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THE FAMILY OF LODATO PROXIMITIES COMPATIBLE WITH A GIVEN TOPOLOGICAL SPACE

W. J. THRON AND R. H. WARREN

Compendium. Let $(X, \mathscr{T})$ be a topological space. By $\mathfrak{M}_1$ we denote the family of all Lodato proximities on X which induce $\mathscr{T}$. We show that $\mathfrak{M}_1$ is a complete distributive lattice under set inclusion as ordering. Greatest lower bound and least upper bound are characterized. A number of techniques for constructing elements of $\mathfrak{M}_1$ are developed. By means of one of these constructions, all covers of any member of $\mathfrak{M}_1$ can be obtained. Several examples are given which relate $\mathfrak{M}_1$ to the lattice $\mathfrak{M}$ of all compatible proximities of Čech and the family $\mathfrak{M}_2$ of all compatible proximities of Efremovič. The paper concludes with a chart which summarizes many of the structural properties of $\mathfrak{M}$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

1. Preliminaries and notation. M. W. Lodato in [**5**] and [**6**] has studied a symmetric generalized proximity structure (see Definition 1.2). Naimpally and Warrack [**8**] have called such a structure a Lodato proximity. We shall also use this name. The closure operator induced by a Lodato proximity satisfies the four Kuratowski closure conditions.

This paper is primarily concerned with a study of the order structure of the family $\mathfrak{M}_1$ of all Lodato proximities which induce the same closure operator on a given set. Lodato characterized the least element in $\mathfrak{M}_1$ and those topological spaces for which $\mathfrak{M}_1 \neq \emptyset$. Sharma and Naimpally [**9**] described the greatest member of $\mathfrak{M}_1$ and have given two methods for constructing members of $\mathfrak{M}_1$.

The symbol $\mathscr{P}(X)$ denotes the power set of X, $|A|$ indicates the cardinal number of the set A, and the triple bar $\equiv$ is reserved for definitions.

Definition 1.1 [**2**]. A topological space $(X,\mathscr{T})$ is called an R_0-space if and only if, given x and y in X such that $x \in [\bar{y}]$, then $y \in [\bar{x}]$.

The R_0-spaces are exactly those spaces for which $\mathfrak{M}_1 \neq \emptyset$. A. S. Davis [**2**] has given a number of characterizations of R_0-spaces. We add one more characterization: A topological space is an R_0-space if and only if each subset of the space is separated from the points which are excluded from its closure. Davis [**2**] claims that if the topology $\mathscr{T}$ on X is isomorphic (as a lattice) to the topology of a T_1-space, then $(X,\mathscr{T})$ is an R_0-space. However, this last statement is false as one notes from the following example: Let X be any infinite

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set and let $\mathscr{C} = [X$, all finite subsets of $X]$. Then $(X, \mathscr{C})$ is a T_1-space where $\mathscr{C}$ is the lattice of closed sets. Let $Y = X \cup [y]$ where $y \notin X$ and let $\mathscr{D} = [Y$, all finite subsets of $X]$. Then $(Y, \mathscr{D})$ is a topological space which does not satisfy the R_0-axiom. We observe that $\mathscr{C}$ and $\mathscr{D}$ are lattice isomorphic.

Definition 1.2 [**7**, p. 1]. Let X be a set. A relation $\mathscr{P}$ on $\mathscr{P}(X)$ is said to define a *Lodato proximity* on X if and only if it satisfies the conditions:
P1: $(A, B) \in \mathscr{P}$ implies $(B, A) \in \mathscr{P}$;
P2: $(A, B \cup C) \in \mathscr{P}$ if and only if $(A, B) \in \mathscr{P}$ or $(A, C) \in \mathscr{P}$;
P3: $(\emptyset, A) \notin \mathscr{P}$ for every $A \subset X$;
P4: $([x], [x]) \in \mathscr{P}$ for all $x \in X$;
P5: $(A, B) \in \mathscr{P}$ and $([b], C) \in \mathscr{P}$ for all $b \in B$ imply $(A, C) \in \mathscr{P}$.

We now list a number of basic results about Lodato proximities which are established in the literature. Let $\mathscr{P}$ be a Lodato proximity on X. The function $c = c(\mathscr{P}) : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by $c(A) \equiv [x : ([x], A) \in \mathscr{P}]$ is a Kuratowski closure operator satisfying the R_0-axiom. If $\mathscr{T}$ is the topology generated by c, then we say that $\mathscr{P}$ induces $\mathscr{T}$ or that $\mathscr{P}$ is compatible with $\mathscr{T}$. More generally, for a relation $\mathscr{S}$ on $\mathscr{P}(X)$, we say $\mathscr{S}$ induces c if for each $A \subset X$, $c(A) = [x : ([x], A) \in \mathscr{S}]$. If $(X, \mathscr{T})$ is an R_0-space, then

$$\mathscr{R}_1 \equiv [(A, B) : \bar{A} \cap \bar{B} \neq \emptyset]$$

is a Lodato proximity on X compatible with $\mathscr{T}$. Let $\mathfrak{M}_1 = \mathfrak{M}_1(X, \mathscr{T})$ be the family of all Lodato proximities on X which induce $\mathscr{T}$ and let $\mathfrak{M}_1$ be ordered by set inclusion. Then $\mathfrak{M}_1$ has a least element $\mathscr{R}_1$ (defined above) and a greatest element

$$\mathscr{W}_1 \equiv \mathscr{R}_1 \cup [(A, B) : \bar{A} \text{ and } \bar{B} \text{ are not finite unions of point closures}].$$

Definition 1.3. Let X be a set. A relation on $\mathscr{P}(X)$ satisfying P1, P2, P3 and P4 of Definition 1.2 is called a *Č-proximity* on X.

Č-proximities have been studied extensively in [**11**]. Every Lodato proximity on X is a Č-proximity on X. By $\mathfrak{M} = \mathfrak{M}(X, c)$ we denote the family of all Č-proximities on the set X which induce the operator c. Clearly $\mathfrak{M}_1(X, \mathscr{T}) \subset \mathfrak{M}(X, c)$ where $\mathscr{T}$ is the topology generated by c when c is a Kuratowski closure operator. Let $\mathfrak{M}$ be partially ordered by set inclusion. Then $\mathfrak{M}$ has a least element

$$\mathscr{R} \equiv [(A, B) : (\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \emptyset]$$

and a greatest element

$$\mathscr{W} \equiv \mathscr{R} \cup [(A, B) : A \text{ and } B \text{ are infinite subsets of } X].$$

A number of the results in this paper are built upon the properties of $(\mathfrak{M}, \subset)$ which are proved in [**11**].

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The following definitions will be useful in the sequel.

Definition 1.4. Let $(L, \leqq)$ be a partially ordered set. If $a, b \in L$, we say *a covers b* or *b is covered by a* when $a > b$ and $a > c > b$ is not satisfied for any $c \in L$. Moreover $(L, \leqq)$ is said to be *covered* if and only if, given $x \in L$ such that there is $y \in L$ satisfying $y > x$, then there is $z \in L$ which covers x and satisfies $z \leqq y$. Also $(L, \leqq)$ is said to be *anticovered* if and only if the dual of $(L, \leqq)$ is covered.

Definition 1.5. Let $(L, \leqq)$ be a partially ordered set. If $(L, \leqq)$ has a least element d, then $a \in L$ is an *atom* if and only if a covers d. Also $c \in L$ is an *antiatom* if and only if c is an atom in the dual of $(L, \leqq)$. Furthermore $(L, \leqq)$ is called *atomic* when each $x \in L$, x not the least element, is the least upper bound of the atoms $\leqq x$. Moreover $(L, \leqq)$ is called *strongly atomic* if and only if, given $a \in L$, the partially ordered set $[b : a \leqq b \in L]$ is atomic. Also $(L, \leqq)$ is *antiatomic* if and only if the dual of $(L, \leqq)$ is atomic.

2. Lodato proximities. In this section we give several characterizations of a Lodato proximity.

THEOREM 2.1. *Let $\mathscr{P}$ be a Č-proximity on X. Then $\mathscr{P}$ is a Lodato proximity on X if and only if $(\bar{A}, \bar{B}) \in \mathscr{P}$ implies $(A, B) \in \mathscr{P}$.*

Proof. In [**7**, p. 5] the authors have proved that if $\mathscr{P}$ is a Lodato proximity, then $(\bar{A}, \bar{B}) \in \mathscr{P}$ implies $(A, B) \in \mathscr{P}$.

