Canad. Math. Bull. Vol. 47 (4), 2004 pp. 573–588

A Generalization of the Turán Theorem and Its Applications

Yu-Ru Liu

Abstract. We axiomatize the main properties of the classical Turán Theorem in order to apply it to a general context. We provide applications in the cases of number fields, function fields, and geometrically irreducible varieties over a finite field.

1 Introduction

Let $m \in \mathbb{N}$ and define $\omega(m)$ to be the number of distinct prime divisors of m. Hardy and Ramanujan [3] proved in 1917 that the normal order of $\omega(m)$ is $\log\log m$. In other words, given any $\epsilon > 0$, we have

$$\#\{m \leq x\,;\, |\omega(m) - \log\log m| > \epsilon \log\log m\} = \mathrm{o}(x).$$

The method they used was rather complicated and seemed difficult to generalize. In 1934, Turán [12] gave a greatly simplified proof of the Hardy-Ramanujan result by showing that

$$\sum_{m\leq x}(\omega(m) - \log\log x)^2 \ll x\log\log x.$$

His proof was essentially probabilistic and concealed in it an elementary sieve method [4]. Because of its simplicity and importance, this result is now known as the Turán Theorem. At the end of [12], Turán also stated that

$$\sum_{m\leq x}(\omega(m) - \log\log x)^2 = x\log\log x + \mathrm{o}(x\log\log x)$$

can be obtained and the proof of it is at [1]. Recently, Saidak [11] improved the Turán Theorem by proving the asymptotic formula

$$\sum_{m\leq x}(\omega(m) - \log\log x)^2 = x\log\log x + Cx + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big),$$

where C is an explicit constant. Indeed, the setting of the Turán Theorem can be generalized. The purpose of this paper is to axiomatize the main properties in order to apply the results in a more general context. We will see applications in Section 4

Received by the editors October 29, 2002.
Research partially supported by NSERC PGSB scholarship and NSERC discovery grant.
AMS subject classification: 11N37, 11N80.

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

in the cases of number fields, function fields, and geometrically irreducible varieties over a finite field.

We now formulate the general setting of the Turán's Theorem. Let P be a set of elements with a map

$$N\colon P \to \mathbb{N}\setminus\{1\}, \quad p \mapsto N(p).$$

Let M be a free abelian monoid generated by elements of P. For each $m \in M$, we write

$$m = \sum_{p\in P} n_p(m)p,$$

with $n_p(m) \in \mathbb{N}\cup\{0\}$ and $n_p(m) \neq 0$ for only finitely many p. We extend the map N on M as follows:

$$N\colon M \longrightarrow \mathbb{N}$$
$$m = \sum_{p\in P} n_p(m)p \longmapsto N(m) := \prod_{p\in P} N(p)^{n_p(m)},$$

i.e., N is a monoid homomorphism from $(M,+)$ to $(\mathbb{N},\cdot)$. Let X be a subset of $\mathbb{N}$ that contains the image $\mathrm{Im}(N(M))$. We choose either $X = \mathbb{N}$ or $X = \{q^{rn}\,;\, n \in \mathbb{N}\cup\{0\}\}$ for some fixed $q, r \in \mathbb{N}\setminus\{1\}$.

Given P, M, and X as above, for each (sufficiently large) $x \in X$, we assume that the following two conditions hold: let $m \in M$ and $p \in P$, we have

(A) (Cardinality of elements) $\sum_{N(m)\le x} 1 = \kappa x + \mathrm{O}(x^{\theta})$, for some $\kappa > 0$ and $0 \le \theta < 1$.

(B) (Cardinality of primes) $\sum_{N(p)\le x} 1 = \mathrm{O}\big(\frac{x}{\log x}\big)$.

For each $m \in M$, we define

$$\omega(m) = \sum_{\substack{p\in P\\ n_p(m)\ge 1}} 1,$$

the number of elements of P that generate m, counted without multiplicity. Then we have a generalization of the Turán Theorem.

Theorem 1 *Given P, M, and X satisfying* (A) *and* (B), *for $x \in X$, we have*

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x \log\log x + Cx + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big).$$

Here κ is the same constant as in (A) *and C is a constant that depends only on P.*

As an immediate corollary of Theorem 1, we obtain a generalization of the Hardy-Ramanujan Theorem on the normal order of $\omega(m)$.

Corollary 1 *Let P, M, and X satisfy* (A) *and* (B). *For $\epsilon > 0$ and $x \in X$, we have*

$$\#\big\{\, m \in M\,;\, N(m) \le x, |\omega(m) - \log\log N(m)| > \epsilon \log\log N(m)\big\} = \mathrm{o}(x).$$

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

2 Technical Lemmas

To prove Theorem 1, we need the following lemmas.

