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LOCALLY IRREDUCIBLE RINGS

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In the study of torsion-free abelian groups of finite rank the notions of irreducibility, field of definition and E-ring have played significant rôles. These notions are tied together in the following theorem of R. S. Pierce:

THEOREM. *Let* R *be a ring whose additive group is torsion free finite rank irreducible and let* Γ *be the centralizer of* QR *as a* $QE(R)$ *module. Then* Γ *is the unique smallest field of definition of* R *. Moreover,* $\Gamma \cap R$ *is an* E*-ring, in fact, it is a maximal* E*-subring of* R *.*

In this paper we consider extensions of Pierce's result to the infinite rank case. This leads to the concept of local irreducibility for torsion free groups.

1. Introduction

A group G (in this paper the word group will always mean torsion-free abelian group) is called *irreducible* if QG $(Q \otimes_Z G)$ is a simple QE $(Q \otimes_Z E)$-module, where E is the ring of endomorphisms of G. These groups have been studied extensively by J. D. Reid [10], [11], [12] and play an important role in the theory of torsion-free groups of finite rank.

Let R be a ring (all rings in this paper have an identity and have a torsion-free additive group). A subfield F of the centre of QR is called a *field of definition* of R if $(F \cap R)x_1 \oplus \ldots \oplus (F \cap R)x_n$ is of

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finite index in R for some F-independent subset $\{x_1,\ldots,x_n\} \subset R$. The concept of field of definition first appeared in [3] and [7] in the study of subrings of simple algebras, and subsequently has appeared frequently in various contexts, (for instance see [6] or [9]).

A ring R is called an *E-ring* if the embedding $x \to x_\ell$ of R into End(R+) is onto. Here x_ℓ means left multiplication by x. Schultz introduced the term E-ring in [14]. A further study of E-rings was made by Bowshell and Schultz in [4]. In spite of their seemingly specialized nature, E-rings have appeared frequently in the literature (see [1], [2], [12], [5], [7]).

In the finite rank case the concepts of irreducibility, field of definition and E-ring are tied together in the following theorem, which first appeared in [7].

THEOREM. *Let R be a (torsion-free reduced) ring of finite rank which is irreducible as an additive group. Let $\Gamma = \mathrm{Hom}_{QE}(QR, QR)$. Then:*

(1) *Γ is a subfield of the centre of QR and Γ is the unique smallest field of definition of R.*

(2) *$\Gamma \cap R$ is an E-ring. In fact, $\Gamma \cap R$ is a maximal E-subring of R.*

It is easy to verify that if R is irreducible, then so is R_p, the localization of R at an integral prime p. In this paper we study torsion free rings R for which each R_p is irreducible. We are able to generalize the above theorem, even in certain infinite rank cases. Our work is based on [3], [7] and [9], which are fundamental references for this paper.

Our notation is fairly standard. Specifically: Z_p, $\hat{Z}_p$, $\hat{Q}_p$ stand for the ring of integers localized at p, the ring of p-adic integers and the field of p-adic numbers, respectively. The symbols $\doteq$ and $\triangleq$ denote quasi-equality and quasi-isomorphism, while the symbols $\oplus$ and $\times$ represent group direct sum and ring direct sum, respectively.

A ring R is called p-local provided $qR = R$ for all primes $q \neq p$. If R is a p-local ring, then $\hat{R}$ denotes $\hat{Z}_p \otimes_Z R$ with the natural ring structure, and $Q\hat{R}$ represents $Q \otimes_Z \hat{R} \simeq \hat{Q}_p \otimes_Z R$. Following [9] , let $L(R)$ be the maximal divisible subgroup of $\hat{R}$. Note that if we regard $Q\hat{R}$ as a QE-module in the natural way, then $L(R)$ is a QE-submodule of $Q\hat{R}$.

1. The local case

Throughout this section R will be a torsion-free p-local reduced ring which is irreducible as an abelian group. In particular, QR is a simple QE-module and $\Gamma = \mathrm{Hom}_{QE}(QR, QR)$ is a division ring. More specifically, Γ can be identified with a subfield of the centre of QR , since the elements of Γ commute with all left and right multiplications by elements of QR . Furthermore, by the Jacobson Density Theorem, QE is a dense subring of $\mathrm{Hom}_\Gamma(QR, QR)$. An important class of irreducible rings is the class of rings R for which QR is a simple Q-algebra. These rings are irreducible since QE contains left and right multiplications by elements of QR .

We start with a technical lemma, which is a modification of Theorem 3.1 of [9].

LEMMA 1.1. *$L(R) = Q\hat{R}(\hat{\Gamma} \cap L(R))$.*

Proof. Let $N = Q\hat{R}(\hat{\Gamma} \cap L(R)) \subset L(R)$. Note that N is a QE-submodule of $L(R)$. Suppose there exists $w \in L(R)\setminus N$. Since $w \in Q\hat{R}$, write $w = \alpha_1 x_1 + \ldots + \alpha_r x_r$, with $\alpha_i \in \hat{Q}_p$ and $x_i \in QR$. We may assume w has been chosen so that r is minimal. Clearly, $\alpha_i \neq 0$, $x_i \neq 0$ for each i . Moreover, since both $L(R)$ and N are $\hat{Q}_p$-modules, we may take $\alpha_1 = 1$.