Assume that $(\bar{A}, \bar{B}) \in \mathscr{P}$ implies $(A, B) \in \mathscr{P}$. To verify P5, suppose $(C, D) \in \mathscr{P}$ and $([d], E) \in \mathscr{P}$ for all $d \in D$. Hence $d \in \bar{E}$ for all $d \in D$, and $D \subset \bar{E}$. P2 implies $(C, \bar{E}) \in \mathscr{P}$. Since $C \subset \bar{C}$, P1 and P2 imply $(\bar{C}, \bar{E}) \in \mathscr{P}$. By our assumption $(C, E) \in \mathscr{P}$.

Definition 2.1. Let X be a set and $\mathscr{P}$ a relation on $\mathscr{P}(X)$. If $A \subset X$, we define $A^* \equiv [x \in X : ([x], A) \in \mathscr{P}]$. We introduce names for the following statements.

P5′: $(A^*, B^*) \in \mathscr{P}$ and $([b]^*, C^*) \in \mathscr{P}$ for all $b \in B$ imply $(A, C) \in \mathscr{P}$.
P6: $(A^*, B^*) \in \mathscr{P}$ implies $(A, B) \in \mathscr{P}$.
P6′: $(A^*, B) \in \mathscr{P}$ implies $(A, B) \in \mathscr{P}$.
P7: $(A, B) \in \mathscr{P}$ and $B \subset C^*$ imply $(A, C) \in \mathscr{P}$.
P7′: $(A, B) \in \mathscr{P}$ and $B^* \subset C^*$ imply $(A, C) \in \mathscr{P}$.
P8: $A^{**} \subset A^*$.

One notes that the definition of A^* is motivated by the fact that if $\mathscr{P}$ is a Lodato proximity, then $A^* = \bar{A}$.

THEOREM 2.2. *Let X be a set and $\mathscr{P}$ a relation on $\mathscr{P}(X)$. Then the following are equivalent:*

(i) *$\mathscr{P}$ is a Lodato proximity on X.*
(ii) *$\mathscr{P}$ satisfies* P1, P2, P3, P4 *and* P6.

(iii) $\mathscr{P}$ *satisfies* P1, P2, P3, P4 *and* P7.
(iv) $\mathscr{P}$ *satisfies* P1, P2, P3, P4, P7′ *and* P8.
(v) $\mathscr{P}$ *satisfies* P1, P2, P3, P4 *and* P5′.
(vi) $\mathscr{P}$ *satisfies* P1, P2, P3, P4 *and* P6′.

Proof. The proof is a straightforward verification. We indicate an easy route. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i). (ii) $\Rightarrow$ (v) $\Rightarrow$ (i). (ii) $\Rightarrow$ (vi) $\Rightarrow$ (iii).

THEOREM 2.3. *In the axiom system in Theorem* 2.2(iv), P7′ *and* P8 *are independent axioms.*

Proof. First we give an example where P1, P2, P3, P4 and P8 hold, but P7′ fails. Let X be the set of real numbers, $\mathscr{T}$ the usual topology on X, $\mathscr{P} = [(A, B) : (\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \emptyset]$ and $A^* = \bar{A}$ for all $A \subset X$.

The following example satisfies P1, P2, P3, P4 and P7′ but not P8. Let $S = [r, s, t]$ and let $\mathscr{P}$ be the relation on $\mathscr{P}(S)$ such that

$$\sim\mathscr{P} = [(\emptyset, A) : A \subset S] \cup [(B, \emptyset) : B \subset S] \cup [([s], [t])].$$

3. Continuous and p-continuous functions.

Definition 3.1 [**7**, p. 8]. A mapping f from a Lodato proximity space $(X, \mathscr{P})$ to a Lodato proximity space $(Y, \mathscr{P}^*)$ is said to be *p-continuous* if and only if $(A, B) \in \mathscr{P}$ implies $(f(A), f(B)) \in \mathscr{P}^*$.

An equivalent formulation of this definition is: f is p-continuous if and only if for all $(C, D) \notin \mathscr{P}^*$ with $C, D \subset Y$, it is true that $(f^{-1}(C), f^{-1}(D)) \notin \mathscr{P}$. It is known [**7**, p. 8] that every p-continuous function is a continuous function with respect to the induced closure operators. In this context the following theorem may be of interest.

THEOREM 3.1. *Let* $(X, \mathscr{T})$ *and* $(Y, \mathscr{U})$ *be* R_0*-topological spaces. Let* $\mathscr{P} \in \mathfrak{M}_1(Y, \mathscr{U})$ *and let* $\mathscr{R}_1$ *be the least element in* $\mathfrak{M}_1(X, \mathscr{T})$. *If* $f : (X, \mathscr{T}) \rightarrow (Y, \mathscr{U})$ *is continuous, then* f *is p-continuous with respect to* $\mathscr{R}_1$ *and* $\mathscr{P}$.

Proof. If $(A, B) \in \mathscr{R}_1$, then $\bar{A} \cap \bar{B} \neq \emptyset$. Since f is continuous,

$$\overline{f(A)} \cap \overline{f(B)} \neq \emptyset.$$

Hence $(f(A), f(B))$ is in $\mathscr{P}$.

There is a fixed completely regular topological (R_0-closure) space $(Y, \mathscr{U})$ with the following property. If $(X, \mathscr{C})$ is a completely regular topological (R_0-closure) space and if $\mathscr{P}$ is a compatible proximity ($\check{C}$-proximity) on X such that all continuous functions from $(X, \mathscr{C})$ to $(Y, \mathscr{U})$ are p-continuous with respect to $\mathscr{P}$ and the smallest compatible proximity ($\check{C}$-proximity) on Y, then $\mathscr{P}$ must be the smallest compatible proximity ($\check{C}$-proximity) on X. For $\check{C}$-proximities this result is proved in [**11**, Theorem 3.4]. By the next theorem we shall show that no such R_0-topological space $(Y, \mathscr{U})$ exists for Lodato proximities. Our proof is constructive.

THEOREM 3.2. *For each* R_0*-topological space* $(Y, \mathscr{U})$ *there is an* R_0*-topological space* $(X, \mathscr{C})$ *and there is* $\mathscr{P}$ *in* $\mathfrak{M}_1(X, \mathscr{C}) - [\mathscr{R}_1]$ *such that each continuous function* $f : (X, \mathscr{C}) \to (Y, \mathscr{U})$ *is p-continuous with respect to* $\mathscr{P}$ *and* $\mathscr{R}^*$. *Here* $\mathscr{R}_1$ *is the least member of* $\mathfrak{M}_1(X, \mathscr{C})$ *and* $\mathscr{R}^*$ *is the least member of* $\mathfrak{M}_1(Y, \mathscr{U})$.

Proof. Given $(Y, \mathscr{U})$, let X be an infinite set such that $|X| > |Y|$. Choose disjoint subsets A, B of X such that $|A| = |B| = \aleph_0$ and $|X - (A \cup B)| = |X|$. Let $\mathscr{C}$ be the following family of subsets of X: X, A, B, any finite subset of X and any finite union of these sets. Then $(X, \mathscr{C})$ is an R_0-topological space where $\mathscr{C}$ is the family of closed sets.

Let $\mathscr{P}$ be the relation on $\mathscr{P}(X)$ defined by:

$$\mathscr{P} = \mathscr{R}_1 \cup [(C, D) : (\bar{C} \supset A \text{ and } \bar{D} \supset B) \text{ or } (\bar{C} \supset B \text{ and } \bar{D} \supset A)].$$

Then $\mathscr{P} \in \mathfrak{M}_1(X, \mathscr{C})$ and $\mathscr{P} \neq \mathscr{R}_1$.

Suppose there exists a continuous function f: $(X, \mathscr{C}) \to (Y, \mathscr{U})$ which is not p-continuous with respect to $\mathscr{P}$ and $\mathscr{R}^*$. Then there are subsets E, F of X such that $(E, F) \in \mathscr{P}$ but $(f(E), f(F)) \notin \mathscr{R}^*$. Thus $\overline{f(E)} \cap \overline{f(F)} = \emptyset$. Since f is continuous, $f(\bar{E}) \cap f(\bar{F}) = \emptyset$ from which it follows that $\bar{E} \cap \bar{F} = \emptyset$. Hence $(E, F) \in (\mathscr{P} - \mathscr{R}_1)$ and consequently $\overline{f(A)} \cap \overline{f(B)} = \emptyset$.