Lemma 1 *Given P, M, and X satisfying* (A) *and* (B), *we have*

$$(1) \sum_{N(p)\le x} \frac{1}{N(p)^{\alpha}} \ll \frac{x^{1-\alpha}}{\log x} \qquad \text{if } 0 \le \alpha < 1,$$

$$(2) \sum_{N(m)\le x} \frac{1}{N(m)^{\alpha}} \ll 1 \qquad \text{if } \alpha > 1.$$

In particular, (2) *implies that*

$$\sum_{N(p)\le x} \frac{1}{N(p)^{\alpha}} \ll 1 \qquad \text{if } \alpha > 1.$$

Proof These results follow from the technique of partial summation [8, p. 17–18].

The next lemma is a generalization of Mertens' theorem [7].

Lemma 2 *Given P, M, and X satisfying* (A) *and* (B), *we have*

$$\sum_{N(p)\le x} \frac{1}{N(p)} = \log\log x + A + \mathrm{O}\Big(\frac{1}{\log x}\Big)$$

for some constant A that depends only on P.

Proof Consider $\sum_{N(m)\le x} \log N(m)$. Applying (A) and partial summation, we have

$$\sum_{N(m)\le x} \log N(m) = \kappa x \log x + \mathrm{O}(x).$$

On the other hand, for $p \in P$, we can write

$$\sum_{N(m)\le x} \log N(m) = \sum_{\substack{N(p)^s\le x\\ s\ge 1}} \Big(\sum_{N(m')\le \frac{x}{N(p)^s}} 1\Big) \log N(p) \quad (\text{here } m' = m - sp)$$

$$= \kappa x \sum_{\substack{N(p)^s\le x\\ s\ge 1}} \frac{\log N(p)}{N(p)^s} + \mathrm{O}\Big(\sum_{\substack{N(p)^s\le x\\ s\ge 1}} \frac{x^{\theta}\log N(p)}{N(p)^{s\theta}}\Big).$$

By Lemma 1, we have

$$\sum_{\substack{N(p)^s\le x\\ s\ge 1}} \frac{\log N(p)}{N(p)^{s\theta}} \ll x^{1-\theta}$$

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

and

$$\sum_{\substack{N(p)^s \le x \\ s \ge 2}} \frac{\log N(p)}{N(p)^s} \ll 1.$$

It follows that

$$\sum_{N(p) \le x} \frac{\log N(p)}{N(p)} = \log x + \mathrm{O}(1).$$

Let $X = \mathbb{N}$ and $z \in \mathbb{N}$. Define

$$S(z) := \sum_{N(p) \le z} \frac{\log N(p)}{N(p)} = \log z + \tau(z), \text{ where } \tau(z) = \mathrm{O}(1).$$

We have

$$\begin{aligned}
\sum_{N(p) \le x} \frac{1}{N(p)} &= \frac{S(x)}{\log x} + \int_2^x \frac{\log t + \tau(t)}{(\log t)^2 t}\, dt \\
&= 1 + \int_2^x \frac{1}{t \log t}\, dt + \int_2^\infty \frac{\tau(t)}{t(\log t)^2}\, dt \\
&\quad - \int_x^\infty \frac{\tau(t)}{t(\log t)^2}\, dt + \mathrm{O}\Big(\frac{1}{\log x}\Big) \\
&= \log\log x + \Big(1 - \log\log 2 + \int_2^\infty \frac{\tau(t)}{t(\log t)^2} dt\Big) + \mathrm{O}\Big(\frac{1}{\log x}\Big).
\end{aligned}$$

If $X = \{q^{rn}\,;\, n \in \mathbb{N} \cup \{0\}\}$, define

$$S'(z) := \sum_{N(p) \le q^{rz}} \frac{\log N(p)}{N(p)} = z \log(q^r) + \tau(z), \text{ where } \tau(z) = \mathrm{O}(1).$$

For $x = q^{rx'}$, we have

$$\begin{aligned}
\sum_{N(p) \le x = q^{rx'}} \frac{1}{N(p)} &= \frac{S'(x')}{\log q^{rx'}} + \int_1^{x'} \frac{t \log q^r + \tau(t)}{t^2 \log q^r}\, dt \\
&= \log\log x + \Big(1 - \log\log q^r + \int_1^\infty \frac{\tau(t)}{t^2 \log q^r}\, dt\Big) \\
&\quad + \mathrm{O}\Big(\frac{1}{\log x}\Big).
\end{aligned}$$

This completes the proof of Lemma 2

Lemma 3 *Given P, M, and X satisfying* (A) *and* (B),

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

(1) *If* $X = \mathbb{N}$, *we have*

$$\sum_{N(p)\le \frac{x}{2}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log x)^2 + A\log\log x + B + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

(2) *If* $X = \{q^{rn} \,;\, n \in \mathbb{N}\cup\{0\}\}$, *we have*

$$\sum_{N(p)\le \frac{x}{q^r}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log x)^2 + A\log\log x + B + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

Here A is the same constant as in Lemma 2 and B is some other constant.

