CORRESPONDENCE.

THE INTEREST QUESTION.

To the Editor of the Assurance Magazine.

SIR,—"What is the proper expression for the amount of £1, with the fractional part of a year's interest?" is not of importance from any differences in the money result of problems of any practical moment being involved in it, but from the diversity of opinion which has existed among modern authors, and the embarrassment which this gives to writers who may have occasion to treat of questions which involve the point in dispute.

Interest is the rent paid for the use of money; it is payable at the expiration of such intervals of time, and is such a proportion of the sum lent, as may be agreed upon between the borrower and the lender. Sometimes the interest is not paid when due, but is added to the principal sum; sometimes the sum lent is repaid, not at the expiration of a period when interest falls due, but at some intermediate time, and it is required to know what, in such cases, should be paid as interest.

For the sake of brevity of expression, I shall assume the period of conversion to be a year, and r to be the interest upon every £1 payable at the end of each year; but the reasoning is not the less general on that account, since any other period of conversion may be substituted with equal propriety, employing r', the amount of interest upon every £1 actually agreed to be paid at the end of each period, instead of r. The assumption is also an appropriate one, since the recent discussions in your *Magazine* have been based upon it.

The general question may be resolved by an application of the following self evident principle: "Interest upon money for equal periods of time will be proportional to the amount lent." It is granted by all writers that the amount to which £1 will accumulate in n years, n being a whole number, is $(1+r)^n$; also, that the amount of £1 at the end of $n+\frac{1}{2}$ years is the amount at the end of n years multiplied by the accumulation of £1 in half a year. The same remark will apply to any other fraction of a year; so that the question is reduced to that of finding the amount to which £1 will accumulate in half a year, in a quarter of a year, &c.—that is, the contract being to pay r interest for every £1 on loan, at the end of a year from the grant of the loan to determine the amount equitably due as interest, if the loan is repaid at the end of half a year, &c.

Let the equitable interest of £1 for half a year, as yet undetermined, be x, or the amount of £1 in half a year be $1+x$; then, by the principle, in the second half year this $1+x$ will accumulate to $(1+x)(1+x)=(1+x)^2$, which should be equal to $1+r$ by the wording of the contract that r is the interest due at the end of the year $\therefore 1+x=(1+r)^{\frac{1}{2}}$. The intention is, to place the lender in the same position at the end of the year as if he then received the stipulated interest r for £1, neither better nor worse. Now any other value for the amount of £1 in half a year violates this condition; for if we assume that the amount is $(1+r)^{\frac{1}{2}}+x'$, where x' is either positive or negative, this amount, improved during the second half year, would give $[(1+r)^{\frac{1}{2}}+x']^2$ as the amount at the end of the year, which cannot be equal to $1+r$ for any value of x' except $x'=o$. The interest of

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£1 for half a year, deducible from this, $(1+r)^{\frac{1}{2}}-1$, differs but very slightly from $\frac{r}{2}$, or half the yearly interest, the first three terms in the development of $(1+r)^{\frac{1}{2}}$ being $1+\frac{r}{2}-\frac{r^2}{8}$: it is less than half the yearly interest, which is but equitable; for the borrower is paying money as interest which is not due, according to the terms of the contract, until the close of another half year.

Having thus established $(1+r)^{\frac{1}{2}}$ as the amount of £1 in half a year, that of $(1+r)^{\frac{1}{4}}$ in a quarter of a year easily follows; for, as before, if $1+x$ be employed to represent this amount, we must have

$$1 : 1+x :: 1+x : (1+r)^{\frac{1}{2}}, \text{ or } 1+x=(1+r)^{\frac{1}{4}}.$$

In a similar manner, the amount of £1 at the end of any other fraction of a year which may occur in practice may be shown to be $(1+r)^p$, p being the fraction of the year.

Now these values of $(1+r)^{\frac{1}{2}}$ and $(1+r)^{\frac{1}{4}}$ have been objected to on various grounds. It has been repeatedly said that they involve the theory of incessant conversions, or that the interest of money must be supposed to accrue and to be added to the principal momentarily. I ask if any such assumption appears in the foregoing demonstrations. The earlier writers on interest appear to have uniformly adopted these values, which result from $(1+r)^n$ as the amount of £1 in n years, whether n be a whole number or a fractional one. Dr. Price was the first who dissented from them, and his object in doing so was to obtain results corresponding to practice—he obtained absurd ones. The same facts with which Dr. Price wished to bring his formulæ in accordance, led Mr. Milne, at a later period, to give a different method. We shall therefore briefly state them.

Dividends in the funds and in public companies, interest on mortgages, &c., are usually payable half yearly; and the half yearly dividend, or interest, is one half the yearly; and the interest upon money repaid during a period of conversion is proportional to the portion of the period elapsed.

