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ON A METHOD FOR CONSTRUCTING BERGMAN KERNELS

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Abstract

We establish a method of constructing kernels of Bergman operators for second-order linear partial differential equations in two independent variables, and use the method for obtaining a new class of Bergman kernels, which we call *modified class E kernels* since they include certain class E kernels. They also include other kernels which are suitable for *global* representations of solutions (whereas Bergman operators generally yield only *local* representations).

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1. Introduction

Let $\Omega = G \times G^* \subset \mathbf{C}^2$, where G is a simply connected domain in the complex z-plane such that $0 \in G$, and G^* is the corresponding domain in the z^*-plane. Consider the differential equation

$$Lu = u_{zz^*} + b(z, z^*)\, u_{z^*} + c(z, z^*)\, u = 0, \tag{1.1}$$

where $b, c \in C^\omega(\Omega)$. Note that (1.1) can be obtained from

$$\Delta w + A(x, y)\, w_x + B(x, y)\, w_y + C(x, y)\, w = 0$$

by setting $z = x + iy$, $z^* = x - iy$ (x, y complex), and that the absence of the u_z-term in (1.1) is no restriction of generality. We exclude the trivial case $c = 0$. All C^ω-solutions of (1.1) in Ω can be locally represented by Bergman operators. Such an operator

$$T_g : C^\omega(D_{\hat{r}}) \to C^\omega(D_{\hat{r}} \times D^*_{r^*})$$

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is defined by

$$(1.2) \qquad (T_g f)(z, z^*) = \int_{-1}^{1} g(z, z^*, t)\, f(z\tau)(2\tau)^{-\frac{1}{2}}\, dt, \quad \tau = \tfrac{1}{2}(1-t^2),$$

where $\hat{r} = \min(r, 2\tilde{r})$, D_λ is an open disc of radius λ about the origin, $g \in C^\omega(\Omega_0 \times J)$, $\Omega_0 = D_r \times D^*_{r^*} \subset \Omega$, and $J = [-1, 1]$.

DEFINITION 1.1. g is called a *Bergman kernel* for L(in Ω_0) if

$$(1.3) \qquad Kg = 2\tau g_{z^*t} - t^{-1} g_{z^*} + 2ztLg = 0 \quad (z, z^*, t) \in \Omega_0 \times J$$

$$(1.4a) \qquad (zt)^{-1} g_{z^*} \in C^0(\Omega_0 \times J),$$

$$(1.4b) \qquad \tau^{\frac{1}{2}} g_{z^*} \to 0 \quad \text{as } t \to \pm 1 \text{ uniformly in } \Omega_0.$$

An operator T_g with such a kernel is called a *Bergman operator* for L (in Ω_0), and f an *associated function* of $u = T_g f$.

THEOREM 1.2 (Bergman (1969)). *If $f \in C^\omega(D_{\hat{r}})$ and T_g is a Bergman operator for L in Ω_0, then $u = T_g f \in C^\omega(\hat{\Omega}_0)$ and $Lu = LT_g f = 0$ in $\hat{\Omega}_0 = D_{\hat{r}} \times D^*_{\hat{r}^*}$.*

By means of Bergman operators one can utilize methods and results of complex analysis for characterizing general properties of solutions of (1.1); in this way one obtains theorems on the location and type of singularities, the coefficient problem, the growth near the boundary of the domain of holomorphy and other basic properties; for some recent developments, see Meister and others (1976). In general, an equation (1.1) being given, there exist various Bergman kernels g. In connection with those applications, the 'simplicity' of g is essential. Hence the development of methods for constructing suitable Bergman kernels is a fundamental problem, which has attracted particular interest during the past decade and is far from being solved, although important special classes of kernels (first and second kind, class E (see below), class P) have been introduced and investigated.

In connection with *Riemann's method*, the determination of the Riemann function of a given equation (1.1) is often accomplished by means of *ordinary* differential equations; see Wood (1976) and Geddes and Mackie (1977). This suggests a similar approach for Bergman kernels, although K in (1.3) is more complicated than the adjoint L^*. A first systematic contribution in that direction was made by Florian (1962, 1965). Relations to the theory of class P operators were later obtained by Kreyszig (1973). In the present paper we show that the method of ordinary differential equations can be developed to include certain operators of class E and other Bergman operators which yield *global* representations of solutions.