Let $T = f^{-1}(\overline{f(A)})$ and $S = f^{-1}(\overline{f(B)})$. Since T and S are disjoint, closed sets, we note that $|X - (T \cup S)| = |X - (A \cup B)|$. Given x in $X - (T \cup S)$, then $f^{-1}(\overline{[f(x)]})$ is a closed, finite subset of $X - (T \cup S)$. Hence $|X - (T \cup S)| \leqq |Y|$. Since $|X| = |X - (T \cup S)|$, we have contradicted $|X| > |Y|$.

THEOREM 3.3. *Let* $(X, \mathscr{T})$ *be an* R_0*-topological space, let* $Y = [0, 1, 2]$ *and let* $\mathscr{C} = [\emptyset, Y, [0], [2], [0, 2]]$. *Thus* $(Y, \mathscr{C})$ *is a topological space with* $\mathscr{C}$ *the family of closed sets. We define* $(A, B) \notin \mathscr{P}$ *if and only if there is a continuous function* $f : (X, \mathscr{T}) \to (Y, \mathscr{C})$ *such that* $A \subset f^{-1}([0])$ *and* $B \subset f^{-1}([2])$. *Then* $\mathscr{P} = \mathscr{R}_1$.

Proof. Let $(A, B) \notin \mathscr{P}$. Then there is a continuous function $f : (X, \mathscr{T}) \to (Y, \mathscr{C})$ such that $A \subset f^{-1}([0])$ and $B \subset f^{-1}([2])$. Since f is continuous, $f^{-1}([0])$ is a closed set. Thus $\bar{A} \subset f^{-1}([0])$. Similarly, $\bar{B} \subset f^{-1}([2])$. Since $f^{-1}([0]) \cap f^{-1}([2]) = \emptyset$, $\bar{A} \cap \bar{B} = \emptyset$. Consequently $(A, B) \notin \mathscr{R}_1$.

Conversely, let $(A, B) \notin \mathscr{R}_1$. Then $\bar{A} \cap \bar{B} = \emptyset$. We define $g : X \to Y$ by

$$g(x) = \begin{cases} 0, & \text{if } x \in \bar{A} \\ 1, & \text{if } x \notin \bar{A} \cup \bar{B} \\ 2, & \text{if } x \in \bar{B}. \end{cases}$$

Clearly g is well-defined and continuous. Hence $(A, B) \notin \mathscr{P}$.

We point out that the image space $(Y, \mathscr{C})$ in the last theorem is not R_0.

4. Construction of Lodato proximities. In this section we characterize the least and greatest elements of $\mathfrak{M}_1$ and describe several techniques for constructing members of $\mathfrak{M}_1$. The most important result is that each atom of $\mathfrak{M}_1$ can be constructed from the least member of $\mathfrak{M}_1$ by using two special bunches.

THEOREM 4.1. *Let* $(X, \mathscr{T})$ *be an* R_0*-topological space. If* $[\mathscr{P}_i : i \in I] \subset \mathfrak{M}_1$, *then* $\bigcup [\mathscr{P}_i : i \in I] \in \mathfrak{M}_1$. *Thus* $\mathfrak{M}_1$ *is a complete lattice with the operator* $\vee = \bigcup$.

Proof. Since $\mathfrak{M}_1 \subset \mathfrak{M}$, it follows that $[\mathscr{P}_i : i \in I] \subset \mathfrak{M}$. From [**11**], $\bigcup [\mathscr{P}_i : i \in I] \in \mathfrak{M}$. So $\mathscr{R} \subset \bigcup [\mathscr{P}_i : i \in I] \subset \mathscr{W}$. By [**11**, Theorem 3.1], $\bigcup [\mathscr{P}_i : i \in I]$ induces $\mathscr{T}$.

Let $(\bar{A}, \bar{B}) \in \bigcup [\mathscr{P}_i : i \in I]$. Then for some i, $(\bar{A}, \bar{B}) \in \mathscr{P}_i$. Since $\mathscr{P}_i \in \mathfrak{M}_1$, by Theorem 2.1 we have $(A, B) \in \mathscr{P}_i$. Therefore

$$(A, B) \in \bigcup [\mathscr{P}_i : i \in I].$$

Hence by Theorem 2.1, $\bigcup [\mathscr{P}_i : i \in I]$ is a Lodato proximity on X.

Since $\mathfrak{M}_1$ has a least element and since $\bigcup [\mathscr{P} : \mathscr{P} \in \mathfrak{M}_1]$ is the greatest element of $\mathfrak{M}_1$, the last statement in the theorem follows from the work of Thron [**10**, pp. 7–10].

THEOREM 4.2. *Let* $(X, \mathscr{T})$ *be an* R_0*-topological space. The greatest element of* $\mathfrak{M}_1$ *is* $\mathscr{W}_1 \equiv \mathscr{R}_1 \cup [(A, B) : \bar{A}$ *and* $\bar{B}$ *are not finite unions of point closures*].

Proof. Since $\mathfrak{M}_1 \subset \mathfrak{M}$, $\mathscr{R}_1 \in \mathfrak{M}$. Thus $\mathscr{R} \subset \mathscr{R}_1 \subset \mathscr{W}$. Clearly $\mathscr{W}_1 \subset \mathscr{W}$. Therefore by [**11**, Theorem 3.1], $\mathscr{W}_1$ induces $\mathscr{T}$ and satisfies P3, P4.

Clearly P1 is satisfied by $\mathscr{W}_1$.

To verify P2, let $(A, B \cup C) \in \mathscr{W}_1 - \mathscr{R}_1$. Then $\bar{A}$ and $\overline{B \cup C}$ are not a finite union of point closures. Since $\bar{B} \cup \bar{C} = \overline{B \cup C}$, $\bar{B}$ or $\bar{C}$ is not a finite union of point closures. So (A, B) or $(A, C) \in \mathscr{W}_1$.

Conversely, let $(A, B \cup C) \notin \mathscr{W}_1$. Then $(A, B \cup C) \notin \mathscr{R}_1$, and so $(A, B) \notin \mathscr{R}_1$ and $(A, C) \notin \mathscr{R}_1$. Also $\bar{A}$ or $\overline{B \cup C}$ is a finite union of point closures. If $\bar{A}$ is a finite union of point closures, then $(A, B) \notin \mathscr{W}_1$ and $(A, C) \notin \mathscr{W}_1$. If $\overline{B \cup C}$ is a finite union of point closures, then $\bar{B}$ is a finite union of point closures. Hence $(A, B) \notin \mathscr{W}_1$. Similarly, $(A, C) \notin \mathscr{W}_1$.

We have shown that $\mathscr{W}_1$ is a $\check{C}$-proximity on X. To verify that $\mathscr{W}_1$ is a Lodato proximity on X we will show that $(\bar{A}, \bar{B}) \in \mathscr{W}_1$ implies $(A, B) \in \mathscr{W}_1$ and appeal to Theorem 2.1. Clearly $(\bar{A}, \bar{B}) \in \mathscr{R}_1$ implies $(A, B) \in \mathscr{R}_1$ since $\mathscr{R}_1 \in \mathfrak{M}_1$. So we suppose that $(\bar{A}, \bar{B}) \in \mathscr{W}_1 - \mathscr{R}_1$. Then $\bar{\bar{A}}$ and $\bar{\bar{B}}$ are not a finite union of point closures. Since $\bar{A} = \bar{\bar{A}}$ and $\bar{B} = \bar{\bar{B}}$, $\bar{A}$ and $\bar{B}$ are not a finite union of point closures. Thus $(A, B) \in \mathscr{W}_1$.

We now show that $\mathscr{W}_1$ is the greatest member of $\mathfrak{M}_1$. Suppose there is $\mathscr{P} \in \mathfrak{M}_1$ and $\mathscr{P} \not\subset \mathscr{W}_1$. Choose $(E, F) \in \mathscr{P} - \mathscr{W}_1$. Consequently $(E, F) \notin \mathscr{R}_1$. Hence $\bar{E}$ or $\bar{F}$ is a finite union of point closures, say $\bar{E}$. So

$$\bar{E} = \bigcup_{i=1}^{n} [\overline{y_i}].$$

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It follows from P1, P2 and induction that for some i, $([\overline{y_i}], \bar{F}) \in \mathscr{P}$. By Theorem 2.1 $([y_i], F) \in \mathscr{P}$. Thus $y_i \in \bar{F}$, and $(E, F) \in \mathscr{R}_1$ which is a contradiction.

Theorem 4.2 was first published by Sharma and Naimpally [**9**]. We arrived at this result independently by the approach presented above.

