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DIHEDRAL GROUPS OF AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

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ABSTRACT. In this note we determine which dihedral subgroups of $\mathrm{GL}_g(\mathbb{C})$ can be realized by group actions on Riemann surfaces of genus $g > 1$.

1**. Introduction.** We study the realizability problem for dihedral groups in $\mathrm{GL}_g(\mathbb{C})$. This is a special case of a more general problem. A group G of analytic automorphisms of a Riemann surface S of genus $g > 1$ can be represented as a subgroup $R(S, G)$ of $\mathrm{GL}_g(\mathbb{C})$ by passing to the induced action on the vector space $\mathbb{V}$ of holomorphic differentials. The problem is to determine those subgroups of $\mathrm{GL}_g(\mathbb{C})$ which are conjugate to $R(S, G)$ for some S and some G. In 1983, I. Kuribayashi proved that an element A of prime order in $\mathrm{GL}_g(\mathbb{C})$ is realizable if and only if A satisfies the "Eichler trace formula" [1]. In 1986 and 1990, I. Kuribayashi and A. Kuribayashi determined all realizable subgroups of $\mathrm{GL}_g(\mathbb{C})$ for $g \leq 5$ (see [2], [3], [4] and [5]). We consider dihedral groups D_{2p}, where p is an odd prime.

MAIN THEOREM. *A dihedral subgroup of order* $2p$ *in* $\mathrm{GL}_g(\mathbb{C})$, *$p$ an odd prime, is realized by an action on a Riemann surface of genus g iff each non-identity element has integer trace less than or equal to* 1.

2**. Some lemmas.** The essential ingredients of the proof are the relationships between group actions on compact connected Riemann surfaces and Fuchsian groups, as well as the Lefschetz Fixed Point Formula. Let D_{2p} be the dihedral group of $2p$ elements and T_p, $T_2 \in D_{2p}$ be two fixed generators of orders p and 2, with the relations $T_p^p = T_2^2 = (T_pT_2)^2 = 1$. Suppose there is an embedding of D_{2p} in $\mathrm{Aut}(S)$. Then we have a faithful representation $R: D_{2p} \longrightarrow \mathrm{GL}_g(\mathbb{C})$, by passing to the space of holomorphic differentials on S. Recall that the genus of S is assumed to be > 1.

We want to characterize such subgroups $R(D_{2p})$. We denote by $D_{2p}(A, B)$ any subgroup of $\mathrm{GL}_g(\mathbb{C})$ generated by A, B with the relation $A^p = B^2 = (AB)^2 = I$. Let $G_i = D_{2p}(A_i, B_i)$, $i = 1, 2$. G_1 and G_2 are said to be *conjugate*, denoted by $G_1 \sim G_2$, if there is $Q \in \mathrm{GL}_g(\mathbb{C})$ such that $Q^{-1}G_1Q = G_2$, and strongly conjugate, denoted by $G_1 \approx G_2$, if $Q^{-1}A_1Q = A_2$ and $Q^{-1}B_1Q = B_2$. A subgroup $D_{2p}(A, B)$ is said to be *realizable* if it is conjugate to some $R(D_{2p})$.

It is well known that the trace of an element of order 2 in $\mathrm{GL}_g(\mathbb{C})$ is an integer, and the trace of an element of order p in $\mathrm{GL}_g(\mathbb{C})$ is an algebraic integer in the cyclotomic field

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$\mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi i}{p}}$. A subgroup G in $\mathrm{GL}_g(\mathbb{C})$ is called an *I-group* if all elements of G have integer traces and furthermore G is called an *IR-group* if each non-identity element has integer trace less than or equal to 1.

Let $X \in D_{2p}(A,B)$ be of order p. Then $X \sim X^{-1}$, and hence $\mathrm{tr}(X) = \mathrm{tr}(X^{-1}) = \overline{\mathrm{tr}(X)}$. Therefore $\mathrm{tr}(X)$ is a real number. Furthermore if $\mathrm{tr}(X)$ is rational, then $\mathrm{tr}(X)$ is an integer.

LEMMA 1. *If some element $X \in D_{2p}(A,B)$ of order p has rational trace, then the group $D_{2p}(A,B)$ is an I-group and all elements of order p in $D_{2p}(A,B)$ are conjugate.*

PROOF. It is clear that $\mathrm{tr}(X) = k + k_1(\zeta + \zeta^{-1}) + \cdots + k_m(\zeta^m + \zeta^{-m})$ $(m = \frac{p-1}{2})$, for some non-negative integers $k, k_1, \ldots, k_m$ with $k + 2(k_1 + \cdots + k_m) = g$. But $\zeta, \ldots, \zeta^{p-1}$ are independent over the rational field $\mathbb{Q}$, so we have $k_1 = \cdots = k_m$, say l. Therefore $\mathrm{tr}(X) = k - l$ is an integer. Also we have some matrix $Q \in \mathrm{GL}_g(\mathbb{C})$ such that $Q^{-1}XQ = A_l$ where

$$A_l = \begin{pmatrix} I_k & & & \\ & \zeta I_l & & \\ & & \ddots & \\ & & & \zeta^{p-1} I_l \end{pmatrix}.$$

