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# Counting Multiple Cyclic Choices Without Adjacencies

Alice McLeod and William Moser

*Abstract.* We give a particularly elementary solution to the following well-known problem. What is the number of $k$-subsets $X \subseteq I_n = \{1, 2, 3, \ldots, n\}$ satisfying "no two elements of $X$ are adjacent in the circular display of $I_n$"? Then we investigate a new generalization (multiple cyclic choices without adjacencies) and apply it to enumerating a class of 3-line latin rectangles.

## 1 Introduction

For $n \geq 1$ let $I_n = \{1, 2, 3, \ldots, n\}$ and let circ $I_n$ denote the display of $I_n$ in a circle, rising order clockwise. When $n \geq 2$ it is clear what is meant by "$x$ is adjacent to $y$ in circ $I_n$." When $n = 1$ we have a seemingly peculiar situation: when you look from 1 in either direction (clockwise or counterclockwise) in circ $I_1$, the first element you see is 1 itself, so let us agree that "1 is adjacent to 1 in circ $I_1$".

Let $(n|k), n \geq 1, k \geq 0$, denote the number of sets $X \subseteq I_n$ such that $|X| = k$ and no elements in $X$ are adjacent in circ $I_n$. Clearly, when $n \geq 1, (n|0) = 1$ (the set $\varnothing$ is counted); when $n \geq 2, (n|1) = n$ (the 1-element subsets of $I_n$ are counted); and $(1|1) = 0$ (because 1 is adjacent to 1 in circ $I_1$).

The numbers $(n|k)$ can be generalized as follows. For given integers $2 \leq n_1 \leq n_2 \leq \cdots \leq n_t, t \geq 1, dk \geq 0$, we define the number of multiple cyclic $k$-choices

$$(n_1, n_2, \ldots, n_t|k) := \sum_{\substack{i_1+i_2+\cdots+i_t=k \\ i_1,i_2,\ldots,i_t \geq 0}} (n_1|i_1)(n_2|i_2)\ldots(n_t|i_t).$$

These count the number of subsets of size $k$ of the set

$$\{1, 2, \ldots, n_1 + n_2 + \cdots + n_t\}$$

satisfying: no integers in a subset are adjacent in the display of these numbers in the $t$ disjoint circles (of sizes $n_1, n_2, \ldots, n_t$)

$$\begin{array}{cc} 1, 2, \ldots, n_1 & \text{in a circle} \\ n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 & \text{in a circle} \\ \vdots & \\ n_1 + n_2 + \cdots + n_{t-1} + 1, \ldots, n_1 + n_2 + \cdots + n_t & \text{in a circle} \end{array}$$

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In §2 we determine the well-known numbers $(n|k)$ in a particularly elementary way and then obtain a new identity which expresses $(n_1, n_2, \dots, n_t|k)$ as a sum of numbers $(m|i)$. In §3 we look at some special cases of this identity.

The Problème des Ménages asks for the (ménage) number $u_n$, $n \geq 2$, of permutations $(x_1, x_2, \dots, x_n)$ of $(1, 2, 3, \dots, n)$ such that the $3 \times n$ array

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{array}$$

is a latin rectangle, *i.e.,* in every column the three integers are distinct.

Consider a permutation $(a_1, a_2, \dots, a_n)$ which has $t \geq 1$ cycles whose lengths are

$$n_1, n_2, \dots, n_t, \quad t \geq 1, \quad 2 \leq n_1 \leq n_2 \leq \cdots \leq n_t, \quad n_1 + n_2 + \cdots + n_t = n.$$

The number of permutations $(x_1, x_2, \dots, x_n)$ such that the array

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{array}$$

is a latin rectangle is the same for all permutations that have the same cycle structure as $(a_1, a_2, \dots, a_n)$. Let $u_{n_1,\dots,n_t}$ denote this number. In §4 we express $u_{n_1,n_2,\dots,n_t}$ as a sum of ménage numbers $u_m$.