Note that $\overline{A \cup B}$ is a finite union of point closures if and only if $\bar{A}$ and $\bar{B}$ are finite unions of point closures. Therefore

$$\mathscr{W}_1 = \mathscr{R}_1 \cup [(A, B) : \overline{A \cup B} \text{ is not a finite union of point closures}].$$

Of course, $\mathscr{W}_1 = \bigcup [\mathscr{P} : \mathscr{P} \in \mathfrak{M}_1]$.

The following theorem gives necessary and sufficient conditions for the greatest element of $\mathfrak{M}$ to be the greatest element of $\mathfrak{M}_1$.

THEOREM 4.3. *Let $(X,\mathscr{T})$ be an R_0-topological space. Then $\mathscr{W} = \mathscr{W}_1$ if and only if $\mathscr{W}_1 \supset [(A, B) : A$ and B are infinite subsets of $X]$. If $(X,\mathscr{T})$ is a T_1-space, then $\mathscr{W} = \mathscr{W}_1$.*

Proof. The proof is an easy verification.

THEOREM 4.4. *Let $(X,\mathscr{T})$ be an R_0-topological space, and let $\mathscr{S}$ be a relation on $\mathscr{P}(X)$. If $\mathscr{R}_1 \subset \mathscr{S} \subset \mathscr{W}_1$, then $\mathscr{S}$ induces $\mathscr{T}$; $\mathscr{S}$ satisfies* P3, P4; *and $A^* = \bar{A}$ for every $A \subset X$.*

Proof. Clearly $\mathscr{R} \subset \mathscr{R}_1$ and $\mathscr{W}_1 \subset \mathscr{W}$. So $\mathscr{R} \subset \mathscr{S} \subset \mathscr{W}$. By [**11**, Theorem 3.1], $\mathscr{S}$ induces $\mathscr{T}$ and satisfies P3 and P4.

Also $y \in A^* \Leftrightarrow ([y], A) \in \mathscr{S} \Leftrightarrow y \in \bar{A}$.

Definition 4.1. Let $(X,\mathscr{T})$ be a topological space and let $A \subset X$. We define $p(A)$ to be the cardinal number of the set of distinct point closures contained in $\bar{A}$.

THEOREM 4.5. *Let $(X,\mathscr{T})$ be a topological space and let $A, B \subset X$.*

(i) $p(A \cup B) \leqq p(A) + p(B)$.

(ii) *If $p(A) \geqq \aleph_0$ or $p(B) \geqq \aleph_0$, then $p(A) + p(B) = p(A \cup B)$.*

Proof. Since the proof is straightforward, it is omitted.

Definition 4.2. Let $(X,\mathscr{T})$ be a topological space and $A \subset X$. We define $A\dagger$ to be the family of all ordered triples of cardinal numbers (λ, μ, ν) for which there is an open cover $\mathscr{U}$ of A such that $\lambda \geqq |\mathscr{U}| \geqq \mu$ and every subcover has cardinality $\geqq \nu$.

Note that A is compact if and only if, whenever $(\lambda, \mu, \nu) \in A\dagger$, then ν is finite.

Definition 4.3. Let $(X,\mathscr{T})$ be an R_0-topological space; let E be a closed subset of X; let λ, μ, ν be infinite cardinal numbers and let $\mathscr{P} \in \mathfrak{M}_1$. We introduce the following notation.

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(i) $\mathscr{P}(E, \mu) \equiv \mathscr{P} \cup [(A, B) : p(\bar{A} \cap E) \geqq \mu$ and $p(\bar{B} \cap E) \geqq \mu]$.
(ii) $\mathscr{P}^0(E, \mu) \equiv \mathscr{W}_1 \cap (\mathscr{P} \cup [(A, B) : p(\bar{A} \cap E) \geqq \mu$ or

$p(\bar{B} \cap E) \geqq \mu])$.

(iii) $\mathscr{P}\{E, \lambda, \mu, \nu\} \equiv \mathscr{P} \cup [(A, B) : (\lambda, \mu, \nu) \in (\bar{A} \cap E)\dagger \cap (\bar{B} \cap E)\dagger]$.
(iv) $\mathscr{P}^0\{E, \lambda, \mu, \nu\} \equiv \mathscr{W}_1 \cap (\mathscr{P} \cup [(A, B) :$

$(\lambda, \mu, \nu) \in (\bar{A} \cap E)\dagger \cup (\bar{B} \cap E)\dagger]$.

THEOREM 4.6. *Each of the following relations is in* $\mathfrak{M}_1$*:*
(i) $\mathscr{P}(E, \mu)$.
(ii) $\mathscr{P}^0(E, \mu)$.
(iii) $\mathscr{P}\{E, \lambda, \mu, \nu\}$.
(iv) $\mathscr{P}^0\{E, \lambda, \mu, \nu\}$.

Proof. Based upon Theorem 4.4 the proof is relatively straightforward and therefore is omitted.

The Lodato proximities in Theorem 4.6 have properties similar to those established in [**11**, Theorems 3.6–3.9]. Since we will not make use of these properties, we omit them.

Theorem 2.3 of [**9**] is a special case of Theorem 4.6(i) when $E = X$ and $\mathscr{P} = \mathscr{R}_1$. Also Theorem 2.1 of [**9**] is a special case of Theorem 4.6(iii) when $E = X$, $\mathscr{P} = \mathscr{R}_1$ and $\mu = \nu = \aleph_0$.

Definition 4.4 [**7**, p. 10]. Let $\mathscr{P}$ be a Lodato proximity on X. A nonempty family $\mathscr{B} \subset \mathscr{P}(X)$ is a *bunch* with respect to $(X, \mathscr{P})$ if and only if $\mathscr{B}$ satisfies the following three axioms:
B1: $(A, B) \in \mathscr{P}$ for all A, B in $\mathscr{B}$;
B2: $A \cup B \in \mathscr{B}$ if and only if $A \in \mathscr{B}$ or $B \in \mathscr{B}$;
B3: $A \in \mathscr{B}$ if and only if $\bar{A} \in \mathscr{B}$.
In [**7**, p. 10] the authors prove that if $\mathscr{P}$ is a Lodato proximity on X and $\mathscr{U}$ is a ultrafilter on X, then

$$b(\mathscr{U}) \equiv [A \subset X : \bar{A} \in \mathscr{U}]$$

is a bunch with respect to $(X, \mathscr{P})$. We note that if the ultrafilter $\mathscr{U}$ contains no point closures, then $b(\mathscr{U})$ contains no point closures. In the light of the next theorem this means that if the ultrafilters $\mathscr{U}, \mathscr{V}$ contain no point closures, then $(b(\mathscr{U}) \times b(\mathscr{V})) \subset \mathscr{W}_1$.

THEOREM 4.7. *Let* $(X, \mathscr{T})$ *be an* R_0*-topological space and let* $\mathscr{A}, \mathscr{B}$ *be bunches with respect to* $(X, \mathscr{P})$ *where* $\mathscr{P} \in \mathfrak{M}_1$. *If* $\mathscr{A} \cup \mathscr{B}$ *contains no point closures, then* $(\mathscr{A} \times \mathscr{B}) \subset \mathscr{W}_1$.

Proof. Suppose there is $(A, B) \in ((\mathscr{A} \times \mathscr{B}) - \mathscr{W}_1)$. Then $\bar{A}$ or $\bar{B}$ is a finite union of point closures, say $\bar{A}$. So

$$\bar{A} = \bigcup_{i=1}^{n} [\overline{a_i}].$$

By B2 and induction, there is i such that $[\overline{a_i}] \in \mathscr{A}$, which is a contradiction.

THEOREM 4.8. *Let $(X, \mathscr{T})$ be an R_0-topological space and let $\mathscr{P} \in \mathfrak{M}_1$. Also let $\mathscr{A}$ and $\mathscr{B}$ be bunches with respect to $(X, \mathscr{P})$. If $(\mathscr{A} \times \mathscr{B}) \subset \mathscr{W}_1$, then $\mathscr{P}' = \mathscr{P} \cup (\mathscr{A} \times \mathscr{B}) \cup (\mathscr{B} \times \mathscr{A})$ is in $\mathfrak{M}_1$.*

Proof. Clearly $\mathscr{P}'$ satisfies P1. P2 follows directly from B2. Since $\mathscr{R}_1 \subset \mathscr{P}' \subset \mathscr{W}_1$, the result follows from Theorems 4.4 and 2.2, and from the fact that P6 is a consequence of B3.