Proof (1) Let $X = \mathbb{N}$. By Lemma 2 and partial summation, we have

$$\sum_{N(p)\le \frac{x}{2}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} = (\log\log 2)\log\log x + A\log\log 2 + \mathrm{O}\Big(\frac{1}{\log x}\Big) + \int_2^{\frac{x}{2}} \frac{\log\log t + A + \mathrm{O}\big(\frac{1}{\log t}\big)}{\log x - \log t}\,\frac{dt}{t}.$$

By elementary integrations, we see that

$$\int_2^{\frac{x}{2}} \frac{dt}{\log t(\log x - \log t)t} \ll \frac{\log\log x}{\log x}$$

and

$$\int_2^{\frac{x}{2}} \frac{1}{\log x - \log t}\,\frac{dt}{t} = \log\log x - \log\log 2 + \mathrm{O}\Big(\frac{1}{\log x}\Big).$$

By change of variables, we write

$$\begin{aligned}\int_2^{\frac{x}{2}} \frac{\log\log t}{\log x - \log t}\,\frac{dt}{t} &= \int_{\log 2}^{\log\frac{x}{2}} \frac{\log\big(\log x(1-\frac{u}{\log x})\big)}{u}\,du\\ &= (\log\log x)^2 - \log\log 2\cdot\log\log x + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big)\\ &\quad + \int_{\frac{\log 2}{\log x}}^{1-\frac{\log 2}{\log x}} \frac{\log(1-s)}{s}\,ds.\end{aligned}$$

Since $\log(1-s) \ll s$ and $\int_0^1 \frac{\log(1-s)}{s}\,ds = \frac{\pi^2}{6}$ for $0 < s < 1$, we have

$$\int_2^{\frac{x}{2}} \frac{\log\log t}{\log x - \log t}\,\frac{dt}{t} = (\log\log x)^2 - \log\log 2\cdot\log\log x - \frac{\pi^2}{6} + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

Combining all the above results, we obtain

$$\sum_{N(p)\le\frac{x}{2}} \frac{1}{N(p)} \log\log\frac{x}{N(p)} = (\log\log x)^2 + A\log\log x - \frac{\pi^2}{6} + O\Big(\frac{\log\log x}{\log x}\Big).$$

(2) For $X = \{q^{rn} ; n \in \mathbb{N}\cup\{0\}\}$, replace z in the above proof by q^{rz}. Using similar arguments as before, we obtain

$$\sum_{N(p)\le\frac{x}{q^l}} \frac{1}{N(p)} \log\log\frac{x}{N(p)} = (\log\log x)^2 + A\log\log x + \Big((\log\log q^r)^2 - \frac{\pi^2}{6}\Big) + O\Big(\frac{\log\log x}{\log x}\Big).$$

Lemma 4 *Given P, M, and X satisfying* (A) *and* (B),

(1) *If* $X = \mathbb{N}$, *we have*

$$\sum_{N(p)\le\frac{x}{2}} \frac{1}{N(p)\log\frac{x}{N(p)}} \ll \frac{\log x}{\log\log x}.$$

(2) *If* $X = \{q^{rn} ; n \in \mathbb{N}\cup\{0\}\}$, *we have*

$$\sum_{N(p)\le\frac{x}{q^r}} \frac{1}{N(p)\log\frac{x}{N(p)}} \ll \frac{\log x}{\log\log x}.$$

Proof (1) Divide $[1, \frac{x}{2}]$ as $I_j = [e^j, e^{j+1}]$. We have

$$\sum_{N(p)\le\frac{x}{2}} \frac{1}{N(p)\log\frac{x}{N(p)}} \le \sum_{j=0}^{\log\frac{x}{2}} \frac{1}{\log\frac{x}{e^{j+1}}} \sum_{e^j<N(p)\le e^{j+1}} \frac{1}{N(p)}$$
$$= \sum_{j=0}^{\log\frac{x}{2}} \frac{1}{(\log x-(j+1))}\Big(\log\frac{j+1}{j} + O\big(\frac{1}{j}\big)\Big).$$

The last inequality follows from Lemma 2. Since $\log(1+\frac{1}{x}) \ll \frac{1}{x}$ for $|x| < 1$, we have

$$\sum_{N(p)\le\frac{x}{2}} \frac{1}{N(p)\log\frac{x}{N(p)}} \ll \sum_{j=1}^{\log\frac{x}{2}} \frac{1}{(\log x - j)}\frac{1}{j}$$
$$= \frac{1}{\log x}\Big(\sum_{j=1}^{\log\frac{x}{2}}\big(\frac{1}{j} + \frac{1}{\log x - j}\big)\Big)$$
$$\ll \frac{\log\log x}{\log x}.$$

(2) The proof is exactly the same as above except replacing all $\frac{x}{2}$ by $\frac{x}{q^r}$.