Since QR is simple over QE we can choose $f \in QE$ so that $f(x_1) = 1$. Then $w' = f(w) = 1 + \alpha_2 f(x_2) + \ldots + \alpha_r f(x_r) \in L(R)$. In particular, since $L(R) \neq Q\hat{R}$, $r \geq 2$. Suppose $w' \in N$. Then $x_1 w' \in N$ and $w - x_1 w' = \alpha_2(x_2 - x_1 f(x_2)) + \ldots + \alpha_r(x_r - x_1 f(x_r))$ belongs to $L(R)\setminus N$, contradicting the minimality of r . Thus, $w' \notin N$.

For all $c \in QR$, $\phi \in QE(R)$, denote

$$\Delta(c,\phi) = \phi(c)w' - \phi(cw') = \sum_{i=2}^{r} \alpha_i[\phi(c)f(x_i) - \phi(cf(x_i)] .$$

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Then $\Delta(c,\phi) \in L(R)$, hence $\Delta(c,\phi) \in N$ by minimality of r. Suppose, for all c, ϕ and i, that $\phi(c)f(x_i) = \phi(cf(x_i))$. Then, by definition of Γ, $f(x_i) \in \Gamma$ for each i. But this implies $w' \in N$, a contradiction. Therefore, there exist $c \in QR$, $\phi \in QE(R)$ and i such that $e = \phi(c)f(x_i) - \phi(cf(x_i)) \neq 0$. Without loss of generality, take $i = r$.

Choose $\theta \in QE$ with $\theta(e) = f(x_r)$. Then $w' - \theta[\Delta(c,\phi)]f(x_r) = 1 + \Sigma_{i=2}^{r-1} \alpha_i y_i$, where $\alpha_i \in \hat{Q}_p$ and $y_i = f(x_i) - \theta[\phi(c)f(x_i) - \phi(cf(x_i))]f(x_r) \in QR$. Since $w' - \theta[\Delta(c,\phi)]f(x_r)$ belongs to $L(R)$, w' also belongs to N by minimality of r. However, $\theta[\Delta(c,\phi)]f(x_r) \in N$ as well, implying $w' \in N$. This final contradiction completes the proof.

For the remainder of this section we make the additional assumption that the ring R has finite p-rank.

The next lemma goes back to Beaumont-Pierce [3]. See also Lady [6], and Pierce-Vinsonhaler [9].

LEMMA 1.2. $QE = \{f \in \mathrm{End}(QR) \mid f[L(R)] \subset L(R)\}$.

Proof. Under the usual identifications, $R = \hat{R} \cap QR$. Moreover, $\hat{R} = L(R) \oplus F$, where F is a finite rank free $\hat{Z}_p$-module (since R has finite p-rank). Therefore, if $f \in \mathrm{End}(QR)$ and $f[L(R)] \subset L(R)$, then $p^k f(\hat{R}) \subset \hat{R}$. This implies $p^k f \in E(R)$. Since $L(R)$ is an $E(R)$-submodule of $\hat{R}$, the equality follows.

The ideas involved in the next theorem have been used repeatedly. See Pierce [7], Lady [6], Bowshell-Schultz [4], Pierce-Vinsonhaler [9].

THEOREM 1.3. *Let R be a reduced p-local ring of finite p-rank, which is irreducible as an abelian group, and let* $\Gamma = \mathrm{Hom}_{QE}(QR,QR)$, $C = \Gamma \cap R$.

Then: (1) $QE = \mathrm{Hom}_\Gamma(QR,QR)$;

(2) $R \doteq (\Gamma \cap R)x_1 \oplus \dots \oplus (\Gamma \cap R)x_n$ *for some* $\{x_1,\dots,x_n\} \subset R$;

(3) Γ *is the smallest field of definition of* R;

(4) C *is an E-ring.*

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Proof. (1) As previously remarked, QE is a dense subring of $\mathrm{Hom}_{\Gamma}(QR,QR)$. To show the reverse inclusion we apply Lemmas 1.1 and 1.2. Let $f \in \mathrm{Hom}_{\Gamma}(QR,QR)$. Then

$$f[L(R)] = f[Q\hat{R}(\hat{\Gamma} \cap L(R)] = f(Q\hat{R})(\hat{\Gamma} \cap L(R)] \subset Q\hat{R}(\hat{\Gamma} \cap L(R)) = L(R) .$$

By Lemma 1.2, $f \in QE(R)$.

(2) Let $0 \neq x \in R$. Then $\Gamma x \oplus M = QR$ for some Γ-submodule M of QR. Define $\theta_x : QR \to \Gamma \subset QR$ by $\theta_x(sx+m) = s$. Then, by (1), $\theta_x \in QE$. Choose a positive integer k such that $k\theta_x \in E(R)$. Let $r = (sx+m) \in R$. Then $k\theta_x(r) = ks \in \Gamma \cap R$. It follows that $R \doteq (\Gamma \cap R)x \oplus M \cap R$. Continue to split off quasi-summands of R in this way. The process must stop after a finite number of steps because R is reduced and of finite p-rank.