THEOREM 4.9. *Let $(X, \mathscr{T})$ be an R_0-topological space and let $\mathscr{P}, \mathscr{P}^* \in \mathfrak{M}_1$ such that there is (C, D) in $\mathscr{P} - \mathscr{P}^*$. Then there are nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X such that $(C, D) \in (\mathscr{F} \times \mathscr{G}) \subset (b(\mathscr{F}) \times b(\mathscr{G})) \subset \mathscr{P}$. Also $\mathscr{F}$ and $\mathscr{G}$ can be chosen so that $\mathscr{F} \cup \mathscr{G}$ contains no point closures.*

Proof. Since $\mathscr{P} \in \mathfrak{M}$, by [**11**, Theorem 3.11] there are nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X such that $(C, D) \in (\mathscr{F} \times \mathscr{G}) \subset \mathscr{P}$. Let $(A, B) \in (b(\mathscr{F}) \times b(\mathscr{G}))$. Then $(\bar{A}, \bar{B}) \in (\mathscr{F} \times \mathscr{G}) \subset \mathscr{P}$. Since $\mathscr{P} \in \mathfrak{M}_1$, $(A, B) \in \mathscr{P}$. Thus $b(\mathscr{F}) \times b(\mathscr{G}) \subset \mathscr{P}$.

Let $(E, G) \in (\mathscr{F} \times \mathscr{G})$. Then $(\bar{E}, \bar{F}) \in (\mathscr{F} \times \mathscr{G})$, since $\mathscr{F}$ and $\mathscr{G}$ are filters. Consequently $(E, F) \in (b(\mathscr{F}) \times b(\mathscr{G}))$, and $\mathscr{F} \times \mathscr{G} \subset b(\mathscr{F}) \times b(\mathscr{G})$.

Suppose there is $x \in X$ such that $[\bar{x}] \in \mathscr{F}$. Then $[x] \in b(\mathscr{F})$ and $([x], D) \in \mathscr{P}$. Thus $x \in \bar{D}$. On the other hand, $([\bar{x}] \cap C) \in \mathscr{F}$. Since $\emptyset \notin \mathscr{F}$, there is $y \in ([\bar{x}] \cap C)$. Because $(X, \mathscr{T})$ is R_0, $x \in [\bar{y}] \subset \bar{C}$. Thus $x \in \bar{C} \cap \bar{D}$, and $(C, D) \in \mathscr{R}_1 \subset \mathscr{P}^*$ which is a contradiction.

It follows from Theorem 4.9 that

$$[\mathscr{R}_1 \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F}))],$$

where $\mathscr{R}_1$ is the least member of $\mathfrak{M}_1$ and $\mathscr{F}$ and $\mathscr{G}$ are ultrafilters containing no point closures, is a base for the lattice $\mathfrak{M}_1$.

THEOREM 4.10. *Let $(X, \mathscr{T})$ be an R_0-topological space and let $\mathscr{P}$ cover $\mathscr{P}^*$ in $\mathfrak{M}_1$. Then there are nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X such that*

$$\mathscr{P} = \mathscr{P}^* \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F})).$$

Proof. Choose $(C, D) \in \mathscr{P} - \mathscr{P}^*$. By Theorem 4.9, there are nonprincipal ultrafilters $\mathscr{F}, \mathscr{G}$ on X such that $(C, D) \in (b(\mathscr{F}) \times b(\mathscr{G})) \subset \mathscr{P}$. So

$$\mathscr{P}^* \subsetneq \mathscr{P}^* \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F})) = \mathscr{P}'.$$

By Theorem 4.8 $\mathscr{P}' \in \mathfrak{M}_1$. Since $\mathscr{P}' \subset \mathscr{P}$ and $\mathscr{P}$ covers $\mathscr{P}^*$, we must have $\mathscr{P}' = \mathscr{P}$.

One might ask if the converse of Theorem 4.10 is true, i.e., if $(X, \mathscr{T})$ is an R_0-topological space and if $\mathscr{P} = \mathscr{P}^* \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F}))$ where $\mathscr{P}^* \in \mathfrak{M}_1$, then is it true that $\mathscr{P}$ covers $\mathscr{P}^*$ in $\mathfrak{M}_1$? The answer is no. To see this, consider Example 6.9 with $n = 3$ in Section 6.

Let the family of bunches with respect to the Lodato proximity space $(X, \mathscr{P})$ be ordered by set inclusion. Given nonempty $A \subset X$, then by Zorn's lemma there is a minimal bunch containing A.

THEOREM 4.11. *Let $(X, \mathscr{P})$ be a Lodato proximity space and let $\mathscr{B}$ be a minimal bunch containing the nonempty subset A. Then there is an ultrafilter $\mathscr{U}$ on X such that $\mathscr{B} = b(\mathscr{U})$.*

Proof. Since $\mathscr{B}$ is a bunch, $\mathscr{B}$ satisfies the hypotheses of Lemma 5.7 in [**8**]. By this lemma there is an ultrafilter $\mathscr{U}$ on X such that $A \in \mathscr{U} \subset \mathscr{B}$. From B3 it follows that $b(\mathscr{U}) \subset \mathscr{B}$. Since $\mathscr{B}$ is minimal and $A \in b(\mathscr{U})$, we have $b(\mathscr{U}) = \mathscr{B}$.

After proving Theorem 4.11, we asked: if the bunch $b(\mathscr{U})$ contains A, then is $b(\mathscr{U})$ always a minimal bunch containing A? The answer is no. To see this fact, consider Example 6.9 with $n = 3$ in Section 6.

THEOREM 4.12. *Let $(X, \mathscr{T})$ be an R_0-topological space, let $\mathscr{P} \in \mathfrak{M}_1$ and let $(C, D) \in \mathscr{W}_1 - \mathscr{P}$. Then there are ultrafilters $\mathscr{F}$, $\mathscr{G}$ on X such that*

$$\mathscr{P}' = \mathscr{P} \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F}))$$

is a minimal member of $\mathfrak{M}_1$ containing both (C, D) and $\mathscr{P}$.

Proof. As a result of Theorem 4.9 there are ultrafilters $\mathscr{U}$, $\mathscr{V}$ on X such that $(C, D) \in (b(\mathscr{U}) \times b(\mathscr{V})) \subset \mathscr{W}_1$. By the above work there are ultrafilters $\mathscr{F}$, $\mathscr{G}$ on X such that $b(\mathscr{F})$, $b(\mathscr{G})$ are minimal bunches containing C, D respectively. Clearly $(b(\mathscr{F}) \times b(\mathscr{G})) \subset \mathscr{W}_1$, and thus by Theorem 4.8, $\mathscr{P}'$ is in $\mathfrak{M}_1$. Since $b(\mathscr{F})$ and $b(\mathscr{G})$ are minimal, it follows that $\mathscr{P}'$ is a minimal member of $\mathfrak{M}_1$ containing both $\mathscr{P}$ and (C, D).

The converse of Theorem 4.12 is false, i.e., one can use Example 6.9 with $n = 3$ to show that there is an R_0-topological space, $\mathscr{P} \in \mathfrak{M}_1$ and (C, D) in $\mathscr{W}_1 - \mathscr{P}$ such that $\mathscr{P}' = \mathscr{P} \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F}))$ is not a minimal member of $\mathfrak{M}_1$ containing both $\mathscr{P}$ and (C, D).

5. The order structure of the family $\mathfrak{M}_1$ of all Lodato proximities compatible with a given R_0-topological space. In Theorem 4.1 we showed that $\mathfrak{M}_1$ is a complete lattice and we characterized least upper bound in $\mathfrak{M}_1$. In this section greatest lower bound is characterized and $\mathfrak{M}_1$ is shown to be a sublattice of the lattice $\mathfrak{M}$ of all compatible $\check{C}$-proximities. Although $\mathfrak{M}_1$ is distributive, it is not nicely behaved since it may lack atoms and antiatoms. This section concludes with several cardinality arguments about $\mathfrak{M}_1$.

THEOREM 5.1. *Let* $(X, \mathscr{T})$ *be an* R_0*-topological space and let*

$$[\mathscr{P}_\alpha : \alpha \in K] \subset \mathfrak{M}_1.$$

Then the greatest lower bound of $[\mathscr{P}_\alpha : \alpha \in K]$ *in* $\mathfrak{M}_1$ *is the greatest lower bound of* $[\mathscr{P}_\alpha : \alpha \in K]$ *in* $\mathfrak{M}$.