## 2 $(n_1, n_2, \dots, n_t|k)$ Is a Sum of Numbers $(m|i)$

For convenience we take $\binom{n}{k} = n!/k!\,(n-k)!$ if $0 \leq k \leq n$, and 0 otherwise. It is well known [5, problem2, p. 222] that when $n \geq 1$ and $k \geq 0$

$$(n|k) = \begin{cases} \frac{n}{n-k}\binom{n-k}{k} & \text{if } n \neq k, \\ 0 & \text{if } n = k. \end{cases}$$

Here is a particularly elementary proof of this for $0 \leq k \leq \frac{n}{2}$, $n \geq 1$. A choice of $k$ integers from $\{1, 2, \dots, n\}$ corresponds to a sequence of $k$ 1's and $n-k$ 0's in a row, or in a circle with one entry capped. We want to count the number of such circular displays of $k$ 1's and $n-k$ 0's in a circle, one entry capped, with every 1 followed (clockwise) by at least one 0. We build and count these displays as follows. Place $n-k$ 0's in a circle, creating $n-k$ boxes (the spaces between the 0's) and color one of the boxes (say blue). The boxes are now distinguishable. Choose $k$ of these boxes ($\binom{n-k}{k}$ choices), place a single 1 into each of the chosen boxes, "cap" one of the $n$ entries ($n$ ways to do this), erase the color and the $n\binom{n-k}{k}$ displays fall into sets each containing $n-k$ congruent displays. Choose one display from each set and we have $\frac{n}{n-k}\binom{n-k}{k}$ displays, precisely those we want.

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By taking $(0|0) = 2$ and $(0|k) = 0$ if $k \geq 1$, (these have no combinatorial meaning) the numbers $(n|k)$, $k \geq 0$, $n \geq 0$ satisfy and are determined by the recurrence

$$(1)\qquad \begin{gathered}(n|k) = (n-1|k) + (n-2|k-1), \quad n \geq 2,\ k \geq 1,\\ (0|0) = 2, (n|0) = 1 \text{ for } n \geq 1, \quad (n|k) = 0 \text{ for } n = 0, 1, k \geq 1.\end{gathered}$$

They are exhibited in Table 1. The initial conditions in boldface.

| $k\backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | **2** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | ... |
| 1 | **0** | **0** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
| 2 | **0** | **0** | 0 | 0 | 2 | 5 | 9 | 14 | 20 | 27 | ... |
| 3 | **0** | **0** | 0 | 0 | 0 | 0 | 2 | 7 | 16 | 30 | ... |

Table 1. $(n|k)$

The recurrence (1) leads to the generating function

$$\sum_{n,k\geq 0} (n|k) x^n z^k = \frac{2-x}{1-x-x^2 z} = \frac{1}{1-\alpha x} + \frac{1}{1-\beta x} = \sum_{n\geq 0} (\alpha^n + \beta^n) x^n,$$

where $\alpha, \beta$ are power series in $z$ satisfying $\alpha + \beta = 1$, $\alpha\beta = -z$. Equating coefficients of $x^n$ we have

$$\sum_{k\geq 0} (n|k) z^k = \alpha^n + \beta^n, \quad n \geq 0, \quad \alpha + \beta = 1, \quad \alpha\beta = -z$$

($\alpha^n + \beta^n$ is a polynomial in $z$).

***Theorem 1*** *Let $t \geq 1, 0 \leq n_1 \leq n_2 \leq \cdots \leq n_t$, $k \geq 0$, $I_t = \{1, 2, \ldots, t\}$, $A^c = I_t - A$ when $A \subseteq I_t$, $s(A) = \sum_{i\in A} n_i$ if $A \neq \varnothing$, $s(\varnothing) = 0$,*

$$m(A) = \min\left(s(A), s(A^c)\right), \quad M(A) = \max\left(s(A), s(A^c)\right).$$

*Then*

$$(n_1, n_2, \ldots, n_t|k) = \sum_{1\in D\subseteq I_t} (-1)^{m(D)} \big(M(D) - m(D)|k - m(D)\big).$$

**Proof** The generating function

$$\begin{aligned}\sum_{k\geq 0} (n_1, n_2, \ldots, n_t|k) z^k &= \sum_{k\geq 0} \sum_{\substack{i_1+\cdots+i_t=k\\ i_1,i_2,\ldots,i_t\geq 0}} (n_1|i_1) z^{i_1} (n_2|i_2) z^{i_2} \cdots (n_t|i_t) z^{i_t}\\ &= \sum_{i_1,i_2,\ldots,i_t\geq 0} (n_1|i_1) z^{i_1} (n_2|i_2) z^{i_2} \cdots (n_t|i_t) z^{i_t}\\ &= \Big(\sum_{i_1\geq 0} (n_1|i_1) z^{i_1}\Big) \Big(\sum_{i_2\geq 0} (n_2|i_2) z^{i_2}\Big) \cdots \Big(\sum_{i_t\geq 0} (n_t|i_t) z^{i_t}\Big)\\ &= (\alpha^{n_1} + \beta^{n_1})(\alpha^{n_2} + \beta^{n_2}) \cdots (\alpha^{n_t} + \beta^{n_t})\end{aligned}$$

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(remember, $\alpha+\beta=1, \alpha\beta=-z$).

