ON BOUNDED SEQUENCES SATISFYING A LINEAR INEQUALITY

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(Received 21st September 1972)

In a recent paper, E. T. Copson (**2**) proves the following result:

Theorem C. *Let* $k_i > 0$ $(i = 1, \ldots, m)$, $k_1 + \ldots + k_m = 1$, *and the real sequence* (a_n) *satisfy the inequality*

$$a_{n+m} \leqq \sum_{i=1}^{m} k_i a_{n+m-i} \quad (n = 0, 1, 2, \ldots). \tag{1}$$

If (a_n) *is bounded, then it must be convergent.*

The direction of the inequality in (1) is clearly immaterial, since we may replace a_n by $-a_n$ and reach the same conclusion. For $m = 1$ this reduces to the classical result that a bounded monotone sequence is convergent. Of the following examples (the first two of which are quoted by Copson) in which the hypothesis $k_i > 0$ does not hold, namely

$$a_{n+4} \leqq \tfrac{1}{2}(a_{n+2} + a_n), \tag{2}$$

$$a_{n+3} \leqq -\tfrac{1}{2}a_{n+2} + \tfrac{3}{4}a_{n+1} + \tfrac{3}{4}a_n, \tag{3}$$

$$a_{n+3} \leqq a_{n+2} + a_{n+1} - a_n, \tag{4}$$

$$a_{n+3} \leqq 4a_{n+2} - 5a_{n+1} + 2a_n, \tag{5}$$

the conclusion of Theorem C is false for (2) and (4), but true for (3) and (5), so that the condition $k_i > 0$ is sufficient but not necessary. It is the purpose of this note to supply a necessary and sufficient condition for the conclusion of Theorem C.

After this paper had been accepted for publication, I discovered in conversation with Professor D. Borwein that he had already considered this problem in a forthcoming paper (**1**). His theorem is somewhat more general, but its specialisation still does not cover examples (4) and (5) above. Also my method is of independent interest in that it is constructive and specifically indicates the form which (a_n) must take (that is, it solves the inequality (1)).

Consider the sequence-to-sequence transformation

$$t_n = p_0 a_n + p_1 a_{n-1} + \ldots + p_l a_{n-l} \equiv \sum_{k=0}^{\infty} p_{nk} a_k \equiv P(a_n) \quad (n = 0, 1, \ldots), \tag{6}$$

where l is a fixed non-negative integer, (a_n) is a complex sequence (we take

$a_i = 0$ for $i<0$) and p_i are arbitrarily given complex numbers (not all zero)—and we can suppose without loss of generality that $p_0 \neq 0$. The associated polynomial is then

$$p(z) = p_0 + p_1 z + \dots + p_l z^l, \quad p(0) \neq 0. \tag{7}$$

For the specialisation to Theorem C we take (a_n), (p_n) real, $l = m-1$,

$$p_0 = 1, p_i = 1-k_1-\dots-k_i \ (i = 1, \dots, m-1), p_i = 0 \ (i \geqq m),$$

and then (1) is equivalent to

$$t_n \leqq t_{n-1} \quad (n \geqq m); \tag{8}$$

if also (a_n) is bounded, then (by (6)) so is (t_n), and hence (by (8)) (t_n) must be convergent. The problem is therefore to determine under what conditions the transformation (6) (which is, apart from a multiplicative constant, a Nörlund summability method associated with the polynomial p) evaluates no bounded divergent sequence.

Returning to the general case, we remark that the matrix $P = (p_{nk})$ defined by (6) is *conservative*, i.e. (t_n) converges whenever (a_n) converges (P evaluates every convergent sequence) since it satisfies the well-known necessary and sufficient conditions (e.g. Peyerimhoff (3, Theorem II.1))

$$\sup_n \sum_{k=0}^{\infty} |p_{nk}| < \infty, \quad \exists \lim_n p_{nk} \ (k = 0, 1, \dots), \quad \exists \lim_n \sum_{k=0}^{\infty} p_{nk}. \tag{9}$$

The first result to be proved is:

Theorem 1. *Let (t_n) be the transform of (a_n), with associated polynomial p, as defined by* (6) *and* (7).