(3) Suppose F is a field contained in the center of QR with $R \doteq (F \cap R)y_1 \oplus \ldots \oplus (F \cap R)y_m$ for some $\{y_1,\ldots,y_m\} \subset R$. Then $H = \mathrm{Hom}_F(QR,QR) \subset QE$. Since QR is a vector space over F we have $F = \mathrm{Hom}_H(QR,QR) \supset \mathrm{Hom}_{QE}(QR,QR) = \Gamma$.

(4) Since $QC = Q(\Gamma \cap R) = \Gamma$ is a field, then C is irreducible. Moreover, as a pure subring of R, C is p-local and of finite p-rank. Let $\Gamma' = \mathrm{Hom}_{QE(C)}(QC,QC)$. By (2), $C = (\Gamma' \cap C)y_1 \oplus \ldots \oplus (\Gamma' \cap C)y_m$ for some $\{y_1,\ldots,y_m\} \subset C$. This, combined with the result (2) for R, implies that Γ' is a field of definition for R. By (3), $\Gamma' \supset \Gamma$. Since we are regarding Γ' as a subring of $QC = \Gamma$, then $\Gamma' = \Gamma$. That is,

$$\Gamma = \Gamma' = \mathrm{Hom}_{QE(C)}(QC,QC) = \mathrm{Hom}_{QE(C)}(\Gamma,\Gamma) .$$

It follows that $QE(C) \subsetneq \mathrm{Hom}_{\Gamma}(\Gamma,\Gamma) = \Gamma$ and, hence, that $E(C) = E(\Gamma \cap R) = \Gamma \cap R$.

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2. The global case

In this section we consider torsion-free reduced rings R for which each localization R_p satisfies the conditions of Section 1: R_p is irreducible and of finite p-rank. We call such a ring *locally irreducible.*

For each prime p, let $\Gamma(p) = \Gamma(R,p) = \mathrm{Hom}_{QE(R_p)}(QR,QR)$, and let $\Gamma = \Gamma(R)$ be the subring of the center of QR generated by $\{\Gamma(p) \mid p \text{ prime}\}$. We will see that in some ways, Γ acts like a smallest field of definition of R. In particular, we have

LEMMA 2.1. *If F is a field of definition of R, then $\Gamma(R) \subset F$.*

Proof. By definition, $\Gamma(p) = \mathrm{Hom}_{QE(R_p)}(QR,QR)$. On the other hand, if F is a field of definition of R then $\mathrm{Hom}_{QE(R)}(QR,QR) \subset F$. Finally, since $QE(R) \subset QE(R_p)$, then $\mathrm{Hom}_{QE(R_p)}(QR,QR) \subset \mathrm{Hom}_{QE(R)}(QR,QR)$. It follows that $\Gamma(p) \subset F$ for all primes p, so that $\Gamma(R) \subset F$.

LEMMA 2.2. *If R is locally irreducible, then $QE(R) \subset \mathrm{Hom}_{\Gamma(R)}(QR,QR)$.*

Proof. Let $f \in QE(R)$. Then for all primes p, $f \in QE(R_p)$, and therefore f commutes with $\Gamma(p)$. It follows that f commutes with $\Gamma(R)$

The next lemma describes the structure of Γ.

LEMMA 2.3. *Let R be locally irreducible and $\Gamma = \Gamma(R)$. Then:*

(1) *there exist primes $p_1,\ldots,p_n$ such that $\Gamma = \Gamma(p_1)\ldots\Gamma(p_n)$ is the subring generated by $\Gamma(p_1),\ldots,\Gamma(p_n)$;*

(2) *$\Gamma \simeq F_1 \times \ldots \times F_m$, where each F_i is a field;*

(3) *if e_i is the central idempotent of QR corresponding to the identity of F_i, then $\Gamma(e_iR) \supset e_i\Gamma = F_i$.*

Proof. (1) Let $p_1,p_2,\ldots$ be a listing of the primes p for which $pR \neq R$. Then $\Gamma(p_1) \subset \Gamma(p_1)\Gamma(p_2) \subset \ldots$ is an ascending chain of $\Gamma(p_1)$ submodules of QR. Since QR is finite dimensional over $\Gamma(p_1)$ by Theorem 1.3, the chain must stabilize. This implies (1).

(2) By (1) we can write $\Gamma = \Gamma(p_1)\dots\Gamma(p_n)$. Let

$$F = \Gamma(p_1) \cap \dots \cap \Gamma(p_n) .$$

Then F is a subfield of each $\Gamma(p_i)$, and a simple argument shows that each $\Gamma(p_i)$ is finite dimensional over F for $1 \le i \le n$. Furthermore, each $\Gamma(p_i)$ is a separable extension of F since $\mathrm{char}(R) = 0$. Thus $T = \Gamma(p_1) \otimes_F \dots \otimes_F \Gamma(p_n)$ is a commutative, separable, finite dimensional algebra over F (see [8], p.188). This implies that T is semisimple and hence a direct product of fields ([8], p.186). However, Γ is a ring epimorphic image of T. Thus $\Gamma \simeq F_1 \times \dots \times F_m$ for some collection of fields $F_1,\dots,F_m$.