All elements of order p are powers of X, so they have the same canonical form and the lemma follows. ∎

LEMMA 2. *Suppose $G_i = D_{2p}(A_i, B_i)$, $i = 1, 2$, are two I-groups. Then the following three conditions are equivalent.*

1. $G_1 \sim G_2$;
2. $G_1 \approx G_2$;
3. $\mathrm{tr}(A_1) = \mathrm{tr}(A_2)$ *and* $\mathrm{tr}(B_1) = \mathrm{tr}(B_2)$.

PROOF. Let $G = D_{2p}(A,B)$ be an I-group. Let $Q \in \mathrm{GL}_g(\mathbb{C})$ be a matrix such that $Q^{-1}AQ = A_l$. Then from the relation $AB = BA^{-1}$ it follow that

$$Q^{-1}BQ = \begin{pmatrix} B_{1,1} & & & \\ & & & B_{2,p} \\ & & \iddots & \\ & B_{p,2} & & \end{pmatrix}$$

where $B_{1,1}$ is a $k \times k$ matrix, and $B_{2,p}, B_{3p-1}, \ldots, B_{p,2}$ are $l \times l$ matrices. We also have $B_{1,1}^2 = I_k$ and $B_{i,j}B_{j,i} = I_l$ since $B^2 = I$. The matrix $Q^{-1}BQ$ can be conjugated to

$$B_{x,y} = \begin{pmatrix} I_x & & & \\ & -I_y & & \\ & & & I_l \\ & & \iddots & \\ & I_l & & \end{pmatrix}$$

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where $x + y = k$, by a matrix R commuting $Q^{-1}AQ$. In fact

$$R = \begin{pmatrix} R_1 & & & & \\ & B_{2,p} & & & \\ & & \ddots & & \\ & & & B_{\frac{p+1}{2},\frac{p+3}{2}} & \\ & & & & I_{\frac{l(p-1)}{2}} \end{pmatrix}$$

where $R_1^{-1}B_{1,1}R_1 = \begin{pmatrix} I_x & \\ & -I_y \end{pmatrix}$. Thus every dihedral I-group $G \approx D_{2p}(A_l, B_{x,y})$. By simple calculation we have $g = x + y + (p-1)l$ and $\mathrm{tr}(A_l) = x + y - l$. The number of I_l blocks in $B_{x,y}$ is $p - 1$, an even number, and therefore $\mathrm{tr}(B_{x,y}) = x - y$. From these equations the equivalences easily follow. ∎

3**. Proof of Main Theorem.** If σ is an automorphism of S of finite order greater than 1, then we have the Lefschetz Fixed Point Formula, $\mathrm{tr}(\sigma) + \overline{\mathrm{tr}(\sigma)} = 2 - \mathrm{Fix}(\sigma)$, where $\mathrm{Fix}(\sigma)$ is the number of fixed points of σ, see [7]. It is easy to deduce

LEMMA 3. *If $D_{2p}(A, B)$ is realizable, then $D_{2p}(A, B)$ is an* IR*-group.*

PROOF. In our case $\mathrm{tr}(\sigma) = \overline{\mathrm{tr}(\sigma)}$, and $\mathrm{Fix}(\sigma)$ is an integer. Hence $D_{2p}(A, B)$ is an I-group, see Lemma 1. Also since $\mathrm{Fix}(\sigma) \geq 0$, we get $D_{2p}(A, B)$ is an IR-group. ∎

Thus we have completed the proof of the necessity condition of Main Theorem.

To any action of D_{2p} on S we can associate a short exact sequence of groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma = \Gamma(g_0; \overbrace{p, \dots, p}^{t}, \overbrace{2, \dots, 2}^{s}) \xrightarrow{\theta} D_{2p} \longrightarrow 1$$

where Γ has generators

$$X_1, \dots, X_{g_0}, Y_1, \dots, Y_{g_0}, A_1, \dots, A_t, B_1, \dots, B_s$$

and relations

$$A_1^p = \cdots = A_t^p = B_1^2 = \cdots = B_s^2 = [X_1, Y_1] \cdots [X_{g_0}, Y_{g_0}] A_1 \cdots A_t B_1 \cdots B_s = 1 \tag{1}$$

By the Riemann-Hurwitz formula

$$(g-1) = 2p(g_0 - 1) + (p-1)t + \frac{ps}{2} \tag{2}$$

we see that s must be even. From the results of Macbeath [6], we see that $\mathrm{Fix}(T_p) = 2t$ and $\mathrm{Fix}(T_2) = s$. Hence if $D_{2p}(A, B)$ is realized by this action then $\mathrm{tr}(A) = 1 - t$ and $\mathrm{tr}(B) = \frac{2-s}{2}$. Conversely, if we have such a short exact sequence and Π is torsion free then we can deduce a group action of D_{2p} on some S of genus g which is given by (2) and $\mathrm{tr}(T_p) = 1 - t$, $\mathrm{tr}(T_2) = \frac{2-s}{2}$. To prove the sufficiency condition of the Main Theorem, we also need the following lemma.