This product has a complete expansion in $2^t$ terms, one term corresponding to each subset $A \subseteq I_t = \{1,2,\dots,t\}$, namely $\alpha^{s(A)}\beta^{s(A^c)}$ ; hence

$$\sum_{k\geq 0}(n_1,n_2,\dots,n_t|k)z^k = \sum_{A\subseteq I_t}\alpha^{s(A)}\beta^{s(A^c)}. \tag{2}$$

The $2^t$ terms of this sum come in $2^{t-1}$ pairs: for each $D \subseteq I_t$ with $1 \in D$

$$\alpha^{s(D)}\beta^{s(D^c)} \text{ is paired with } \alpha^{s(D^c)}\beta^{s(D)}$$

and now

$$\sum_{A\subseteq I_t}\alpha^{s(A)}\beta^{s(A^c)} = \sum_{1\in D\subseteq I_t}\left(\alpha^{s(D)}\beta^{s(D^c)} + \alpha^{s(D^c)}\beta^{s(D)}\right).$$

We can simplify this sum. For any $D \subseteq I_t$ with $1 \in D$,

$$\begin{aligned} \alpha^{s(D)}\beta^{s(D^c)} + \alpha^{s(D^c)}\beta^{s(D)} &= (\alpha\beta)^{m(D)}\left(\alpha^{M(D)-m(D)} + \beta^{M(D)-m(D)}\right) \\ &= (-z)^{m(D)}\sum_{k\geq 0}\left(M(D)-m(D)|k\right)z^k \\ &= (-1)^{m(D)}\sum_{k\geq 0}\left(M(D)-m(D)|k\right)z^{k+m(D)} \\ &= (-1)^{m(D)}\sum_{k\geq 0}\left(M(D)-m(D)|k-m(D)\right)z^k. \end{aligned} \tag{3}$$

Now, from (2) and (3),

$$\begin{aligned} \sum_{k\geq 0}(n_1,n_2,\dots,n_t|k)z^k &= \sum_{1\in D\subseteq I_t}(-1)^{m(D)}\sum_{k\geq 0}\left(M(D)-m(D)|k-m(D)\right)z^k \\ &= \sum_{k\geq 0}\Big(\sum_{1\in D\subseteq I_t}(-1)^{m(D)}\left(M(D)-m(D)|k-m(D)\right)\Big)z^k. \end{aligned}$$

Equate coefficients of $z^k$ and we have completed the proof of Theorem 1. ■

## 3 Special Cases of Theorem 1

In the case $t=2$ of Theorem 1, $0 \leq n_1 \leq n_2, k \geq 0, I_2 = \{1,2\}$,

$$(n_1,n_2|k) = \sum_{1\in D\subseteq I_2}(-1)^{m(D)}\left(M(D)-m(D)|k-m(D)\right).$$

The table below shows all the information we need to simplify this:

| $D$ | $D^c$ | $s(D)$ | $s(D^c)$ | $m(D)$ | $M(D)$ |
|---|---|---|---|---|---|
| $\{1,2\}$ | $\varnothing$ | $n_1+n_2$ | $0$ | $0$ | $n_1+n_2$ |
| $\{1\}$ | $\{2\}$ | $n_1$ | $n_2$ | $n_1$ | $n_2$ |

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and we have

$$(4)\qquad (n_1, n_2|k) = (n_1 + n_2|k) + (-1)^{n_1}(n_2 - n_1|k - n_1), \quad 0 \le n_1 \le n_2.$$

This identity was established by Moser and Pollack [3].