(3) This is a routine calculation using the definitions.

To study the relationship between Γ and R, it often suffices, by Lemma 2.3, to assume Γ is a field. We make this reduction whenever it is feasible.

The following simple example shows that even if R is of finite rank, locally irreducible and $\Gamma(R)$ is a field, $\Gamma(R)$ need not be a field of definition for R.

EXAMPLE. Let A be the subgroup of Q generated by $\{1/p \mid p \text{ is a prime}\}$, and let $R = Z \oplus A$ with ring structure defined by $(m,a)(n,b) = (mn, mb+na)$. Then, for each prime p, $R_p \cong Z_p \oplus Z_p$ is irreducible, and $\Gamma(p) = Q \oplus (0)$. Thus, $\Gamma(R) = Q \oplus (0)$. Note that $\Gamma(R)$ is not a field of definition of R. Indeed, R has no field of definition. In this example, $QE(R)$ is the ring of lower triangular 2×2 rational matrices, while $\mathrm{Hom}_\Gamma(QR,QR)$ is the ring of all 2×2 rational matrices. Compare with Theorem 1.3 (1).

In the remainder of this section we show that $\Gamma(R) \cap R$ is an E-ring in any case, and that, with an additional assumption, $\Gamma \cap R$ is a quasi-summand of R. For the sake of convenience we denote

$$\mathrm{supp}(R) = \{p \in Z \mid p \text{ is prime and } pR \neq R\}.$$

Let $C = C(R) = \Gamma \cap R$, and, for each $p \in \mathrm{supp}(R)$, let

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$C(p) = \Gamma(p) \cap R$. Plainly, C is the pure subring of the centre of R generated by $\{C(p) \mid p \in \mathrm{supp}(R)\}$. Moreover, by Theorem 1.3, for each $p \in \mathrm{supp}(R)$, $C(p)$ is an E-ring and $R_p \simeq [C(p)_p]^n$ for some $n = n(p)$. We next show C is an E-ring.

THEOREM 2.4. *Let R be a locally irreducible ring. Then $C = C(R)$ is an* E*-ring.*

Proof. Let $\phi : C \to C$ be an endomorphism of C with $\phi(1) = 0$. We will show that $\phi = 0$. It is an easy exercise to verify that this implies C is an E-ring (or see [4]). For a given prime $p \in \mathrm{supp}(R)$, regard ϕ as an endomorphism of $C_p \subset R_p$. Note that C_p is a $C(p)_p$-submodule of R_p, which is quasi-equal to a free $C(p)_p$ module. If π is (quasi-) projection onto one of the free cyclic summands of R_p, then $\pi\phi(C(p)_p) = 0$, since $\pi\phi(1) = 0$ and $C(p)_p$ is an E-ring. This implies $\phi(C(p)) = 0$ for each prime $p \in \mathrm{supp}(R)$.

Now let $q \neq p$ be primes in $\mathrm{supp}(R)$ and $0 \neq x \in C(q)$. Then, with π as above, $a \to ax \to \pi\phi(ax)$ induces an endomorphism θ of $C(p)_p$. Moreover, $\theta(1) = 0$ since $\phi(x) \in \phi(C(q)) = 0$. Since $C(p)_p$ is an E-ring, $\theta = 0$. It follows that $\pi\phi(C(p)C(q)) = 0$, and hence that $\phi(C(p)C(q)) = 0$. An induction argument shows $\phi(C(p_1)\ldots C(p_k)) = 0$ for any primes $p_1,\ldots,p_k$. Hence $\phi(C) = 0$ and C is an E-ring.

We next consider the question of finding a necessary and sufficient condition for C to be a quasi-summand of R. We start with a simple lemma from commutative ring theory.

LEMMA 2.5. *Let C be a Dedekind domain. Suppose $A \supset B$ are torsion free* C*-algebras and P is a prime in C with A_P/B_P P-bounded. If B/PB contains no nilpotent ideals, then $A_P = B_P$.*

Proof. By assumption we can write $P^n A_P \subset B_P$ for some $n > 0$. Consider $I = PA_P \cap B_P$, an ideal in B_P containing PB_P. Then $\bar{I} = I/PB_P$ is an ideal in B_P/PB_P with $(\bar{I})^n = 0$. By assumption, we have $\bar{I} = 0$. That is, $PA_P \cap B_P = PB_P$. However, PC_P is a principal ideal since C is Dedekind. Thus, $PB_P = PA_P \cap B_P$ implies $B_P = A_P$.

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PROPOSITION 2.6. *Let* S *be a torsion-free reduced algebra over the Dedekind domain* C *such that* C *is pure in* S *and*

(1) QS *and* QC *are fields,*

(2) C *has finite* p-*rank for all integral primes* p ;

(3) S_p *is finitely generated over* C_p *for all integral primes* $p \in \operatorname{supp}(S)$.