(a) *If p has no zero on the unit circle, (a_n) is bounded, and (t_n) is convergent, then (a_n) is convergent.*

(b) *If p has a zero $1/\lambda$ on the unit circle, then there exists a bounded divergent (a_n) for which (t_n) is convergent. If $\lambda \neq 1$, the choice may be made so that $t_n = 0$ $(n \geqq l)$.*

Proof. Writing the formal power series

$$t(z) = \sum_{n=0}^{\infty} t_n z^n, \quad a(z) = \sum_{n=0}^{\infty} a_n z^n,$$

(6) is a Cauchy product, obtained from $t(z) = p(z)a(z)$. If we also denote

$$\bar{p}(z) = 1/p(z) = \sum_{n=0}^{\infty} \bar{p}_n z^n \quad (\text{since } p(0) \neq 0)$$

we then have $a(z) = \bar{p}(z)t(z)$, so that

$$a_n = \sum_{k=0}^{n} \bar{p}_{n-k} t_k \equiv \sum_{k=0}^{\infty} p_{nk}^{-1} t_k \equiv P^{-1}(t_n) \quad (n = 0, 1, \dots), \tag{10}$$

where $P^{-1} = (p_{nk}^{-1})$ is the (unique two-sided) inverse matrix of P.

https://doi.org/10.1017/S0013091500015297 Published online by Cambridge University Press

Suppose the factorisation of $p(z)$ is

$$p(z) = p_0(1-\lambda_1 z)^{m_1}\dots(1-\lambda_r z)^{m_r}.$$

Then $\bar{p}(z)$ is expressible in partial fractions, the fractions corresponding to a factor $(1-\lambda z)^m$ being of the form

$$\frac{\kappa_1}{1-\lambda z}+\frac{\kappa_2}{(1-\lambda z)^2}+\dots+\frac{\kappa_m}{(1-\lambda z)^m}.$$

Take a typical partial fraction

$$\frac{\kappa_\rho}{(1-\lambda z)^\rho} = \kappa_\rho \sum_{k=0}^{\infty}\binom{\rho+k-1}{k}\lambda^k z^k;$$

its contribution to a_n (given by (10)) is $\kappa_\rho a'_n$, where

$$a'_n = \sum_{k=0}^{n}\binom{\rho+n-k-1}{n-k}\lambda^{n-k}t_k. \tag{11}$$

(10) and (11) hold without any restrictions on (a_n) and (t_n) other than the basic relation (6) between them.

Suppose now that (t_n) converges. If $|\lambda|<1$ then (a'_n) converges, and so the complete contribution to (a_n), corresponding to such a λ, is a convergent sequence.

On the other hand, if $|\lambda|>1$, it is easily verified that

$$c'_n \equiv \sum_{k=n+1}^{\infty}\binom{\rho+n-k-1}{n-k}\lambda^{n-k}t_k \equiv \sum_{k=0}^{\infty} h_{nk}t_k \tag{12}$$

is a conservative (indeed, regular) sequence-to-sequence transformation, since its matrix (h_{nk}) satisfies (9). Hence if (t_n) converges, then (c'_n) converges, and moreover

$$\sum_{k=0}^{\infty}\binom{\rho+n-k-1}{n-k}\lambda^{n-k}t_k$$

$$= \frac{\lambda^n}{(\rho-1)!}\sum_{k=0}^{\infty}(n-k+1)\dots(n-k+\rho-1)\lambda^{-k}t_k = \lambda^n q'(n), \tag{13}$$

where $q'(n)$ is a polynomial of degree $\rho-1$. Combining (11), (12), (13), we get

$$a'_n = \lambda^n q'(n) - c'_n$$

(compare this derivation with the one given in Peyerimhoff (3, pp. 17-18) for the case $p(z) = 1-2z$), and we can apply the same argument to all the partial fractions corresponding to a given λ, adding the results together to obtain the overall contribution to a_n.

(*a*) If $|\lambda_j| \neq 1$ $(j = 1, \dots, r)$, then all possible cases are covered by the arguments detailed above. More precisely, if

$$|\lambda_j|>1 \text{ for } j = 1, \dots, s,\ |\lambda_j|<1 \text{ for } j = s+1, \dots, r,$$

it follows that, when (t_n) converges, a_n is of the form

$$a_n = q_1(n)\lambda_1^n + \ldots + q_s(n)\lambda_s^n + c_n,$$

where $q_j(n)$ is a polynomial of maximum degree $m_j - 1$ $(j = 1, \ldots, s)$ and (c_n) converges. Consequently, provided there are no zeros of $p(z)$ on the unit circle, (a_n) is either convergent or unbounded when (t_n) converges. In other words, if (a_n) is bounded, then it must be convergent.

(*b*) Suppose that p has a (simple or multiple) zero $1/\lambda$ on the unit circle. Writing

$$t(z) = p(z)a(z) = q(z)(1-\lambda z)a(z),$$

$q(z)$ is then a polynomial and hence defines a conservative matrix Q (in the same way that $p(z)$ defines P), and (t_n) is the Q-transform of the sequence $(a_n - \lambda a_{n-1})$. If $\lambda \neq 1$ then $a_n = \lambda^n$ is a bounded divergent sequence, and

$$a_n - \lambda a_{n-1} = 0 \ (n \geqq 1);$$

hence $t_n = 0$ $(n \geqq l)$. If $\lambda = 1$ a suitable choice is

$$(a_n) = (1, 0, 1, \tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \tfrac{2}{3}, \tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{3}{4}, \tfrac{2}{4}, \tfrac{1}{4}, 0, \tfrac{1}{4}, \ldots);$$

then (a_n) is a bounded divergent sequence, and $a_n - a_{n-1} \to 0$; since Q is conservative, it follows that (t_n) converges. This completes the proof.