2. Ordinary differential equations for modified class E kernels

For a Bergman kernel

(2.1) $$g = g(q), \quad q = q(z, z^*, t), \quad (z, z^*, t) \in \Omega \times J$$

we immediately have from (1.3)

LEMMA 2.1. *A Bergman kernel (2.1) for L in Ω satisfies*

(2.2a) $$rg'' + sg' + g = 0, \quad ' = d/dq \quad (z, z^*, t) \in \Omega \times J,$$

where

(2.2b) $$r = [(zct)^{-1} \tau q_t + c^{-1} q_z] \, q_{z^*},$$

(2.2c) $$s = (zct)^{-1}(\tau q_{z^*t} - \tfrac{1}{2} t^{-1} q_{z^*}) + c^{-1}(q_{zz^*} + b q_{z^*}).$$

The main problem now is the determination of functions q such that (2.2) becomes an *ordinary differential equation* with $q(z, z^*, t)$ as the independent variable. We shall give a solution of this problem for $g(q)$ with

(2.3) $$q(z, z^*, t) = \exp\left[\sum_{\mu=0}^{m} q_\mu(z, z^*) \, t^\mu\right], \quad m \in \mathbf{N}.$$

If $g(q) = q$ with q as in (2.3), then T_g is called an *operator of class E* and its kernel g a *kernel of class E.* A given equation (1.1) is said to *admit* an operator of class E if C^ω-solutions of the equation can be obtained by the use of such an operator; similarly for other classes of operators. Necessary and sufficient conditions for (1.1) to admit an operator of class E were obtained by Kreyszig (1955). We shall now determine conditions on q in (2.3) such that (2.2) becomes an Euler equation

(2.4) $$\alpha q^2 g'' + \beta q g' + g = 0$$

and characterize the operators L in (1.1) which admit Bergman operators with such simple kernels. The latter include certain operators of the first kind (see Bergman (1969), p. 12) as well as operators of class E (called *operators of exponential type* in Bergman (1969), p. 31). These new Bergman operators and their kernels are said to be *of class E_M*; we also call them *modified class E operators and kernels*, respectively.

THEOREM 2.2. *q in (2.3) satisfies (2.2b) with $r = \alpha q^2$ and (2.2c) with $s = \beta q$ if and only if $\beta = \alpha$. The corresponding solutions*

(2.5) $$g(q) = A_1 \cos \delta q + A_2 \sin \delta q \quad (\delta^2 = \alpha^{-1})$$

of (2.4) are Bergman kernels for L with

(2.6) $$b = 0, \quad c = q_1 q_{1z^*}/2\alpha z,$$

and the coefficients of q are of the form

$$\text{(a)} \qquad q_0(z) = \sum_{\nu=0}^{\rho} \tilde{d}_\nu z^\nu, \quad \tilde{d}_\nu = \text{const}, \ \rho = \left[\frac{m}{2}\right],$$

$$\text{(2.7) (b)} \quad q_{2\mu+1}(z, z^*) = \frac{(-4)^\mu (\mu!)^2}{(2\mu+1)!} \sum_{\nu=\mu}^{\sigma} \binom{\nu}{\mu} a_\nu z^{\nu+\frac{1}{2}}, \quad \mu = 0, \dots, \sigma,$$

$$a_0 = a_0(z^*), \quad a_\nu = \text{const if } \nu > 0, \quad \sigma = [\tfrac{1}{2}(m-1)],$$

$$\text{(c)} \qquad q_{2\mu}(z) = (-1)^\mu \sum_{\nu=\mu}^{\rho} \binom{\nu}{\mu} \tilde{d}_\nu z^\nu, \quad \mu = 1, \dots, \rho.$$

3. Proof of Theorem 2.2

(2.2c) can be written

$$2\tau q_{z^*t} - t^{-1} q_{z^*} + 2zt(q_{zz^*} + bq_{z^*} - \beta cq) = 0.$$

Substituting (2.3), dividing by q and abbreviating the exponent in (2.3) by $p(z, z^*, t)$ we have

$$2\tau(p_{z^*t} + p_{z^*} p_t) - t^{-1} p_{z^*} + 2zt(p_{zz^*} + p_z p_{z^*} + bp_{z^*} - \beta c) = 0.$$

Let $[j]$ denote the equation obtained from this by equating the coefficient of t^j to zero. From $[-1]$ we have $q_{0z^*} = 0$. Hence by [1],

$$c = (q_1 q_{1z^*} + q_{2z^*})/2\beta z. \tag{3.1}$$

[0] is an identity. $[j]$ with $j = m+2, \dots, 2m+1$ is

$$\sum_{\nu=0}^{m} E_{j+\nu} q_{m-\nu, z^*} = 0,$$

$$E_k = \left[2z\frac{\partial}{\partial z} - (k-m-1)\right] q_{k-m-1} + (k-m+1)\, q_{k-m+1}, \tag{3.2}$$