Then S *is finitely generated over* C .

Proof. If $p \in \operatorname{supp}(S)$, (3) implies that S_p is quasi-equal to a finite rank free C_p-module. It follows that S has finite p-rank for each prime $p \in \operatorname{supp}(S)$. Furthermore, S_P is equal to a finite rank free C_P-module for each prime P of C , since such a P must contain an integral prime $p \in \operatorname{supp}(S)$, and C_P is a PID .

Let B be the integral closure of C in QS . Then B is a Dedekind domain which is finitely generated as a C-module, with $QB = QS$ ([13], p.46). It follows that $\bar{S} = BS$ is quasi-equal to S . To see this note that $I = \{x \in C | x\bar{S} \subset S\}$ is a non-zero ideal of C since B is finitely generated over C . Thus, I contains an integer since QC is a field.

We will show $\bar{S}/B$ is bounded, hence finite. Let P be a prime in C and consider $\bar{S}_P/B_P$. By the first paragraph of the proof and the definition of B , $\bar{S}_P \doteq S_P \doteq B_P$ are equal to free C_p-modules. Therefore $\bar{S}_P/B_P$ is P-bounded. If the ring B_P/PB_P is semi-simple, then $\bar{S}_P/B_P$ is zero by Lemma 2.5. However, B_P/PB_P is semi-simple if and only if P is unramified in B , that is, PB is a product of distinct prime ideals of B . This is true for almost all primes P in C by a well-known result from ring theory ([13], p.62). Thus, $\bar{S}_P/B_P$ is non-zero for at most finitely many primes $P_1,\ldots,P_k$ in C . Since $\bar{S}_P/B_P$ is P-bounded for $P = P_i$, $1 \le i \le k$, there exist integers $e_1,\ldots,e_k$ such that $P_1^{e_1}\ldots P_k^{e_k}\cdot\bar{S} \subset B$. However, the ideal $P_1^{e_1}\ldots P_k^{e_k}$ contains an integer, so that $\bar{S}/B$ is bounded. Thus, $S \doteq \bar{S} \doteq B$ is finitely generated over C .

Let R be locally irreducible and let $\Gamma(R) = F_1 \times \ldots \times F_m$, $R \doteq e_1R \oplus \ldots \oplus e_mR$ be as in Lemma 2.3. Note that $C(R) \doteq e_1C(R) \oplus \ldots \oplus e_mC(R)$. Let $\bar{C}(R) = \overline{e_1C(R)} \oplus \ldots \oplus \overline{e_mC(R)}$, where $\overline{e_iC(R)}$ denotes the integral closure of the subring $e_iC(R)$ in the field F_i . We now can state a theorem giving a sufficient condition, in the global case, for $C(R)$ to be a quasi-summand of R .

THEOREM 2.7. *Let R be locally irreducible and assume that $\bar{C}(R) \doteq C(R)$. Then $C(R)$ is a quasi-summand of R .*

Proof. Denote $C = C(R)$, $\bar{C} = \bar{C}(R)$. It suffices to assume that $QC = F$, F a field, since $e_1C \oplus \ldots \oplus e_mC$ is a quasi-summand of R if and only if each e_iC is a quasi-summand of e_iR . In view of the assumption that $\bar{C} \doteq C$, no harm is done, up to quasi-isomorphism, by assuming $\bar{C} = C$, that is, C is integrally closed in F . Let I be a non-zero ideal in C . Then, as before, I contains an integer and, since C has finite p-rank for all p , we have that C/I is finite. Thus, C is Noetherian, therefore Dedekind.

Next we show that the Beaumont-Pierce Principal Theorem, proved in [3] for torsion free rings of finite rank, holds for the locally irreducible torsion free reduced ring R , provided $C = \bar{C}$ (or, more generally, if $\bar{C} \doteq C$) .

Since QR is a finite dimensional algebra over $QC = F$, by the Wedderburn Principal Theorem, $QR = S^* \oplus N^*$, where S^* is a semisimple subalgebra of QR and N^* is the nil radical of QR . Let $S = S^* \cap R$, and $N = N^* \cap R$. We show that $R/S \oplus N$ is finite. Following [3], let $S_1 = \{x \in S^* | x+n \in R \text{ for some } n \in N^*\}$. It is easy to check that $S \subset S_1 \subset S^* = QS$ and that $R/S \oplus N \cong S_1/S$. Thus, it suffices to prove that S_1/S is finite.

We have enough machinery at our disposal to bypass the computations employed in [3] to establish that S_1/S is finite. Write $S^* = M_1 \times \ldots \times M_j$ where each M_i is a full matrix algebra over a division algebra D_i . Up to quasi-isomorphism, it is enough to consider

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the case where $S \subset S_1 \subset S^* = M$, a matrix algebra over a division ring D . Since S_1 and S are full subrings of the simple algebra M , S_1 and S are finitely generated over their centres, K_1 and K respectively ([7]). Thus, since $QK_1 = QK$ is a field, the rings S_1 and S are quasi-equal to free modules over K_1 and K , respectively. It therefore suffices to show that K_1/K is finite. To see this, apply Proposition 2.6 to conclude that K_1 and K are both finitely generated C-modules. Thus K_1/K is finite and $R \doteq S \oplus N$. Moreover, it follows that $C \subset S$, since $C \doteq C \cap S \oplus C \cap N$ and $C \cap N = 0$.