Proof. Let $\mathscr{P}$ be the greatest lower bound of $[\mathscr{P}_\alpha : \alpha \in K]$ in $\mathfrak{M}$. If we show $\mathscr{P} \in \mathfrak{M}_1$, then the result follows because $\mathfrak{M}_1 \subset \mathfrak{M}$. Since $\mathscr{P} \in \mathfrak{M}$, we shall show that $(\bar{A}, \bar{B}) \in \mathscr{P}$ implies $(A, B) \in \mathscr{P}$ and appeal to Theorem 2.1 to obtain $\mathscr{P} \in \mathfrak{M}_1$.

Let $(\bar{A}, \bar{B}) \in \mathscr{P}$ and let

$$A = \bigcup [A_i : i \in I], \qquad B = \bigcup [B_j : j \in J]$$

where I, J are finite sets. Hence $\bar{A} = \bigcup [\bar{A}_i : i \in I]$ and $\bar{B} = \bigcup [\bar{B}_j : j \in J]$. By [**11**, Theorem 4.1] there are i, j such that $(\bar{A}_i, \bar{B}_j) \in \bigcap [\mathscr{P}_\alpha : \alpha \in K]$. Since $[\mathscr{P}_\alpha : \alpha \in K] \subset \mathfrak{M}_1$, an application of Theorem 2.1 leads to

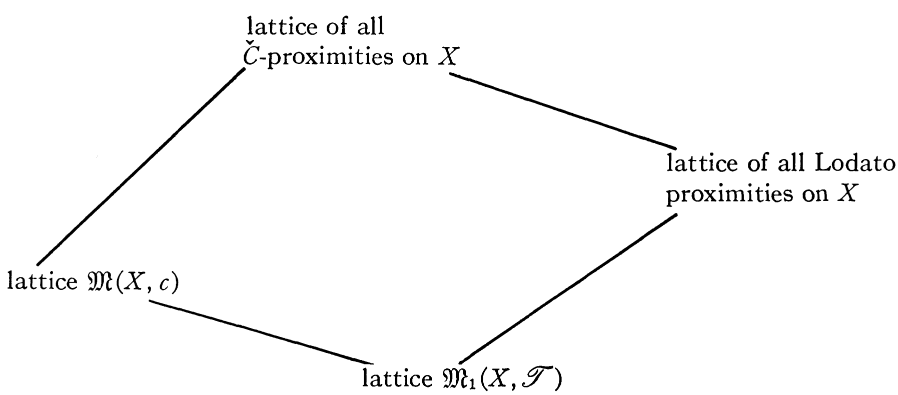
$$(A_i, B_j) \in \bigcap [\mathscr{P}_\alpha : \alpha \in K].$$

By [**11**, Theorem 4.1], $(A, B) \in \mathscr{P}$.

We note that greatest lower bound in $\mathfrak{M}$ is characterized in [**11**, Theorem 4.1]. This same characterization is applicable to greatest lower bound in $\mathfrak{M}_1$.

Using the techniques in the proof of Theorem 5.1, one easily verifies that the lattice of all Lodato proximities on a set X is a sublattice of the lattice of all $\check{C}$-proximities on X (no compatibility requirement in either lattice). Similarly, one easily shows that $\mathfrak{M}_1(X, \mathscr{T})$ is a sublattice of the lattice of all Lodato proximities on X. Also $\mathfrak{M}_1$ is a sublattice of $\mathfrak{M}$.

Let $(X, \mathscr{T})$ be an R_0-topological space and let c be the closure operator induced by $\mathscr{T}$. Then the following diagram shows the sublattice structure.



THEOREM 5.2. $\mathfrak{M}_1$ *is distributive and consequently, modular.*

Proof. By [**11**, Theorem 4.4], $\mathfrak{M}$ is distributive. Since $\mathfrak{M}_1$ is a sublattice of $\mathfrak{M}$ and since every sublattice of a distributive lattice is distributive, the first

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portion is proved. Because a distributive lattice is modular [**1**, pp. 12, 13], the theorem is proved.

THEOREM 5.3. *The lattice* $\mathfrak{M}_1$ *may have no atoms and no antiatoms, even when* $|\mathfrak{M}_1| > 1$. *Thus* $\mathfrak{M}_1$ *may lack the following properties: atomic, antiatomic, strongly atomic, covered and anticovered.*

Proof. This result follows from two examples in the next section. In Example 6.10, $\mathfrak{M}_1$ has no atoms and $|\mathfrak{M}_1| = \aleph_0$. Thus $\mathfrak{M}_1$ may be neither atomic, strongly atomic nor covered.

In Example 6.2 $\mathfrak{M}_1$ has no antiatoms and $|\mathfrak{M}_1| > 1$. Thus $\mathfrak{M}_1$ may be neither antiatomic nor anticovered.

THEOREM 5.4. *Let X be an infinite set and n a positive integer. Then there is a* T_1-*topology* $\mathscr{T}$ *on X such that* $\mathfrak{M}_1(X,\mathscr{T})$ *is linearly ordered and has exactly n elements. Furthermore, there are examples of topological spaces for which* $|\mathfrak{M}_1| = \aleph_0$, $2^{\aleph_0}$, *or* $2^{2^{\aleph_0}}$.

Proof. The first result follows from Examples 6.1 and 6.9 in Section 6. In Example 6.10, $|\mathfrak{M}_1| = \aleph_0$. Extending Example 6.10 by transfinite induction, one obtains a topological space that $|\mathfrak{M}_1| = 2^{\aleph_0}$. In Example 6.2, $|\mathfrak{M}_1| = 2^{2^{\aleph_0}}$ when the underlying set in the space has cardinality $\aleph_0$.

THEOREM 5.5. *Let* $\mathscr{T}$, $\mathscr{U}$ *be* R_0-*topologies on a set X. Let* $\mathscr{R}_1$, $\mathscr{R}_1^*$ *be the least members of* $\mathfrak{M}_1(X,\mathscr{T})$, $\mathfrak{M}_1(X,\mathscr{U})$ *respectively. Then* $\mathscr{R}_1 \subset \mathscr{R}_1^*$ *if and only if* $\bar{A} \subset \hat{A}$ *for every* $A \subset X$. *Here* $\bar{A}$, $\hat{A}$ *denote the closure of A with respect to* $\mathscr{T}$, $\mathscr{U}$ *respectively.*

Proof. Assume $\mathscr{R}_1 \subset \mathscr{R}_1^*$. Suppose there is $t \in (\bar{A} - \hat{A})$. Then $([t], A) \in \mathscr{R}_1$. By assumption, $([t], A) \in \mathscr{R}_1^*$. Thus there is $y \in ([\hat{t}] \cap \hat{A})$. Since $(X, \mathscr{U})$ is R_0, $t \in [\hat{y}] \subset \hat{A}$ which is a contradiction.

Conversely, assume $\bar{A} \subset \hat{A}$ for every $A \subset X$. If $(C, D) \in \mathscr{R}_1$, then $\bar{C} \cap \bar{D} \neq \emptyset$. Since $\bar{C} \subset \hat{C}$ and $\bar{D} \subset \hat{D}$, $\hat{C} \cap \hat{D} \neq \emptyset$. Thus $(C, D) \in \mathscr{R}_1^*$.

THEOREM 5.6. *Let* $\mathscr{T}$, $\mathscr{U}$ *be* R_0-*topologies on a set X. Let* $\mathscr{R}_1$, $\mathscr{R}_1^*$ *be the least members of* $\mathfrak{M}_1(X,\mathscr{T})$, $\mathfrak{M}_1(X,\mathscr{U})$ *respectively. If* $\mathscr{R}_1 \subset \mathscr{R}_1^*$, *then*

$$|\mathfrak{M}_1(X,\mathscr{U})| \leqq |\mathfrak{M}_1(X,\mathscr{T})|.$$

Proof. If $\mathscr{F}$ is an ultrafilter on X, we designate $b(\mathscr{F}) = [A \subset X : \bar{A} \in \mathscr{F}]$ and $b^*(\mathscr{F}) = [A \subset X : \hat{A} \in \mathscr{F}]$. Throughout this proof $\bar{A}$, $\hat{A}$ denote the closure of A with respect to $\mathscr{T}$, $\mathscr{U}$ respectively.

We begin by showing that if $\mathscr{F}$, $\mathscr{G}$ are nonprincipal ultrafilters on X which contain no point closures with respect to $\mathscr{U}$, then

$$\mathscr{R}_1 \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F}))$$

is in $\mathfrak{M}_1(X,\mathscr{T})$. Suppose $\mathscr{F}$ contains a point closure with respect to $\mathscr{T}$. Then there is $x \in X$ such that $[\bar{x}] \in \mathscr{F}$. Since $\mathscr{R}_1 \subset \mathscr{R}_1^*$, Theorem 5.5 implies

$[\bar{x}] \subset [\hat{x}]$. Because $\mathscr{F}$ is a filter, $[\hat{x}] \in \mathscr{F}$ which is a contradiction. Similarly, $\mathscr{G}$ contains no point closures with respect to $\mathscr{T}$. Hence

$$\mathscr{R}_1 \cup (b(\mathscr{F}) \times b(\mathscr{G})) \cup (b(\mathscr{G}) \times b(\mathscr{F}))$$

is in $\mathfrak{M}_1(X, \mathscr{T})$.