$$q_\mu = 0 \quad \text{if } \mu > m.$$

We show that $q_{mz^*} = 0$. Suppose not. Then

$$E_j = 0, \quad j = m+2, \dots, 2m+1, \tag{3.3}$$

from $[j]$ stepwise, beginning with $j = 2m+1$, proceeding in descending order and, in each step, using $E_i = 0$ with $i = j+1, \dots, 2m+1$. Equation $[m+1]$ is

$$(F-m) q_{mz^*} + 2zq_{mzz^*} + \sum_{\nu=1}^{m} E_{m+1+\nu} q_{m-\nu, z^*} = 0, \tag{3.4}$$

$$F = 2(q_2 + zq_{0z} + zb).$$

Here $2zq_{mzz^*} = mq_{mz^*}$ by differentiating $E_{2m+1} = 0$, so that $F = 0$ by (3.3) and (3.4). With $F = 0$ and (3.3), equation $[m]$ becomes simply

$$q_1 q_{mz^*} - (m-1) q_{m-1,z^*} + 2zq_{m-1,zz^*} = 0. \tag{3.5}$$

Here $2zq_{m-1,zz^*} = (m-1) q_{m-1,z^*}$ by differentiating $E_{2m} = 0$. Hence $q_1 q_{mz^*} = 0$ by (3.5). By the assumption that $q_{mz^*} \neq 0$ we have $q_1 = 0$. Hence by (3.2) and $F = 0$, from $[m-1]$ we finally obtain $q_{mz^*} = 0$, contradicting $q_{mz^*} \neq 0$. The same method proves

$$q_{jz^*} = 0, \quad j = 3, \ldots, m-1, \tag{3.6}$$

stepwise in descending order, for each j obtaining a contradiction to $q_{jz^*} \neq 0$ from $[j-1]$. Next we note that (3.3) with $j = m+2, \ldots, 2m-1$ is equivalent to

$$2zq_{jz} - jq_j + (j+2) q_{j+2} = 0, \quad j = 1, \ldots, m-2, \tag{3.7}$$

and remains valid; indeed, this now follows stepwise from $[m+1], [m], \ldots, [4]$, in this order. $q_1 = 0$ was obtained from $q_{mz^*} \neq 0$ which is false, so that $q_1 \neq 0$ becomes possible. (3.6) is valid in both cases. In the case $q_1 \neq 0$ it gives $q_{2z^*} = 0$ by [2]. Hence by (3.1),

$$c = \begin{cases} q_1 q_{1z^*}/2\beta z & \text{if } q_1 \neq 0, \quad \text{(a)} \\ q_{2,z^*}/2\beta z & \text{if } q_1 = 0. \quad \text{(b)} \end{cases} \tag{3.8}$$

Since $c = 0$ is excluded (see Section 1), $F = 0$ now follows from [3] if $q_1 = 0$ and from [2] if $q_1 \neq 0$. From $F = 0$,

$$b = -dq_0/dz - q_2/z. \tag{3.9}$$

Furthermore, by integrating $E_{2m+1} = 0$ and $E_{2m} = 0$,

$$q_m = k_m z^{m/2}, \quad q_{m-1} = k_{m-1} z^{(m-1)/2}.$$

Starting from this and using (3.7), we find that

$$q_1(z, z^*) = a_0(z^*) z^{\frac{1}{2}} + \sum_{\nu=1}^{\sigma} a_\nu z^{\nu+\frac{1}{2}}, \tag{3.10}$$

which proves (2.7b) with $\mu = 0$. Solving (3.7) algebraically for q_{j+2}, we obtain (2.7b) with $\mu = 1, \ldots, \sigma$, stepwise from (3.10). Formula (2.7c) is obtained similarly, first for $\mu = 1$ from (3.7), and then for $\mu = 2, \ldots, \rho$ by the transformed form of (3.7), as before. We now also see why $q_{jz^*} = 0, j = 3, \ldots, m$, implies $a_\nu = \text{const}$, $\nu = 1, \ldots, \sigma$, in (3.10), as shown.

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We now turn to (2.2b) with $r = \alpha q^2$. Substitution of (2.3) gives

$$\tau \sum_{\mu=1}^{m} q_{\mu z^*} t^{\mu} \sum_{\nu=1}^{m} \nu q_{\nu} t^{\nu-1} + zt \sum_{\mu=1}^{m} q_{\mu z^*} t^{\mu} \sum_{\nu=0}^{m} q_{\nu z} t^{\nu} = \alpha zct. \tag{3.11}$$

We equate the coefficients of each power of t on both sides, denoting by $\{j\}$ the equation corresponding to t^j. Then $\{1\}$ is $q_1 q_{1z^*} = 2\alpha zc$. Hence $\beta = \alpha$ by (3.8a). Since $q_1 = 0$ would yield the trivial case $c = 0$, (3.8b) is excluded and the second formula in (2.6) is proved. Furthermore, $\{2\}$ and $q_{1z^*} \neq 0$ give

$$zq_0'(z) + q_2(z) = 0.$$

This implies $b = 0$ by (3.9) and also entails (2.7a), which now follows from (2.7c) with $\mu = 1$. The other equations $\{3\}, \ldots, \{m+2\}$ are equivalent to (3.7) and (3.3) with $j = 2m, 2m+1$, so that they do not cause new conditions. Theorem 2.2. is proved.

