant 2\left(\sqrt{2} - 1\right) \approx 0.828.$$

We next use Theorem 1 to prove Corollary 1. Let $K$ now be a convex set with $G(K^o, \Gamma) = 1$. Without loss of generality we may assume that the lattice point contained in $K^o$ is the origin $O$. Let $\Gamma'$ be the sublattice of $\Gamma$ with fundamental cell having vertices $(0, \pm 1)$, $(\pm 1, 0)$. We first note that $G(K^o, \Gamma') = 0$ (see Figure 2). Hence letting $A'$ and $r'$ be the area and the inradius respectively of $K$ measured in the scale of $\Gamma'$, and applying (1) to $K$ with respect to $\Gamma'$, we have

$$(2r' - 1)A' \leqslant 2\left(\sqrt{2} - 1\right),$$

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with equality when and only when $K$ is congruent to the square of Figure 2. Since $\Gamma'$ is a rotation of $\Gamma$ scaled by a factor of $\sqrt{2}$, $A' = (1/2)A$ and $r' = \left(1/\sqrt{2}\right)r$ where $A$ and $r$ are the area and the inradius respectively of $K$ measured in the scale of the integral lattice $\Gamma$. Hence

$$\left(2.\frac{1}{\sqrt{2}}r - 1\right)\frac{A}{2} \leqslant 2\left(\sqrt{2}-1\right).$$

Simplifying, we get

$$\left(2r - \sqrt{2}\right)A \leqslant 4\left(2-\sqrt{2}\right) \approx 2.343,$$

with equality when and only when $K$ is congruent to the square of Figure 2.

## 3. Proof of Theorem 2

We first note that if $r \leqslant 1/2$, inequalities (3) and (4) are trivially true. Hence we need only consider those cases for which $1/2 < r \leqslant \sqrt{2}/2$.

To prove (3), we first consider $A \leqslant 1$. Since $r > 1/2$, we have $A > \pi/4$ and so

$$(2r-1)\,|A-1| = (2r-1)(1-A) < \left(\sqrt{2}-1\right)\left(1-\frac{\pi}{4}\right) < \frac{1}{2}.$$

Hence we may assume that $A > 1$. Using the same arguments as those given in Section 2, it suffices to consider a set $K$ where $K$ is a rhombus of the type described in Figure 3. Let $Q(r)$ denote such a rhombus with inradius $r$. From (6) we have

$$(2r-1)\,|A-1| = (2r-1)(A-1) = \frac{1}{2r+1} < \frac{1}{2}.$$

Taking the infinite strip to be the limit of $Q(r)$ as $r$ tends to $1/2$, it is seen that the stated bound is best possible.

To prove (4), we first consider $p \leqslant 4$. Since $r > 1/2$, we have $p > \pi$ and so

$$(2r-1)\,|p-4| = (2r-1)(4-p) < \left(\sqrt{2}-1\right)(4-\pi) < 2.$$

Hence we may assume that $p > 4$. We note further that if $K$ is a convex polygon, $K$ may be partitioned into triangles by joining each vertex of $K$ to an in-centre of $K$. Summing the areas of these triangles gives

$$A \geqslant \frac{1}{2}pr, \tag{7}$$

with equality when and only when every edge of $K$ touches the unique incircle. Since any compact convex set may be approximated by a convex polygon, this inequality is

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valid for all compact convex sets in the plane. By combining inequality (7) with (3) and noting that $r > 1/2$, we have

$$(2r-1)\,|p-4| = (2r-1)(p-4) \leqslant (2r-1)\left(\frac{2A}{r}-4\right) \leqslant 4(2r-1)(A-1) \leqslant 4.\frac{1}{2} = 2,$$

obtaining (4). As before, taking the infinite strip to be the limit of $Q(r)$ as $r$ tends to $1/2$, the stated bound is best possible.

Finally, to prove (5), we note that $(w-1)(d-1) \leqslant 1$ with equality when and only when $K$ is a triangle of the type shown in Figure 4 (see [2]).

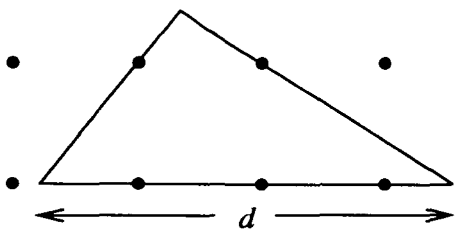


Figure 4.

Since $w \geqslant 2r$, we have

$$(2r-1)(d-1) \leqslant (w-1)(d-1) \leqslant 1.$$

Taking the infinite strip to be the limit of a sequence of triangles of the type shown in Figure 4 as $w$ tends to $2r$, it can be seen that the stated bound is best possible.

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