Let $\mathscr{P} \in \mathfrak{M}_1(X, \mathscr{U})$. If $(C, D) \in \mathscr{P} - \mathscr{R}_1^*$, then by Theorem 4.9 there are nonprincipal ultrafilters $\mathscr{F}_C$, $\mathscr{G}_D$ on X such that

$$(C, D) \in (\mathscr{F}_C \times \mathscr{G}_D) \subset (b^*(\mathscr{F}_C) \times b^*(\mathscr{G}_D)) \subset \mathscr{P}$$

and $\mathscr{F}_C \cup \mathscr{G}_D$ contains no point closures with respect to $\mathscr{U}$.

We define $g : \mathfrak{M}_1(X, \mathscr{U}) \to \mathfrak{M}_1(X, \mathscr{T})$ by

$$g(\mathscr{P}) = \bigcup [\mathscr{R}_1 \cup (b(\mathscr{F}_C) \times b(\mathscr{G}_D)) \cup (b(\mathscr{G}_D) \times b(\mathscr{F}_C)) :$$
$(C, D) \in \mathscr{P} - \mathscr{R}_1^*$ and $\mathscr{F}_C$, $\mathscr{G}_D$ are nonprincipal ultrafilters on X such that $(C, D) \in (\mathscr{F}_C \times \mathscr{G}_D) \subset (b^*(\mathscr{F}_C) \times b^*(\mathscr{G}_D)) \subset \mathscr{P}$ and $\mathscr{F}_C \cup \mathscr{G}_D$ contains no point closures with respect to $\mathscr{U}$].

By the above work, $g(\mathscr{P}) \in \mathfrak{M}_1(X, \mathscr{T})$.

To verify that g is 1 : 1, let $\mathscr{P}, \mathscr{P}' \in \mathfrak{M}_1(X, \mathscr{U})$ and $\mathscr{P} \neq \mathscr{P}'$. Hence there is (E, F) in $\mathscr{P} - \mathscr{P}'$ or $\mathscr{P}' - \mathscr{P}$, say in $\mathscr{P} - \mathscr{P}'$. Thus

$$(E, F) \in (\mathscr{F}_E \times \mathscr{G}_F) \subset (b(\mathscr{F}_E) \times b(\mathscr{G}_F)) \subset g(\mathscr{P}).$$

On the other hand, $(E, F) \notin \mathscr{P}'$ implies $(E, F) \notin \mathscr{R}_1^*$. Since $\mathscr{R}_1 \subset \mathscr{R}_1^*$, $(E, F) \notin \mathscr{R}_1$. If $(C, D) \in \mathscr{P}' - \mathscr{R}_1^*$, then $b^*(\mathscr{F}_C) \times b^*(\mathscr{G}_D) \subset \mathscr{P}'$ implies $(E, F) \notin (b^*(\mathscr{F}_C) \times b^*(\mathscr{G}_D))$. We will show that for each ultrafilter $\mathscr{F}$ on X, $b(\mathscr{F}) \subset b^*(\mathscr{F})$. It follows that $(E, F) \notin (b(\mathscr{F}_C) \times b(\mathscr{G}_D))$. We conclude that $(E, F) \notin g(\mathscr{P}')$. So $g(\mathscr{P}) \neq g(\mathscr{P}')$.

Lastly, we show that if $\mathscr{F}$ is an ultrafilter on X, then $b(\mathscr{F}) \subset b^*(\mathscr{F})$. Let $S \in b(\mathscr{F})$. Then $\bar{S} \in \mathscr{F}$. Since $\mathscr{R}_1 \subset \mathscr{R}_1^*$, Theorem 5.5 implies $\bar{S} \subset \hat{S}$. Since $\mathscr{F}$ is a filter, $\hat{S} \in \mathscr{F}$. Thus $S \in b^*(\mathscr{F})$.

6. Examples. Each of the following examples is an R_0-topological space and illustrates an embedding of $\mathfrak{M}_1$ in $\mathfrak{M}$. When the space is also completely regular, we will comment about the embedding of $\mathfrak{M}_2$ in $\mathfrak{M}_1$ and $\mathfrak{M}$. Here $\mathfrak{M}_2$ denotes the family of all compatible Efremovič proximities partially ordered by set inclusion.

We begin by collecting some known results. In an R_0-topological space, $\mathfrak{M}$ has least element $\mathscr{R}$ and greatest element $\mathscr{W}$, and $\mathfrak{M}_1$ has least element $\mathscr{R}_1$ and greatest element $\mathscr{W}_1$. Furthermore $\mathscr{R} \subset \mathscr{R}_1 \subset \mathscr{W}_1 \subset \mathscr{W}$. In a completely regular topological space $\mathfrak{M}_2$ has a least element which we denote by $\mathscr{R}_2$. Moreover $\mathscr{R}_1 \subset \mathscr{R}_2 \subset \mathscr{W}_1$. In a completely regular space, $\mathscr{R}_1 = \mathscr{R}_2$ if and only if the space is normal. In a completely regular, locally compact space $\mathfrak{M}_2$ has a greatest element which we designate as $\mathscr{W}_2$. Also $\mathscr{R}_2 \subset \mathscr{W}_2 \subset \mathscr{W}_1$.

Example 6.1. A completely regular, normal, locally compact space such that $\mathscr{R} = \mathscr{R}_1 = \mathscr{R}_2 = \mathscr{W}_2 = \mathscr{W}_1 \neq \mathscr{W}$: Let X be an infinite set and let Y, Z

be a partition of X into two infinite sets. Let $\mathscr{T} = [\emptyset, X, Y, Z]$. By Theorem 4.3, there are no T_1-spaces satisfying the conditions of this example.

Example 6.2. A completely regular, normal, locally compact topology on an infinite set X such that $\mathscr{R} = \mathscr{R}_1 = \mathscr{R}_2 \neq \mathscr{W}_2 = \mathscr{W}_1 = \mathscr{W}$, $\mathfrak{M} = \mathfrak{M}_1 \neq \mathfrak{M}_2$, [atoms of $\mathfrak{M}$] = [atoms of $\mathfrak{M}_2$], $\mathfrak{M}_1$ has no antiatoms, $|\mathfrak{M}| = 2^{2^{|X|}} = |\mathfrak{M}_2|$ and $|\mathfrak{M} - \mathfrak{M}_2| \geqq \aleph_0$: Let $\mathscr{T}$ be the discrete topology on X.

Example 6.3. A T_1-space which is not T_2 and not completely regular such that $\mathscr{R} \neq \mathscr{R}_1 = \mathscr{W}_1 = \mathscr{W}$: Let X = [real numbers $x : x = 2$ or $0 \leqq x \leqq 1$]. If $A \subset X$ then we define

$$c(A) = \begin{cases} A \text{ if } A \text{ is finite} \\ \bar{A} \cup [2] \text{ if } A \text{ is infinite.} \end{cases}$$

Here $\bar{A}$ represents the relative closure in X with respect to the usual topology on the set of real numbers. It is easily verified that c is a Kuratowski closure operator on $\mathscr{P}(X)$.

Example 6.4. A completely regular, normal, locally compact space such that $\mathscr{R} \neq \mathscr{R}_1 = \mathscr{R}_2 \neq \mathscr{W}_2 \neq \mathscr{W}_1 = \mathscr{W}$, $|\mathfrak{M} - \mathfrak{M}_1| \geqq \aleph_0$, $|\mathfrak{M}_1 - \mathfrak{M}_2| \geqq \aleph_0$ and $|\mathfrak{M}_2| \geqq \aleph_0$: Let X be the set of real numbers and $\mathscr{T}$ the usual topology on X.

Example 6.5. A completely regular, normal, locally compact space such that $\mathscr{R} = \mathscr{R}_1 = \mathscr{R}_2 \neq \mathscr{W}_2 = \mathscr{W}_1 \neq \mathscr{W}$, $|\mathfrak{M} - \mathfrak{M}_1| \geqq \aleph_0$, $|\mathfrak{M}_1 - \mathfrak{M}_2| \geqq \aleph_0$, and $|\mathfrak{M}_2| \geqq \aleph_0$: Let the space be the topological sum of the spaces in Examples 6.1 and 6.2. By Theorem 4.3, there are no T_1-spaces satisfying the conditions of this example.