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

3 Proof of Theorem 1

Now, we are ready to prove Theorem 1. Our goal is to get an asymptotic formula for

$$\sum_{N(m)\le x}(\omega(m)-\log\log x)^2=\sum_{N(m)\le x}\omega^2(m)-2\log\log x\sum_{N(m)\le x}\omega(m)+(\log\log x)^2\sum_{N(m)\le x}1.$$

By (A), the third term is

$$\kappa x(\log\log x)^2+\mathrm{O}(x^\theta(\log\log x)^2).$$

By Lemmas 1 and 2, the sum of the second term is equal to

$$\begin{aligned}\sum_{N(m)\le x}\omega(m)&=\sum_{N(p)\le x}\sum_{\substack{N(m)\le x\\ n_p(m)\ge 1}}1\\&=\kappa x\sum_{N(p)\le x}\frac{1}{N(p)}+\mathrm{O}\Bigl(x^\theta\sum_{N(p)\le x}\frac{1}{N(p)^\theta}\Bigr)\\&=\kappa x\log\log x+A\kappa x+\mathrm{O}\Bigl(\frac{x}{\log x}\Bigr).\end{aligned}$$

Now, we consider

$$\begin{aligned}\sum_{N(m)\le x}\omega^2(m)&=\sum_{\substack{N(p)N(q)\le x\\ p\ne q}}\sum_{\substack{N(m)\le x\\ n_p(m),n_q(m)\ge 1}}1+\sum_{N(p)\le x}\sum_{\substack{N(m)\le x\\ n_p(m)\ge 1}}1\\&=\sum_{N(p)N(q)\le x}\sum_{N(m')\le\frac{x}{N(p)N(q)}}1-\sum_{N(p)\le x^{1/2}}\sum_{N(m'')\le\frac{x}{N(p)^2}}1\\&\quad+\kappa x\log\log x+A\kappa x+\mathrm{O}\Bigl(\frac{x}{\log x}\Bigr).\end{aligned}$$

Here $m'=m-p-q$ and $m''=m-2p$.

The first sum of the last equation is

$$\sum_{N(p)N(q)\le x}\sum_{N(m')\le\frac{x}{N(p)N(q)}}1=\kappa x\sum_{N(p)N(q)\le x}\frac{1}{N(p)N(q)}+\mathrm{O}\Bigl(x^\theta\sum_{N(p)N(q)\le x}\frac{1}{N(p)^\theta N(q)^\theta}\Bigr).$$

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

If $X = \mathbb{N}$, Lemmas 2, 3, and 4 imply that

$$\begin{aligned}
\sum_{N(p)N(q)\leq x} \frac{1}{N(p)N(q)} &= \sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)} \Big(\sum_{N(q)\leq \frac{x}{N(p)}} \frac{1}{N(q)} \Big) \\
&= \sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)} \Big(\log\log \frac{x}{N(p)} + A + \mathrm{O}\big(\frac{1}{\log \frac{x}{N(p)}}\big) \Big) \\
&= \sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)} \log\log \frac{x}{N(p)} \\
&\quad + A\Big(\log\log \frac{x}{2} + A + O\big(\frac{1}{\log x}\big) \Big) \\
&\quad + \mathrm{O}\Big(\sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)} \frac{1}{\log \frac{x}{N(p)}} \Big) \\
&= (\log\log x)^2 + 2A\log\log x + A^2 + B + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).
\end{aligned}$$

Moreover, by Lemmas 1 and 2, we have

$$\begin{aligned}
\sum_{N(p)N(q)\leq x} \frac{1}{N(p)^\theta N(q)^\theta} &= \sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)^\theta} \Big(\sum_{N(q)\leq \frac{x}{N(p)}} \frac{1}{N(q)^\theta} \Big) \\
&\ll \sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)^\theta} \frac{\big(\frac{x}{N(p)}\big)^{1-\theta}}{\log x} \\
&\ll \frac{x^{1-\theta}}{\log x} \sum_{N(p)\leq \frac{x}{2}} \frac{1}{N(p)} \ll \frac{x^{1-\theta}\log\log x}{\log x}.
\end{aligned}$$

By replacing $\frac{x}{2}$ by $\frac{x}{q^r}$, we obtain the same results for $X = \{q^{rn}\ ;\ n \in \mathbb{N} \cup \{0\}\}$. Hence, we have

$$\sum_{N(p)N(q)\leq x}\ \sum_{N(m')\leq \frac{x}{N(p)N(q)}} 1 = (\log\log x)^2 + 2A\log\log x + A^2 + B + \mathrm{O}\Big(\frac{\log\log x}{\log x}\Big).$$

Now, consider

$$\begin{aligned}
\sum_{N(p)\leq x^{1/2}}\ \sum_{N(m'')\leq \frac{x}{N(p)^2}} 1 &= \sum_{N(p)\leq x^{1/2}} \Big(\frac{\kappa x}{N(p)^2} + \mathrm{O}\big(\frac{x^\theta}{N(p)^{2\theta}}\big) \Big) \\
&= \kappa x \sum_{p\in P} \frac{1}{N(p)^2} - \kappa x \sum_{N(p)>x^{1/2}} \frac{1}{N(p)^2} \\
&\quad + \begin{cases} \mathrm{O}\big(x^\theta \frac{x^{\frac{1}{2}(1-2\theta)}}{\log x}\big) & \text{if } 0 \leq \theta < 1/2, \\ \mathrm{O}(x^\theta) & \text{if } \theta \geq 1/2. \end{cases}
\end{aligned}$$

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

By (B) and partial summation, we have

$$\sum_{N(p)>x^{1/2}} \frac{1}{N(p)^2} \ll \frac{1}{\sqrt{x}\log x}.$$

Combining all the above results, we obtain

$$\sum_{N(m)\le x} \omega^2(m) = \kappa x(\log\log x)^2 + (2A+1)\kappa x\log\log x + \Big(A - \sum_{p\in P}\frac{1}{N(p)^2} + A^2 + B\Big)\kappa x + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big).$$

It follows that

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x\log\log x + \Big(A - \sum_{p\in P}\frac{1}{N(p)^2} + A^2 + B\Big)\kappa x + \mathrm{O}\Big(\frac{x\log\log x}{\log x}\Big),$$

which completes the proof of Theorem 1.