To complete the proof of Theorem 2.7, we must show that C is a quasi-summand of S . As above, reduce to the case that $C \subset S \subset S^* = M$, M a full matrix algebra. Let $\Delta = \mathrm{Hom}_{QE(S)}(QS,QS)$. Then Δ is the unique smallest field of definition for S ([7]). Since multiplication by elements of $F = QC$ commutes with $QE(R) \supset QE(S)$, then $F \subset \Delta$. But, by the first part of the proof, S is finitely generated over C , so that F is a field of definition for S . Hence, $\Delta \subset F$, so $\Delta = F$. Thus $S \doteq (\Delta \cap S)^t = (F \cap S)^t = C^t$ for some positive integer t . Note that we have actually established a little more than was required: namely that, in the general case, $QC = \Delta_1 \times \dots \times \Delta_j$, with Δ_i the smallest field of definition for $M_i \cap R$, $1 \le i \le j$.

COROLLARY 2.8. *Let* R *be as in Theorem* 2.7. *Then* $C(R)$ *is a maximal* E *subring of* R .

Proof. By Theorems 2.4 and 2.7, C is an E-ring which is a (pure) quasi-summand of R . If B is a subring of R with $B \supset C$, then C is a pure quasi-summand of B . It follows that B cannot be an E-ring, since pure quasi-summands of an E-ring must be fully invariant ideals in that ring ([4]), and $1 \in C$.

COROLLARY 2.9. *Let* R *be a torsion-free ring of finite rank which is locally irreducible. Then* $C(R)$ *is a quasi-summand of* R .

Proof. In the finite rank case each F_i of Lemma 2.3 is an algebraic number field. It is well known that, in this case, $\bar{C}(R) \doteq C(R)$.

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3. An infinite rank example

In this section we construct an example to show that the assumption that $\bar{C} \doteq C$ in Theorem 2.7 cannot be removed completely.

LEMMA 3.1. *There exists an infinite set of primes* $S = \{p_1, p_2, \ldots\}$ *such that for all* $i \neq j$, p_i *is a square* mod p_j *and such that* $p_i > i(i+1)/2$ *for all* i.

Proof. Let $p_1 = 5$ aand assume $p_1, \ldots, p_{n-1}$ have been chosen such that each $p_i \equiv 1 \pmod 4$ and such that, for all $i \neq j$, p_i is a square mod p_j. Moreover, assume that $p_i > i(i+1)/2$ for $i \leq n-1$.

The sequence $4k(p_1, \ldots, p_{n-1}) + 1$ contains an infinite number of primes. Let p_n be a prime in this sequence with $p_n > n(n+1)/2$. Note that $p_n \equiv 1 \pmod{p_i}$ is a square mod p_i for $i \leq n-1$. Since also $p_n \equiv 1 \pmod 4$, quadratic reciprocity applies and each p_i is a square mod p_n.

Henceforth, S will denote the set of primes $\{p_1, p_2, \ldots\}$ satisfying the conditions of Lemma 3.1. Let $\{x_j, y_j \mid 1 \leq j < \infty\}$ be a set of algebraically independent elements over Q. For each prime p we will identify this set with a subset of $\hat{Z}_p$ which is algebraically independent over Z_p in the following way. For each j, let c_j and d_j be fixed integers. Choose a set $\{\alpha_{pj}, \beta_{pj} \mid 1 \leq j < \infty\}$ in $\hat{Z}_p$ of elements algebraically independent over Z_p. Identify x_j with $c_j + p\alpha_{pj}$ and y_j with $d_j + p\beta_{pj}$. Note that, for all p, $\{x_j, y_j \mid 1 \leq j < \infty\}$ is algebraically independent in $\hat{Z}_p$, and $x_j \equiv c_j, y_j \equiv d_j \pmod{p\hat{Z}_p}$. We will eventually impose additional requirements on c_j, d_j.

Let $K = Q[\{x_j, y_j, \sqrt{p_j}\}]$ be the ring generated by the set of all x_j, y_j, and $\sqrt{p_j}(p_j \in S)$. For each $p \in S$, apply Hensel's Lemma to identify $\sqrt{p_j}$, $p_j \neq p$, with an element of $\hat{Z}_p$. We can combine this with

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our previous identifications of x_j, y_j to obtain an embedding of K into $\hat{Q}_p \oplus \hat{Q}_p\sqrt{p}$.

We now define a ring R by defining the localizations R_p for each prime p . For $p \notin S$, let

$$R_p = Z_p[\{x_j, y_j, \sqrt{p_j} \mid 1 \leq j < \infty\}] .$$

For $p \in S$, let $R_p = K \cap (\hat{Z}_p \oplus \hat{Z}_p p\sqrt{p})$. Then $R = \cap_p R_p$. Note that Z_p is pure in R_p for each prime p . It follows that p-height(1) = 0 in R for each prime p .