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LEMMA 4. *Assume that $D_{2p}(A, B)$ is an* IR*-group. Then $\frac{1}{2p}\big(g + (p-1)\operatorname{tr}(A) + p\operatorname{tr}(B)\big)$ is a non-negative integer.*

PROOF. This is an easy calculation. Let A, B be of forms A_l, $B_{x,y}$, as in the proof of Lemma 2.

$$\begin{aligned} g + (p-1)\operatorname{tr}(A) + p\operatorname{tr}(B) &= x + y + (p-1)l + (p-1)(x+y-l) + p(x-y) \\ &= p(x+y) + p(x-y) \\ &= 2px. \end{aligned}$$

Thus $\frac{1}{2p}\big(g + (p-1)\operatorname{tr}(A) + p\operatorname{tr}(B)\big) = x$ is a non-negative integer. ■

Now we can complete the proof of the Main Theorem.

PROOF OF MAIN THEOREM. Let $t = 1 - \operatorname{tr}(A)$, $s = 2 - 2\operatorname{tr}(B)$, and

$$g_0 = \frac{1}{2p}\big(g + (p-1)\operatorname{tr}(A) + p\operatorname{tr}(B)\big).$$

We define an epimorphism $\theta\colon \Gamma(g_0; \overbrace{p, \dots, p}^{t}, \overbrace{2, \dots, 2}^{s}) \longrightarrow D_{2p}$ as follows:

CASE 1. If $\operatorname{tr}(A) \le 0$ and $\operatorname{tr}(B) \le 0$, then $t \ge 1$ and $s \ge 2$. We define

$$\theta(A_i) = T_p^{a_i}, \quad \theta(B_j) = T_p^{b_j} T_2 \quad \text{and} \quad \theta(X_k) = \theta(Y_k) = 1$$

where a_i, b_j are integers with $1 \le a_i \le p-1$ and $\sum_{i=1}^{t} a_i + \sum_{j=1}^{s} (-1)^{s+1} b_j \equiv 0 \pmod p$.

CASE 2. If $\operatorname{tr}(A) = 1$ and $\operatorname{tr}(B) \le -1$, then $t = 0$ and $s \ge 4$. We let

$$\theta(B_i) = T_p^{b_i} T_2 \quad \text{and} \quad \theta(X_j) = \theta(Y_j) = 1,$$

where b_i are integers (not all the same) with $0 \le b_i \le p-1$ and $\sum_{i=1}^{s} (-1)^i b_i \equiv 0 \pmod p$.

CASE 3. If $\operatorname{tr}(A) \le 0$ and $\operatorname{tr}(B) = 1$, then $t \ge 1$, $s = 0$, and $g_0 \ge 1$. We set

$$\theta(A_i) = T_p^{a_i}, \quad \theta(X_j) = T_p^{c_j} \quad \text{and} \quad \theta(Y_j) = T_2,$$

where a_i, c_j are integers with $1 \le a_i \le p-1$ and $\sum_{i=1}^{t} a_i + 2\sum_{j=1}^{g_0} c_j \equiv 0 \pmod p$.

CASE 4. If $\operatorname{tr}(A) = 1$, $\operatorname{tr}(B) = 0$, then $t = 0$ and $s = 2$, and $g_0 \ge 1$. We define

$$\theta(B_1) = \theta(B_2) = T_2 \quad \text{and} \quad \theta(X_i) = \theta(Y_i) = T_p.$$

CASE 5. If $\operatorname{tr}(A) = 1$ and $\operatorname{tr}(B) = 1$, then $t = 0$, $s = 0$, and $g_0 \ge 2$. We define

$$\theta(X_1) = \theta(Y_1) = T_p \quad \text{and} \quad \theta(X_i) = \theta(Y_i) = T_2 \quad (\text{for } i = 2, \dots, g_0).$$

It is easy to check that θ is a well defined epimorphism in all cases. Let $\Pi = \operatorname{Ker}(\theta)$. We get a short exact sequence of Fuchsian groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma(g_0; \overbrace{p, \dots, p}^{t}, \overbrace{2, \dots, 2}^{s}) \xrightarrow{\theta} D_{2p} \longrightarrow 1.$$

It is also easy to check that Π is torsion free and then there is an action of D_{2p} on some S, by Lemma 2, which realizes $D_{2p}(A, B)$. ■

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COROLLARY 1. *The minimal genus of D_{2p} is $p-1$.*

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