4. Further properties of modified class E kernels

From (2.7a) and (2.7c) it can be seen that the sum of the terms in the exponent of q in (2.3) containing even powers of t may be arranged in powers of $z\tau$, so that (2.3) then becomes

$$q(z, z^*, t) = \exp\left[\sum_{\mu=0}^{\sigma} q_{2\mu+1}(z, z^*) t^{2\mu+1} + \sum_{\nu=0}^{\rho} \hat{d}_{\nu}(z\tau)^{\nu}\right], \tag{4.1}$$

where $\hat{d}_{\nu} = 2^{\nu} \tilde{d}_{\nu}$. This is useful in simplifying kernels for a given L, particularly for obtaining *minimal kernels* for L, that is, Bergman kernels $g(q)$ with q of the form (2.3) and of minimum degree in t. (For minimal kernels in other classes of Bergman operators and their application, see Kracht and Schröder (1973).)

The class E_M includes operators which are not of class E. Indeed, this holds for certain operators of the first kind as well as others. A simple example will be given below. On the other hand, we have the following remarkable fact.

PROPOSITION 4.1. *If L in (1.1) admits a Bergman operator of class E_M, it also admits a Bergman operator of class E.*

PROOF. Suppose that an operator L admits an operator T_g of class E_M, whose kernel we denote by g. From (2.6) and (2.7c) we see that c does not depend on $q_{2\mu}(z, z^*)$, $\mu = 0, \ldots, \rho$. Hence by choosing $\tilde{d}_{\nu} = 0$, $\nu = 0, \ldots, \rho$, we obtain from g another kernel

$$\tilde{g}(q) = A_1 \cos \delta p_1 + A_2 \sin \delta p_1$$

for the operator L under consideration; here, p_1 denotes the first sum on the right-hand side of (4.1). Clearly, we can take $A_2 = 0$ and $A_1 = 2$, so that

$$\tilde{g}(q) = \exp(i\delta p_1) + \exp(-i\delta p_1). \tag{4.2}$$

Substituting this into (1.2) and setting $\tilde{t} = -t$ in the integral corresponding to the second term on the right-hand side of (4.2), we obtain a representation of u by an operator of class E, and the proposition is proved.

EXAMPLE 4.2. We call $\Delta w + \gamma(x, y) w = 0$ or

$$L_0 u = u_{zz^*} + c(z^*) u = 0 \tag{4.3}$$

the *generalized Helmholtz equation.* L_0 admits operators of class E_M. The simplest of them has the kernel

$$g(z, z^*, t) = \cos q_1(z, z^*) t, \quad q_1(z, z^*) = a(z^*) z^{\frac{1}{2}}. \tag{4.4}$$

This operator is not of class E. It is of the first kind if and only if $a(0) = 0$. For instance, we see that this holds for the classical Helmholtz equation

$$\Delta w + k^2 w = 0 \quad \text{or} \quad u_{zz^*} + \tfrac{1}{4} k^2 u = 0. \tag{4.5}$$

This generalizes a result by Florian (1962) for (4.5) which he obtained in a different way. We further note that the operator T_g with kernel (4.4) maps $f_n(z) = z^n$ onto solutions of (4.3) which are essentially Bessel functions; more precisely, from (1.2) and a well-known integral formula (see Watson (1966), p. 25) we have

$$u_n(z, z^*) = (T_g f_n)(z, z^*) = \sqrt{(\pi)}\, \Gamma(n + \tfrac{1}{2}) z^n q_1(z, z^*)^{-n} J_n(q_1(z, z^*)).$$

We finally mention that the operator $\tilde{T}_g$ obtained by the process of reduction in the proof of Proposition 4.1 has the property that each solution

$$u_n(z, z^*) = (\tilde{T}_g f_n)(z, z^*), \quad f_n(z) = z^n, \quad n = 0, 1, \ldots, \tag{4.6}$$

satisfies an ordinary linear differential equation with $x = (z + z^*)/2$ as the independent variable, of order independent of n and not exceeding $m + 1$. This follows from a result by Kreyszig (1956) and is of interest in applying the Fuchs–Frobenius theory for characterizing singularities of solutions. We conjecture that a similar result holds for operators T_g of class E_M, but the order of the equation may be larger in this case (although still independent of n).

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