It must be borne in mind that the period of conversion is that fixed period of time at the end of which interest becomes due according to the contract. If the contract be to pay r' interest at the end of every half year, the period of conversion is half a year, not a year: what then is the meaning of the term 'yearly interest,' in such cases? A year is the unit of time adopted in financial matters. Incomes, expenses, &c., are referred to this standard; and since the amount of money which goes into the pocket of the lender in each year is of more importance than the times of the year at which he receives it, a more tangible idea is conveyed in the expression "£100 a year, payable half yearly or quarterly," than in "£50 every half year," or "£25 every quarter of a year." The slightest consideration will however show, that the term 'yearly interest' is applied in such cases inaccurately. The yearly interest is what could be obtained from the use of £1 in a year; and if $1+r'$ could be produced in half a year, can it be denied that at the end of the year this would have accumulated to $(1+r')^2=1+2r'+r'^2$? The yearly interest would therefore be $2r'+r'^2$, not $2r'$; but there is no more scientific propriety in this use of the term than in that of biennial or triennial rate.

Dr. Price gave some theorems to show the increase in the value of annuities certain, when the annuity, instead of being payable at the end of each year, was payable in half yearly or quarterly instalments. In obtaining the value of annuities payable half yearly, he takes $1+\frac{r}{2}$ as the amount of £1 in half a year, r being what is above called the yearly interest; the value of such an annuity may be found by the same formula, which gives the value of yearly annuities, when properly modified; and the excess of these values above those of yearly annuities, at r yearly interest, was given by him as the solution of his problem. But there is an evident shifting of the hypothesis in this solution: for if $1+\frac{r}{2}$ could be obtained at the end of every half year for £1, it is quite evident that £1 could be made to produce $\left(1+\frac{r}{2}\right)^2=1+r+\frac{r^2}{4}$ at the end of a year; and it is therefore with a yearly annuity at the rate $r+\frac{r^2}{4}$ that he should have compared his results. The absurd conclusion to which he came was, that although annuities payable by instalments were more valuable on that account, yet that the excess of value diminished when the annuities were for longer than a certain period, and vanished for perpetual annuities. Great men usually have zealous followers, who defend all that has been advanced by them: one of Dr. Price's school states—"The difference, however, between the value of annuities payable yearly and at shorter intervals, is known to be continually lessening, in proportion to the length of the term; till at last, when the term is extended to a perpetuity, those values become the same, whether the payments are made yearly or momently."

Mr. Milne, when treating of annuities payable by instalments, avoids Dr. Price's error of shifting the hypothesis from one yearly rate to another, but concludes the interest of £1 for a fraction of a year to be the same fraction of the yearly interest. He arrives at this conclusion in a very summary manner: dividing all interest into simple and compound, and defining compound interest to be the addition of the interest to the principal at the end of the year, he remarks, that until the end of a year compound interest cannot come into operation; that therefore the interest for any fraction of a year must be simple interest, or be the same fraction of the yearly interest. His definitions are manifestly defective, as assuming the question in dispute. The terms 'simple' and 'compound interest' are objectionable, but it is of no use to quarrel with old established names.

Mr. Milne's introduction of discontinuity, in the formula representing the accumulation of money, was attended with its usual consequences of rendering solutions more complicated in some cases, and of increasing the number of problems. These consequences should have caused him to be very careful in making any change, and, as he was leaving the law of mathematical continuity for the sake of a refinement, which, even if correct, could have little or no practical effect, to make sure that the solutions which he gave were in strict accordance with his theory. They are not.

The present value of a sum of money—say, £1—receivable $n+\frac{1}{2}$ years hence, is the reciprocal of the amount to which it will accumulate in that time; or, by Mr. Milne's theory, $\frac{1}{(1+r)^n\left(1+\frac{r}{2}\right)}$, which is less than $\frac{1}{(1+r)^{n+\frac{1}{2}}}$;

and it is for the sake of the difference that Mr. Milne thought proper to give his new solutions. Instead of the above, however, he employs $\frac{1+\frac{r}{2}}{(1+r)^{n+1}}$, which is greater than $\frac{1}{(1+r)^{n+\frac{1}{2}}}$; so that, setting out with the intention of lowering the present value of sums, he ends by raising them. The solutions of some of his problems were simplified by this change of formula, and this must have been the inducement for the change; but Mr. Griffith Davies has since given solutions of the problems of life annuities payable by instalments, upon Mr. Milne's principles, which are much more simple than Mr. Milne's, although they are in strict accordance with those principles.