Remark We restrict $X = \mathbb{N}$ or $X = \{q^{rn}\ ;\ n \in \mathbb{N}\cup\{0\}\}$ in our general setting to obtain Theorem 1. If we allow X to be any subset of $\mathbb{N}$, we can still get a weaker result

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x\log\log x + \mathrm{O}(x)$$

by using a similar method. If we replace condition (B) by a much weaker condition,

$$(\mathrm{B}')\ \sum_{N(p)\le x} \frac{1}{N(p)} = \log\log x + \mathrm{O}(1),$$

with condition (A), we obtain

$$\sum_{N(m)\le x} (\omega(m) - \log\log x)^2 = \kappa x\log\log x + \mathrm{o}(x\log\log x).$$

4 Applications of the General Setting.

In this section, we provide some examples where the general setting applies. Thus analogues of the Turán Theorem hold in these cases.

Example 1 In the case of rational number, let P be the set of primes of $\mathbb{N}$ and $M = \mathbb{N}$. Take $N\colon M \to \mathbb{N}$ to be the identity map and choose $X = \mathbb{N}$. Conditions (A) and (B) are satisfied with $\kappa = 1$. Hence, Theorem 1 implies the classical Turán Theorem and we recover the asymptotic formula of Saidak [11].

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

Example 2 Let $K/\mathbb{Q}$ be a number field of degree $[K:\mathbb{Q}]$ and $\mathcal{O}_K$ its ring of integers. Let P be the set of prime ideals of $\mathcal{O}_K$ and M the set of ideals of $\mathcal{O}_K$. Take $N\colon M \to \mathbb{N}$ to be the standard norm map, *i.e.*, $\mathfrak{m} \mapsto N(\mathfrak{m}) := |\mathcal{O}_K/\mathfrak{m}|$ and choose $X = \mathbb{N}$. For $\mathfrak{m} \in M$, it was proved by Weber that [13]

$$\sum_{N(\mathfrak{m})\le x} 1 = \kappa x + \mathrm{O}\big(x^{1-1/[K:\mathbb{Q}]}\big) \text{ where } \kappa = \frac{2^{r_1}(2\pi)^{r_2} hR}{\omega\sqrt{|d_K|}},$$

with

$$\begin{aligned} r_1 &= \text{number of real embeddings of } K,\\ 2r_2 &= \text{number of complex embeddings},\\ h &= \text{class number},\\ R &= \text{regulator},\\ \omega &= \text{number of roots of unity},\\ d_K &= \text{discriminant of } K. \end{aligned}$$

Notice that there are at most $[K:\mathbb{Q}]$ many prime ideals $\mathfrak{p}$ lying above $p\,\mathcal{O}_K$ for a prime p. Hence, the Chebyshev Theorem 2 [8, p. 36–37] implies (B). Prachar [9] proved in 1952 that

$$\sum_{N(\mathfrak{m})\le x} (\omega(\mathfrak{m}) - \log\log x)^2 \ll x\log\log x.$$

Theorem 1 implies his result with a stronger estimate.

In the examples of function fields and varieties, to verify conditions (A) and (B), it suffices to get the cardinalities of elements of P and M with fixed image in $\mathbb{N}$. Using elementary geometric sums and integration techniques, we have

Lemma 5 *Let P, M, X be defined as before with $X = \{q^{rn}\,;\, n \in \mathbb{N}\cup\{0\}\}$. Define*

$$a_d := \#\big\{\, m \in M\,;\, N(m) = q^{rd}\big\}\,,\ d \in \mathbb{N}\cup\{0\}$$

and

$$b_d := \#\big\{\, p \in P\,;\, N(p) = q^{rd}\big\}\,,\ d \in \mathbb{N}.$$

(1) *If for all $d \in \mathbb{N}\cup\{0\}$,*

$$a_d = \kappa' q^{rd} + \mathrm{O}(q^{rd\theta}) \text{ for some } \kappa' > 0 \text{ and } 0 < \theta < 1,$$

we have

$$\sum_{N(m)\le x} 1 = \frac{\kappa' q^r}{q^r - 1}x + \mathrm{O}(x^{\theta}).$$

(2) *If for all* $d \in \mathbb{N}$,

$$b_d = \frac{q^{rd}}{d} + \mathrm{O}\big(q^{rd\delta}\big) \text{ for some } 0 < \delta < 1,$$