When $n_1 = n_2 = m \ge 0$, the Moser-Pollack identity (4) simplifies to

$$(m, m|k) = (2m|k) + (-1)^m(0|k - m), \quad m \ge 0,$$

so that

$$(m, m|k) = (2m|k), \quad m \ge 0, k \ne m,$$

$$(m, m|m) = (2m|m) + (-1)^m 2 = 2 + (-1)^m 2 = \begin{cases} 4 & \text{if } 0 \le m \text{ is even,} \\ 0 & \text{if } 1 \le m \text{ is odd.} \end{cases}$$

## 4 Ménage Identities

Using $[i, j]$ to denote the property "the integer $i$ is in the $j$th place", $u_n$ is the number of permutations possessing none of the properties

$$[1,1]\ [1,2]\ [2,2]\ [2,3] \cdots [n-1,n-1]\ [n-1,n]\ [n,n]\ [n,1]\ .$$

Since two of these properties are consistent if and only if they are not adjacent when the $2n$ properties are arranged in a circle (so that $[1,1]$ follows $[n,1]$), the Principle of Inclusion and Exclusion yields

$$u_n = \sum_{0 \le k \le n} (-1)^k (2n|k)(n-k)!, \quad n \ge 2.$$

This is of course well known [1, p. 14].

Now let $u_{m,n}$, $(0 \le m \le n)$ denote the number of permutations of $\{1, 2, \dots, m+n\}$ discordant with the two permutations

$$\begin{matrix} 1 & 2 & 3 & \cdots & m & m+1 & m+2 & \cdots & m+n \\ m & 1 & 2 & \cdots & m-1 & m+n & m+1 & \cdots & m+n-1. \end{matrix}$$

Clearly the number of such permutations is

$$\begin{aligned} u_{m,n} &= \sum_{k \ge 0} (-1)^k (2m, 2n|k)(m+n-k)! \\ &= \sum_{k \ge 0} (-1)^k (2m+2n|k)(m+n-k)! \\ &\qquad + \sum_{k \ge 2m} (-1)^k (2n-2m|k-2m)(n+m-k)! \\ &= u_{m+n} + \sum_{j \ge 0} (-1)^j (2n-2m|j)(n-m-j)! \\ &= u_{m+n} + u_{n-m}. \end{aligned}$$

The generalization is contained in the following theorem.

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**Theorem 2** *For $t \geq 2, 2 \leq n_1 \leq n_2 \leq \cdots \leq n_t$, and $n_1 + n_2 + \cdots + n_t = n$,*

$$u_{n_1,n_2,\ldots,n_t} = \sum_{1 \in D \subseteq I_t} u_{M(D)-m(D)} = \sum_{1 \in D \subseteq I_t} u_{n-m(D)},$$

*where*

$$M(D) = \max(s(D), s(D^c)), \quad m(D) = \min(s(D), s(D^c)),$$
$$s(D) = \sum_{i \in D} 2n_i, \quad s(D^c) = \sum_{i \in D^c} 2n_i.$$

**Proof** By the Principle of Inclusion and Exclusion

$$\begin{aligned}
u_{n_1,n_2,\ldots,n_t} &= \sum_{k \geq 0} (-1)^k (2n_1, 2n_2, \ldots, 2n_t \,|\, k)(n-k)! \\
&= \sum_{k \geq m(D)} (-1)^k \sum_{1 \in D \subseteq I_t} (-1)^{m(D)} (M(D) - m(D) \,|\, k - m(D))(n-k)! \\
&= \sum_{1 \in D \subseteq I_t} (-1)^{m(D)} \sum_{k \geq m(D)} (-1)^k (M(D) - m(D) \,|\, k - m(D))(n-k)! \\
&= \sum_{1 \in D \subseteq I_t} \sum_{j \geq 0} (-1)^j (M(D) - m(D) \,|\, j)(n - m(D) - j)! \\
&= \sum_{1 \in D \subseteq I_t} \sum_{j \geq 0} (-1)^j (2(n - m(D)) \,|\, j)(n - m(D) - j)! \\
&= \sum_{1 \in D \subseteq I_t} u_{n-m(D)}.
\end{aligned}$$

∎

This identity, in the form

$$u_{n_1,n_2,\ldots,n_t} = \sum u_{n_1 \pm n_2 \pm \cdots \pm n_t},$$

where the sum is over the $2^{t-1}$ possible assignments of + and − signs, with the understanding that $u_0 = 2$, $u_1 = -1$ and $u_{-n} = u_n$, was known to Touchard [6]. It was proved by "symbolic operator" methods (see [2]) and used by Riordan [4] to give a remarkably attractive formula for the number of $3 \times n$ latin rectangles.

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*Department of Mathematics and Statistics*
*McGill University*
*Montreal, Quebec*
*H3A2K6*

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