LEMMA 3.2. *The integral domain R defined above is an E-ring. Moreover, as an abelian group R is homogenous of type equal to the type of Z .*

Proof. It is easy to check that, for $p \in S$, R_p is irreducible of p-rank 2 and $\Gamma(p) = Q[\{x_j, y_j, \sqrt{p_j} \mid 1 \leq j < \infty, p \neq p_j \in S\}]$ (refer to Section 2). For $p \notin S$, R_p is a free Z_p-module and $\Gamma(p) = Q$. Thus, $\Gamma(R) = K = QR$. By Theorem 2.4, R is an E-ring.

To see that R is homogeneous of type equal to the type of Z , pick $0 \neq a \in R$. Since $a \in K$ there exists a positive integer m with $ma = \sum g_i h_i$, where the sum is finite, $g_i \in Z[\{x_j, y_j \mid 1 \leq j < \infty\}]$ and $h_i \in Z[\{\sqrt{p_j} \mid 1 \leq j < \infty\}]$. Let $\bar{g}_i \in Z$ be g_i evaluated at $x_j = c_j, y_j = d_j$. Note that for $p \in S$, $ma \equiv \sum \bar{g}_i h_i \bmod pR$. Let $b = \sum \bar{g}_i h_i \in Z[\{\sqrt{p_j} \mid 1 \leq j < \infty\}] \subset R$. Since b is algebraic over Z , there exists $f(x) = f_0 + f_1 + \ldots + f_n x^n \in Z[x]$ with $f(b) = 0$ and $f_0 \neq 0$. Then $f_0 = -b(f_1 + \ldots + f_{n-1}b^{n-1})$, and the p-height of b in R is less than or equal to the p-height of f_0 in R for all p . Thus, in R , type $b \leq$ type f_0 = type Z . Since for all $p \in S$,$ma \equiv b \pmod{pR}$, the p-height of ma in R is 0 for almost all $p \in S$. For $p \notin S$, R_p is a free Z_p-module. It follows that the p-height of ma in R is 0 for almost all $p \notin S$. Finally, since R is p-reduced for all primes

p , the p-height of ma in R is finite for all p . We may conclude that type a = type ma = type Z .

EXAMPLE 3.3. Let R be the integral domain of 3.2. Then there is an R-algebra A such that

(1) A has rank 2 as an R-module.

(2) A is an E-ring.

(3) $C(A) = R$.

(4) $C(A)$ is not a quasi-summand of A .

Proof. Define a multiplication on $QR \oplus QR$ by $(r_1,r_2)(s_1,s_2) = (r_1s_1+r_2s_2, r_1s_2+r_2s_1)$. It is easy to check that this product gives an associative R-algebra structure on $QR \oplus QR$. Let A be the R-subalgebra of $QR \oplus QR$ generated by $R \oplus R$ and $\{\sqrt{p_j}(x_j,y_j) \mid 1 \le j < \infty\}$, where $S = \{p_1,p_2,\ldots\}$ from above. For $p_i \in S$, A_{p_i} is the ring generated by $R_{p_i} \oplus R_{p_i}$ and $\sqrt{p_i}(x_i,y_i)$, so that $p_iA_{p_i} \subset R_{p_i} \oplus R_{p_i} \subset A_{p_i}$. Since $\Gamma(R) = QR$, it is immediate that $\Gamma(A) = QR \oplus 0$. It is a straightforward calculation to show that $C(A) = \Gamma(A) \cap A = R \oplus 0$. For convenience, we identify R with $R \oplus 0$ in A .

Recall that $x_j \equiv c_j \pmod{pR}$, $y_j \equiv d_j \pmod{pR}$ for all primes p , where c_j, $d_j \in Z$. We now show that c_j, d_j may be chosen so that A is an E-ring. Let $K_1 = Q[\{\sqrt{p_j} \mid 1 \le j < \infty\}]$, $R_1 = K_1 \cap R$. Then R_1 is a countable pure subring of R . List all pairs $(a_{1k},b_{1k}) \in R_1 \oplus R_1$, $1 \le k$ where p-height $(a_{1k},b_{1k}) = 0$ in $R_1 \oplus R_1$ for all $p \in S$. Choose c_1, $d_1 \in Z$ so that $c_1b_{11} - d_1a_{11} \not\equiv 0 \pmod{p_1R_1}$.

Let $K_2 = K_1[x_1,y_1]$, $R_2 = K_2 \cap R$. Then R_2 is a countable pure subring of R containing R_1 . List pairs $(a_{2k},b_{2k}) \in (R_2 \oplus R_2) - (R_1 \oplus R_2)$ where p-height$(a_{2k},b_{2k}) = 0$ in $R_2 \oplus R_2$ for all $p \in S$. Choose $c_2,d_2 \in Z$ so that $c_2b_{ij} - d_2a_{ij} \not\equiv 0 \pmod{p_2R_2}$ for ij = 11, 12 or 21.