we have

$$\sum_{N(p) \leq x} 1 = \mathrm{O}\Big(\frac{x}{\log x}\Big).$$

Example 3 Let $\mathbb{F}_q[t]$ be the ring of 1-variable polynomials over a finite field $\mathbb{F}_q$. Take P to be the set of monic irreducible polynomials in $\mathbb{F}_q[t]$ and M the set of monic polynomials. We define the map N as follows:

$$N \colon M \to \mathbb{N}, m := m(t) \mapsto q^{\deg\, m(t)},$$

where $\deg m(t)$ is the degree of the polynomial $m(t)$. Since $\mathrm{Im}(N(M))$ only contains non-negative powers of q, we take $X = \{q^n \,;\, n \in \mathbb{N} \cup \{0\}\}$. In this case, we have [10, p. 6]

$$a_d = q^d$$

and

$$b_d = \frac{q^d}{d} + \mathrm{O}(q^{\frac{d}{2}}).$$

These satisfy the assumptions of Lemma 5 with $r = 1$. Hence, conditions (A) and (B) are verified and we have an analogue of the Turán Theorem in $\mathbb{F}_q[t]$.

Example 4 Let $V/\mathbb{F}_q$ be a geometrically irreducible variety of dimension r in a projective space. Let P be the set of closed points of $V/\mathbb{F}_q$, which is in bijection with the set of orbits of $V(\overline{\mathbb{F}_q})$ under the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ [5, p. 259]. For each $p \in P$, we define $\deg p$ to be the length of the corresponding orbit. The monoid of effective 0-cycles M of $V/\mathbb{F}_q$ is defined by

$$M = \Big\{ m = \sum_{p \in P} n_p(m) p \,;\, n_p(m) \in \mathbb{N} \cup \{0\}, n_p(m) \neq 0 \text{ for only finitely many } p \Big\}.$$

For $m \in M$, we define

$$\deg m = \sum_{p \in P} n_p(m) \deg p.$$

The map N is defined by

$$N \colon M \to \mathbb{N}, \quad m \mapsto q^{r \deg m}.$$

We take $X = \{q^{rn} \,;\, n \in \mathbb{N} \cup \{0\}\}$. The zeta function of $V/\mathbb{F}_q$ is defined by

$$Z(T) = \exp\Big(\sum_{n=1}^{\infty} \frac{|V(\mathbb{F}_{q^n})|}{n} T^n \Big).$$

Let a_d and b_d be defined as in Lemma 5. Using the fact that [5, p. 259]

$$|V(\mathbb{F}_{q^n})| = \sum_{d|n} d b_d,$$

we have

$$Z(T) = \prod_{d=1}^{\infty} (1 - T^d)^{-b_d} = \sum_{d=1}^{\infty} a_d T^d.$$

It was proved by Lang and Weil [6] in 1954 that

$$|V(\mathbb{F}_{q^n})| = q^{rn} + \mathrm{O}\big(q^{(r-\frac{1}{2})n}\big).$$

Applying the Möbius inversion formula, we get

$$\begin{aligned} d b_d &= \sum_{n|d} \mu\big(\frac{d}{n}\big) \big(q^{rn} + \mathrm{O}(q^{(r-\frac{1}{2})n})\big) \\ &= q^{rd} + \mathrm{O}\big(d q^{(r-\frac{1}{2})d}\big) \end{aligned}$$

Hence, we have

$$b_d = \frac{q^{rd}}{d} + \mathrm{O}(q^{(r-\frac{1}{2})d}).$$

The computation of a_d is much more involved. Using the result of Lang-Weil, we have

$$Z(T) = \exp(-\log(1 - q^r T)) \exp\Big(\sum_{n=1}^{\infty} \frac{\mathrm{O}(q^{(r-\frac{1}{2})n})}{n} T^n\Big).$$

From the theory of the l-adic cohomology of Grothendieck [2], we can write

$$Z(T) = \Big(\frac{1}{1 - q^r T}\Big) \frac{f_1(T) f_3(T) \cdots f_{2r-1}(T)}{f_0(T) f_2(T) \cdots f_{2r-2}(T)},$$

where $f_i(T)$ are polynomials. Write

$$f_i(T) = \prod_{j=1}^{B_i} (1 - \omega_{i,j} T),$$

where B_i is the i-th Betti number and $\omega_{i,j}$ are eigenvalues of the ith cohomology group. By taking logarithms on both expressions of $Z(T)$, we have

$$\sum_{i,j} (-1)^i \omega_{i,j}^n = \mathrm{O}\big(q^{(r-\frac{1}{2})n}\big).$$

Since there are only finitely many $\omega_{i,j}$ and the big O notation above is independent from n, we have

$$|\omega_{i,j}| \leq q^{r-\frac{1}{2}},$$

for all i, j.

To consider the coefficients a_d of $Z(T)$, we need the following lemmas.