But it may be said, that the allowance of simple interest for fractions of a year is a fact which should be taken account of in solving such problems as depend upon it. Why is it a fact? If the results differ much from equity, which without any question gives $(1+r)^{\frac{1}{2}}$ as the amount of £1 in half a year, &c., the sooner the legislature and the public are enlightened and custom changed, the better; but in reality the difference is very slight, the results easily calculated, and may be understood by all. "Rough and ready" is the maxim in such cases, a notable example of which is given by Sir R. Peel's rate for the income tax—it is at 7*d.* in the pound, or £2. 18*s.* 4*d.* per cent. It is obvious that 3 per cent. was aimed at, and which would have been obtained from Parliament as easily as £2. 18*s.* 4*d.* per cent.; but perhaps one half of the assessors and payers of the tax would have found a difficulty in calculating 3 per cent. on £187, whilst anyone could tell to how much 187 sevenpences would amount. For the sake, therefore, of this convenience, he sacrificed £140,000 of revenue, and perhaps was wise in so doing. It may moreover be stated confidently, that if customary interest differed much from the true interest, the financiers of public companies could make the custom operate in their favour.

The allowance of simple interest for fractions of a year makes money a little more productive; if therefore investments do not yield simple interest so often as the problems suppose, there will be a further defect in these so called practical solutions. Suppose a Company whose business is to sell annuities payable by quarterly instalments, the values of which are deduced upon the supposition that upon three out of the four annual instalments simple interest will be paid on withdrawing them from investment: the advantage thereby gained, separately small, would upon the whole of the business amount to a sum worth consideration; and, having been allowed to the purchasers of annuities, must be realized by the Company, to avoid loss. Now it is evident that it would not be realized to anything like the extent supposed in the formulæ. The instalments are not withdrawn from investment individually, as in them it is supposed to be; but investments usually remain a considerable number of terms of conversion untouched, and are changed usually at the close of a period. The advantages therefore derivable from the custom of simple interest, which can only arise when investments are withdrawn at other times than the end of a period of conversion, can only be a small proportion of that supposed in the formulæ.

I do not object to the employment of $1+\frac{r}{n}$ as the amount of £1 in $\frac{1}{n}$th of a year in the solutions of problems, when the employment of $(1+r)^{\frac{1}{n}}$

would give results troublesome to calculate: higher powers than the first of a small quantity are continually being rejected in mathematical approximations, and its propriety in these problems is proved by our knowledge of the insignificant amount of the error in a sufficient number of cases to warrant an induction to all other cases.

A few words regarding Mr. Sang's tables. It is evident that I believe him to have used the proper expression for the interest of £1 in half a year; but I think, as there were differences of opinion as to what is the proper expression, he should have stated which he had employed: also, that his tables would have been more useful and much more used, if the value of assurances involved in them had been found as payable at the end of the year in which death occurs.

WILLIAM ORCHARD.

THE RESULTS TO BE LOOKED FOR ON TOSSING A DYNAMICALLY TRUE COIN.

To the Editor of the Assurance Magazine.

SIR,—Will you allow me, as a reader of the *Assurance Magazine*, to offer one or two observations with reference to the subject of a paragraph, "on tossing a dynamically true coin," which appeared in the July number of that periodical?

I would remark, in the first place, that it does not appear to me that the terms of the hypothesis are inconsistent. We must, I think, allow it to be theoretically possible that a dynamically true coin, when tossed, *may* turn up head a hundred times successively, although on the other hand it must be admitted that the hypothesis is purely casuistical, and supposes a coincidence of conditions, the probability of the occurrence of which is so small that there is a moral certainty that such coincidence will never actually obtain.

The argument adduced to prove that if a dynamically true coin has been tossed and has turned up head a hundred times successively, the probability of the next throw is in favour of tail, appears to me to be fallacious *ab initio;* for it is an untenable assumption that "in any given number of trials with such a coin, it is probable that the number of heads turned up will equal the number of tails." If *two* trials, for example, are to be made, it is clearly as likely that there will be either two heads or two tails, as that there will be an equal number of heads and tails; and therefore that in this, the most favourable case, there are no odds in favour of the latter result.

The origin of this assumption was probably a hasty deduction from the evidently true proposition, that if a dynamically true coin be tossed a certain number of times, the probability that there will be a given numerical excess of heads over tails is equal to the probability that there will be the same excess of tails over heads. This is true, however small the excess, and therefore when it is zero. The proposition, in this its limit form, has been hastily taken to be—"it is probable that there will be an equal number of heads and tails"; whereas it really is—"it is as probable that there will be the same number of heads and tails, as that there will be the same number of tails and heads." To apply a Johnsonian phrase, this is a conclusion wherein nothing is concluded.