Example 6.6. A completely regular, normal, locally compact space such that $\mathscr{R} \neq \mathscr{R}_1 = \mathscr{R}_2 \neq \mathscr{W}_2 \neq \mathscr{W}_1 \neq \mathscr{W}$, $|\mathfrak{M} - \mathfrak{M}_1| \geqq \aleph_0$, $|\mathfrak{M}_1 - \mathfrak{M}_2| \geqq \aleph_0$, and $|\mathfrak{M}_2| \geqq \aleph_0$: The space is the topological sum of the spaces in Examples 6.1 and 6.4. By Theorem 4.3, there are no T_1-spaces satisfying the conditions of this example.

Example 6.7. A space which is not completely regular such that $\mathscr{R} \neq \mathscr{R}_1 = \mathscr{W}_1 \neq \mathscr{W}$: The space is the topological sum of the spaces in Examples 6.1 and 6.3. By Theorem 4.3, there are no T_1-spaces satisfying the conditions of this example.

Example 6.8. A completely regular, normal, locally compact space such that $\mathscr{R} = \mathscr{R}_1 = \mathscr{R}_2 = \mathscr{W}_2 = \mathscr{W}_1 = \mathscr{W}$: Let $\mathscr{T}$ be the indiscrete topology on a nonempty set, which is a T_1-space when the set is a singleton.

In the preceding examples we have shown that the eight possible equalities and inequalities between $\mathscr{R}$, $\mathscr{R}_1$, $\mathscr{W}_1$ and $\mathscr{W}$ can occur. In the following examples we shall show variations in the embedding of $\mathfrak{M}_1$ in $\mathfrak{M}$.

Example 6.9. A T_1-space which is not T_2 and not completely regular such that $\mathfrak{M}_1$ is a chain with n elements (n an integer > 1) and $\mathscr{R} = \mathscr{R}_1 \neq \mathscr{W}_1 = \mathscr{W}$: Let X be an infinite set.

We partition X into two infinite sets X_1, X_2.
We partition X_2 into two infinite sets X_3, Y_1.
We partition X_3 into two infinite sets X_4, Y_2.

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We partition X_{n-1} into two infinite sets X_n, Y_{n-2}.
We note that when $n = 2$, there is exactly one partition, that of X into X_1, X_2. Let $\mathscr{C}_n$ be the following family of subsets of X: any finite set, any X_i $(i = 1, \ldots, n)$ and any finite union of these sets. Then $(X, \mathscr{C}_n)$ is the desired topological space where $\mathscr{C}_n$ is the family of closed sets in the space.

Example 6.10. A T_1-space which is not T_2 and not completely regular such that $\mathfrak{M}_1$ is a chain with no atoms, $|\mathfrak{M}_1| = \aleph_0$ and $\mathscr{R} = \mathscr{R}_1 \neq \mathscr{W}_1 = \mathscr{W}$: Let X be an infinite set and partition X into four infinite sets X_0, X_1, Y_1, Y_2.
Let $X_2 = Y_2 \cup X_1$ and partition Y_1 into two infinite sets Y_3, Y_4.
Let $X_3 = Y_4 \cup X_2$ and partition Y_3 into two infinite sets Y_5, Y_6.

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Let $X_n = Y_{2n-2} \cup X_{n-1}$ and partition Y_{2n-3} into two infinite sets Y_{2n-1}, Y_{2n}.

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Let $\mathscr{C}$ be the following family of subsets of X: any finite set, any X_i, X and any finite union of these sets. Then $(X, \mathscr{C})$ is the desired topological space where $\mathscr{C}$ is the family of closed sets in the space.

It is easy to extend Example 6.9 (in a manner similar to Example 6.10) so that $\mathfrak{M}_1$ will be a chain with no antiatoms.

Example 6.11. An R_0-space which is not T_1 and not completely regular such that $\mathfrak{M}_1 = [\mathscr{R}_1, \mathscr{W}_1]$ and $\mathscr{R} = \mathscr{R}_1 \neq \mathscr{W}_1 \neq \mathscr{W}$: The space is the topological sum of the spaces in Examples 6.1 and 6.9 for $n = 2$.

Example 6.12. A T_1-space which is not T_2 and not completely regular such that $\mathfrak{M}_1 = [\mathscr{R}_1, \mathscr{W}_1]$ and $\mathscr{R} \neq \mathscr{R}_1 \neq \mathscr{W}_1 = \mathscr{W}$: Let X be an infinite set. Choose s, t in X such that $s \neq t$. Partition $X - [s, t]$ into three infinite sets R, S, T. Let $X_1 = S \cup [s]$, $X_2 = T \cup [t]$ and $X_3 = R \cup [s, t]$. Let $\mathscr{C}$ be the following family of subsets of X: any finite set, $X_i \cup$ any finite set $(i = 1, 2, 3)$, $X_i \cup X_j \cup$ any finite set $(i, j = 1, 2, 3)$ and X. Then $(X, \mathscr{C})$ is the desired space where $\mathscr{C}$ is the family of closed sets in the space.

Example 6.13. An R_0-space which is not T_1 and not completely regular such that $\mathfrak{M}_1 = [\mathscr{R}_1, \mathscr{W}_1]$ and $\mathscr{R} \neq \mathscr{R}_1 \neq \mathscr{W}_1 \neq \mathscr{W}$: The space is the topological sum of the spaces in Examples 6.8 (when the underlying set is infinite) and 6.12.

We note that Examples 6.9, 6.11, 6.12 and 6.13 show that if $\mathfrak{M}_1 = [\mathscr{R}_1, \mathscr{W}_1]$ where $\mathscr{R}_1 \neq \mathscr{W}_1$, then the four possible equalities and inequalities between $\mathscr{R}$ and $\mathscr{R}_1$ and between $\mathscr{W}_1$ and $\mathscr{W}$ can occur.

7. Summary. The following table is a summary of some of the properties of

$$\mathfrak{X} = \mathfrak{M}, \mathfrak{M}_1 \text{ and } \mathfrak{M}_2.$$

	Is $\mathfrak{X}$ nonempty?	Is $\mathfrak{X}$ determined by closed sets?	Is $\mathfrak{X}$ a lattice?	Is join set theoretic union?
$\mathfrak{M}$	Yes, if and only if the space is R_0-closure	Yes, if and only if $\mathfrak{M} = \mathfrak{M}_1$	Yes	Yes
$\mathfrak{M}_1$	Yes, if and only if the space is R_0-topological	Yes	Yes	Yes
$\mathfrak{M}_2$	Yes, if and only if the space is completely regular topological	Yes	Yes, if the space is locally compact; it is an open question to give necessary and sufficient conditions on the space for $\mathfrak{M}_2$ to be a lattice	Join may not exist; when it does exist, then examples are known where join is not union

	Is $\mathfrak{X}$ strongly atomic?	Is $\mathfrak{X}$ atomic?	Is $\mathfrak{X}$ antiatomic?	Is $\mathfrak{X}$ covered?	Is $\mathfrak{X}$ anticovered?
$\mathfrak{M}$	Yes	Yes	No, if and only if $\lvert\mathfrak{M}\rvert > 1$	Yes	No, if and only if $\lvert\mathfrak{M}\rvert > 1$
$\mathfrak{M}_1$	In some cases yes and in some cases no	In some cases yes and in some cases no	In some cases yes and in some cases no	In some cases yes and in some cases no	In some cases yes and in some cases no
$\mathfrak{M}_2$	Yes	Yes	Open question	Yes	Open question

For the rest of this table, if $\mathfrak{X} = \mathfrak{M}_2$, then we require that $\mathfrak{M}_2$ be a lattice.

	Is $\mathfrak{X}$ complete?	Is $\mathfrak{X}$ distributive?	Is $\mathfrak{X}$ modular?
$\mathfrak{M}$	Yes	Yes	Yes
$\mathfrak{M}_1$	Yes	Yes	Yes
$\mathfrak{M}_2$ when it is a lattice	Yes, if and only if the space is locally compact	Yes, if and only if $\lvert\mathfrak{M}_2\rvert \leqslant 2$, see [4]	Yes, if and only if $\lvert\mathfrak{M}_2\rvert \leqslant 5$, see [4]

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University of Colorado,
Boulder, Colorado;
Aerospace Research Laboratories,
Wright-Patterson AFB, Ohio