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

Lemma 6 *Let $Z(T)$ be the zeta function of a geometrically irreducible variety $V/\mathbb{F}_q$ of dimension r. We define*

$$H(T) = Z(T)(1 - q^r T) = \frac{f_1(T) f_3(T) \cdots f_{2r-1}(T)}{f_0(T) f_2(T) \cdots f_{2r-2}(T)} = \sum_{i=0}^{\infty} c_i T^i.$$

Then we have

$$c_i \ll q^{(r-\frac{1}{2})i} i^s,$$

where $s = B_0 + B_2 + \cdots + B_{2r-2} - 1$.

Proof If i is odd, we write

$$f_i(T) = \sum_{j=0}^{\infty} c_{i,j} T^j.$$

Since $f_i(T)$ is a polynomial, it follows that

$$|c_{i,j}| \ll 1.$$

If i is even, we write

$$\frac{1}{f_i(T)} = \frac{1}{\prod_{j=1}^{B_i}(1 - \omega_{i,j} T)} = \sum_{j=0}^{\infty} c_{i,j} T^j.$$

For a fixed i, the largest absolute value of $c_{i,j}$ appears when all $\omega_{i,j}$ are the same. Notice that the coefficient of T^j of the rational function

$$\frac{1}{(1 - \omega T)^B} = (1 + \omega T + \omega^2 T^2 + \cdots + \omega^j T^j + \cdots)^B$$

is $\leq (j+1)^{B-1} |\omega|^j$. Hence, by the above upper bound of $|\omega_{i,j}|$, we have

$$c_{i,j} \ll j^{B_i - 1} q^{(r-\frac{1}{2})j}.$$

Notice that for α, β, and $a \in \mathbb{R}$, suppose $|d_j| \ll j^{\alpha} q^{aj}$, $|e_k| \ll k^{\beta} q^{ak}$ for all $j, k \in \mathbb{N} \cup \{0\}$. Write

$$\Big(\sum_{j=0}^{\infty} d_j T^j\Big)\Big(\sum_{k=0}^{\infty} e_k T^k\Big) = \sum_{s=0}^{\infty} c_s T^s.$$

Then we have

$$|c_s| \ll q^{as} s^{\alpha+\beta+1}.$$

It follows that the coefficient c_i of T^i of $H(T)$ is bounded by

$$c_i \ll q^{(r-\frac{1}{2})i} i^s,$$

where $s = B_0 + B_2 + \cdots + B_{2r-2} - 1$.

https://doi.org/10.4153/CMB-2004-056-7 Published online by Cambridge University Press

Lemma 7 *Let c_i be the coefficient of T^i of $H(T)$ defined in Lemma 6. For $z \in \mathbb{N} \cup \{0\}$, define*

$$C(z) = \sum_{i \le z} \frac{c_i}{q^{(r-\frac{1}{2})i}}.$$

For any $\epsilon > 0$, we have

$$\sum_{i=0}^{d} \frac{c_i}{q^{ri}} = \kappa' + O\Big(\frac{1}{q^{(\frac{1}{2}-\epsilon)d}}\Big),$$

where $\kappa' = \sum_{z=0}^{\infty} C(z)\Big(\frac{1}{q^{\frac{1}{2}z}} - \frac{1}{q^{\frac{1}{2}(z+1)}}\Big)$.

Proof By Lemma 6, we have

$$\frac{c_i}{q^{(r-\frac{1}{2})i}} \ll i^s.$$

It implies that

$$C(z) \ll z^{s+1}.$$

Using partial summation, we obtain

$$\begin{aligned}\sum_{i=0}^{d} \frac{c_i}{q^{ri}} &= \frac{C(d)}{q^{\frac{1}{2}d}} - \sum_{z=0}^{d-1} C(z)\Big(\frac{1}{q^{\frac{1}{2}(z+1)}} - \frac{1}{q^{\frac{1}{2}z}}\Big) \\ &= \kappa' + O\Big(\frac{d^{s+1}}{q^{\frac{1}{2}d}} + \sum_{z=d}^{\infty} z^{s+1}\Big(\frac{1}{q^{\frac{1}{2}z}} - \frac{1}{q^{\frac{1}{2}(z+1)}}\Big)\Big).\end{aligned}$$

For any $\epsilon > 0$, choose z_0 large enough such that $z^{s+1} \le q^{\epsilon z}$ for $z \ge z_0$. Then for $d \ge z_0$, we have

$$\begin{aligned}\frac{d^{s+1}}{q^{\frac{1}{2}d}} + \sum_{z=d}^{\infty} z^{s+1}\Big(\frac{1}{q^{\frac{1}{2}z}} - \frac{1}{q^{\frac{1}{2}(z+1)}}\Big) &\le \frac{1}{q^{(\frac{1}{2}-\epsilon)d}} + \sum_{z=d}^{\infty} \frac{1}{q^{(\frac{1}{2}-\epsilon)z}} \\ &\ll \frac{1}{q^{(\frac{1}{2}-\epsilon)d}}.\end{aligned}$$

This completes the proof of this Lemma.

Now, we write

$$Z(T) = H(T)\frac{1}{1-q^rT} = \Big(\sum_{i=0}^{\infty} c_i T_i\Big)\Big(\sum_{j=0}^{\infty} q^{rj}T^j\Big) = \sum_{d=0}^{\infty} a_d T^d.$$

Hence, we have

$$a_d = \sum_{i=0}^{d} c_i q^{r(d-i)}$$

By Lemma 7, we obtain the following theorem.