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Inductively define $K_n = K_{n-1}[x_{n-1}, y_{n-1}]$, $R_n = K_n \cap R$, and list the pairs (a_{nk}, b_{nk}) in $(R_n \oplus R_n) - (R_{n-1} \oplus R_{n-1})$ with p-height = 0 for all $p \in S$. Choose integers c_n, d_n so that $c_n b_{ij} - d_n a_{ij} \neq 0 \pmod{p_n R_n}$ for $1 \le i < n$, $1 \le j \le n-i+1$. Note that there are $n(n+1)/2$ such pairs (i,j) . Therefore the choice of c_n, d_n is easy since p_n was chosen larger than $n(n+1)/2$. In fact we can take $c_n = 1$. Then observe that, for each pair of indices ij , there is at most one choice of d_n for which $0 \le d_n < p_n$ and $b_{ij} - d_n a_{ij} \in p_n R$. Since the number of index pairs is $n(n+1)/2 < p_n$, there exists at least one choice of d_n with $b_{ij} - d_n a_{ij} \notin p_n R$ for all ij .

With this choice of c_j, d_j , the ring A becomes an E-ring. To see this, suppose $\phi : A \to A$ satisfies $\phi(1) = 0$. It suffices to show $\phi = 0$. Since $C(A) = R$, ϕ is R-linear (Lemma 2.2). Let $\phi(0,1) = (a,b) \in A \subset QR \oplus QR$. Then $\phi(r,s) = s(a,b)$ for all $(r,s) \in A$. Thus, $\phi(\sqrt{p_j}(x_j, y_j)) = \sqrt{p_j} y_j (a,b) \in A$ for all $1 \le j$. Let m be a positive integer such that $ma, mb \in R$. Then $ma\sqrt{p_j}(y_j, x_j) = ma\sqrt{p_j}(x_j, y_j)(0,1) \in A$. Subtraction yields $(0, m\sqrt{p_j}(ay_j - bx_j)) \in A$. Hence, $m\sqrt{p_j}(ay_j, bx_j) \in R$. Let e be the largest integer dividing ma and mb in R and write $ma = ea'$, $mb = eb'$. Choose j large enough so that $p_j > e$ and $(a', b') = (a_{ik}, b_{ik})$ for some $1 \le i \le j$, $1 \le k \le j-i+1$. We may also assume that the fixed elements a', b' belong to $Q[\{\sqrt{p_r}, x_r, y_r \mid r < j\}]$. Then $\sqrt{p_j}(may_j - mbx_j) \in R$ implies p_j divides $may_j - mbx_j$ in R . Hence p_j divides $a'y_j - b'x_j$ in R , and therefore divides $a'd_j - b'c_j = a_{ik}d_j - b_{ik}c_j$, a contradiction to the choice of c_j, d_j .

We have shown that A is an E-ring with $C(A) = R \neq A$. In particular, $C(A)$ cannot be a quasi-summand of A . This follows, as in the proof of Corollary 2.8, from the fact that any pure quasi-summand of an E-ring is a fully invariant ideal in that ring ([4]). But $C(A)$ cannot be an ideal since $1 \in C(A)$.

References

[1] D. M. Arnold, "Strongly homogeneous torsion-free groups of finite rank", *Proc. Amer. Math. Soc.* 56 (1976), 67-72.

[2] D. M. Arnold, R. S. Pierce, J. D. Reid, C. I. Vinsonhaler, W. J. Wickless, "Torsion-free abelian groups of finite rank projective as modules over their endomorphism rings", *J. Algebra* 71 (1981), 1-10.

[3] R. A. Beaumont and R. S. Pierce, "Torsion-free rings", *Illinois J. Math.* 5 (1961), 61-98.

[4] R. Bowshell and P. Schultz, "Unital rings whose additive endomorphisms commute", *Math. Ann.* 228 (1977), 197-214.

[5] P. A. Krylov, "Strongly homogeneous torsion-free abelian groups", *Siberian Math. J.* 24 (1983), 77-84.

[6] L. Lady, "A seminar on splitting rings for torsion-free modules over Dedekind domains", Lecture Notes in Mathematics 1006 Springer-Verlag (1983), 1-49.

[7] R. S. Pierce, "Subrings of simple algebras", *Michigan Math. J.* 7 (1960), 241-243.

[8] R. S. Pierce, "Associative Algebras", *Graduate Texts in Mathematics*, Springer-Verlag 88 (1982).

[9] R. S. Pierce and C. Vinsonhaler, "Realizing central division algebras", *Pacific J. Math.* 109 (1983), 165-177.

[10] J. D. Reid, "On the ring of quasi-endomorphisms of a torsion-free group", *Topics in Abelian Groups*, Chicago, 1963, 51-68.

[11] J. D. Reid, "On rings on groups", *Pacific J. Math.* 53 (1974), 229-237.

[12] J. D. Reid, "Abelian groups finitely generated over their endomorphism rings" Lecture Notes in Mathematics 874 Springer-Verlag (1981), 41-52.

[13] I. Reiner, "Maximal Orders", *London Math. Soc. Monographs*, Academic Press (1975).

[14] P. Schultz, "The endomorphism ring of the additive group of a ring", *J. Austral. Math. Soc.* 15 (1973), 60-69.

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