Remark 1. If p has *real coefficients* and a zero $1/\lambda$ on the unit circle, then P always evaluates a *real* bounded divergent sequence (a_n). For if $\lambda = 1$ the argument just given applies, while if $\lambda \neq 1$ the choice $a_n = \operatorname{Re} \lambda^n$ makes $t_n = 0$ $(n \geqq l)$.

Remark 2. Note that the application of Theorem 1 to the original problem depends on (8), namely the ultimate monotonicity of (t_n). In Theorem 1(*b*) we were able to ensure that $t_n = 0$ $(n \geqq l)$ if p has a zero on the unit circle different from 1. However, if 1 is a zero of p, the construction of a bounded divergent (a_n), yielding a monotone (t_n), may not be possible; for instance, the monotonicity of $(a_n - a_{n-1})$ implies the ultimate monotonicity of (a_n), so that in the simple example

$$a_{n+2} \leqq 2a_{n+1} - a_n, \quad p(z) = 1 - z,$$

the boundedness of (a_n) implies its convergence, despite the zero of p on the unit circle. Thus as a supplement to Theorem 1 we now prove:

Theorem 2. *Let* $p(z) = (1-z)^m q(z)$, *where* m *is a positive integer and the polynomial* q *has no zeros on the unit circle. If* (a_n) *is bounded and* (t_n) *is real and ultimately monotone (and hence convergent), then* (a_n) *is convergent.*

Proof. If $p(z) = (1-z)^m q(z)$, then $t_n = P(a_n) = \Delta^m Q(a_n)$, where Q is the conservative matrix defined by q, and $\Delta u_n = u_n - u_{n-1}$, $\Delta^2 u_n = \Delta(\Delta u_n)$, etc. If q has no zeros on the unit circle then, by Theorem 1(*a*), Q evaluates no bounded divergent sequence. Now we assume (by hypothesis) that (a_n) is bounded and

(t_n) ultimately monotone. Then using (m times) the fact that (Δu_n) ultimately monotone implies (u_n) ultimately monotone, we see that $(Q(a_n))$ is ultimately monotone and hence (being bounded) converges. But since Q cannot evaluate a bounded divergent sequence, (a_n) must also converge.

Recalling the equivalence of (1) and (8), the following theorem now gives the complete solution to the original problem. The sufficiency part follows from Theorems 1(a) and 2, and the necessity part from Theorem 1(b) and Remark 1.

Theorem 3. *Let* k_i $(i = 1, \ldots, m)$ *be real,* $k_1 + \ldots + k_m = 1$,

$$p_i = 1 - k_1 - \ldots - k_i \ (i = 1, \ldots, m-1), \ p(z) = 1 + p_1 z + \ldots + p_{m-1} z^{m-1},$$

and let the real sequence (a_n) *satisfy the inequality*

$$a_{n+m} \leqq \sum_{i=1}^{m} k_i a_{n+m-i} \quad (n = 0, 1, \ldots).$$

Then in order that the boundedness of (a_n) *shall always imply its convergence, it is necessary and sufficient that* $p(z)$ *shall have no zeros in the set*

$$C = \{z: |z| = 1, z \neq 1\}.$$

As Copson (**2**, p. 163) points out, the hypothesis $k_i > 0$ $(i = 1, \ldots, m)$ in Theorem C suffices to imply that the only zeros of p are outside the unit circle (this makes the transformation (6) equivalent to convergence), and Theorem C therefore follows from Theorem 3. The application of Theorem 3 to the examples (2), (3), (4), (5) is immediate.

Remark 3. A natural generalisation of the problem would be to replace (1) by

$$a_{n+1} \leqq \sum_{i=0}^{n} k_{ni} a_i \quad (n > n_0);$$

this is equivalent to $t_{n+1} \leqq t_n$ $(n > n_0)$, where

$$h_{ni} = 1 - \sum_{v=i}^{n} k_{vi}; \ t_0 = a_0, \ t_{n+1} = a_{n+1} + \sum_{i=0}^{n} h_{ni} a_i \quad (n \geqq 0). \tag{14}$$

If the transformation (14) is mercerian (i.e. equivalent to convergence) then boundedness of (a_n) would imply its convergence. For example (see Rado (**4**, modification of Theorem 3)), it would suffice that (h_{ni}) should be conservative and

$$\limsup_{n \to \infty} \sum_{i=0}^{n} |h_{ni}| < 1.$$

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https://doi.org/10.1017/S0013091500015297 Published online by Cambridge University Press