Theorem 2 *Let $V/\mathbb{F}_q$ be a geometrically irreducible variety of dimension r. Let P be the set of closed points and M the set of effective 0-cycles. We define the map $N\colon M \to \mathbb{N}$, $m \mapsto q^{r \deg m}$. For any $\epsilon > 0$, we have*

$$(1)\ a_d = \#\{m \in M\,;\, \deg m = d\} = \kappa' q^{rd} + \mathrm{O}(q^{(r-\frac{1}{2}+\epsilon)d}),$$

where κ' is the same constant as in Lemma 7. *We also have*

$$(2)\ b_d = \#\{p \in P\,;\, \deg p = d\} = \frac{q^{rd}}{d} + \mathrm{O}(q^{d(r-\frac{1}{2})}).$$

Theorem 2 and Lemma 5 imply that condition (A) and (B) are satisfied in this setting. Thus we obtain an analogue of the Turán Theorem for a geometrically irreducible variety.

Remark 1 By Lemma 5 and Theorem 2, we have

$$\sum_{N(m)\le x} 1 = \frac{\kappa' q^r}{q^r - 1} x + \mathrm{O}(x^{1-\frac{1}{2r}+\epsilon}).$$

We see from the above proof that the x^ϵ term can be replaced by $\log x$. If we apply the fact from the cohomology theory that

$$|\omega_{i,j}| \le q^{\frac{i}{2}},$$

where $\omega_{i,j}$ are the eigenvalues of the ith cohomology group, we can improve the above estimation to

$$\sum_{N(m)\le x} 1 = \frac{\kappa' q^r}{q^r - 1} x + \mathrm{O}(x^{1-\frac{1}{r}} \log x).$$

This is a similar result to the case of number fields where $r = [K:\mathbb{Q}]$ except the extra $\log x$ factor. It will be nice if we can eliminate it.

Remark 2 In the case of smooth projective curve $C/\mathbb{F}_q$, M is the set of effective divisors. Using Weil's result on the zeta function of C [5](Ch VIII), we have

$$a_d = \kappa' q^d + \mathrm{O}(1).$$

Moreover, the constant κ can be written explicitely. We have

$$\kappa = \frac{\kappa' q}{q-1} = \frac{h}{q^g}\Big(\frac{q}{q-1}\Big)^2,$$

where h is the order of $\mathrm{Pic}^0(C/\mathbb{F}_q)$ and g is the genus of C. It will be an interesting projective to study κ and express it explicitely in terms of geometric objects in a general case.

Acknowledgements This paper is part of my Ph.D. thesis at Harvard. I would like to thank my thesis advisor, Prof. B. Mazur, for many important suggestions about this work. He has inspired me in so many ways and it is a great privilege to be his student. I also would like thank Prof. R. Murty for introducing to me probabilistic number theory and for being always on my side. His encouragement and unlimited support are the main sources for me to complete my Ph.D. work. Finally, I would like to thank S. Mohit, C. Y. Kao and W. Kuo for their comments about this paper. Especially, I owe my thanks to Satya for many useful discussions related to this work.

References

[1] P. D. T. A. Elliott, *Probabilistic number theory, I, II*. Springer-Verlag, New York, 1979.
[2] R. Hartshorne, *Algebraic geometry*. Springer Verlag, New York, 1977.
[3] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of number n.* Quar. J. Pure Appl. Math. **48**(1917), 76–97
[4] Y.-R. Liu and M. R. Murty, *The Turán sieve method and some of its applications.* J. Ramanujan Math. Soc. **14**(1999), 21–35.
[5] D. Lorenzini, *An invitation to arithmetic geometry*. Graduate Studies in Mathematics, 9, American Mathematics Society, Providence, RI, 1996.
[6] S. Lang & A. Weil, *Number of points of varieties in finite fields.* Am. J. Math. **76**(1954), 819–827.
[7] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie.* J. Reine Angew. Math. **78**(1874), 46–62.
[8] M. R. Murty, *Problems in analytic number theory*. Springer-Verlag, New York, 2001.
[9] K. Prachar, *Verallgemeinerung eines satzes von Hardy and Ramanujan auf algebraische zahlkörper*. Monatsh. Math. **56**(1952), 229–232.
[10] M. Rosen, *Number theory in Function Fields*. Lecture notes given at K.A.I.S.T., Taejon, Korea, 1994.
[11] F. Saidak, *On a theorem of Hardy-Ramanujan-Turán I*, to appear.
[12] P. Turán, *On a theorem of Hardy and Ramanujan*. J. London Math. Soc. **9**(1934), 274–276.
[13] H. Weber, *Über zahlengruppen in algebraischen körpern*. Math. Ann. **49**(1897), 83–100.

Department of Pure Mathematics
University of Waterloo
Waterloo, ON
N2L 3G1
e-mail: yrliu@math.uwaterloo.ca