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On the continued fraction algorithm

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The fact that continued fractions can be described in terms of Farey sections is used to obtain a generalised continued fraction algorithm. Geometrically, the algorithm transfers the continued fraction process from the real line R to an arbitrary rational line l in R^n. Arithmetically, the algorithm provides a sequence of simultaneous rational approximations to a set of n real numbers $\theta_1, \ldots, \theta_n$ in the extreme case where all of the numbers are rationally dependent on 1 and (say) θ_1. All but a finite number of best approximations are given by the algorithm.

1. Farey section and continued fractions

Farey sections have been used to study approximation problems in complex number fields (Cassels, Ledermann and Mahler [1], see also Mahler [5]). Recently Szekeres has exploited the connection between continued fractions and Farey sections to obtain a multidimensional approximation algorithm (Szekeres [6]). The present work arose out of investigations of the behaviour of the Szekeres algorithm.

For each positive integer N, the N-th Farey section F_N consists of the naturally ordered sequence of all reduced fractions $\frac{a}{b}$ $(b > 0)$ with $b \le N$. (An integer n is regarded as $\frac{n}{1}$.) We use the following properties of F_N :

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The necessary and sufficient condition that the fractions $\frac{a}{b}$, $\frac{c}{d}$ *of* F_N *be consecutive is that* $|ad-bc| = 1$ *and the fraction* $\frac{a+c}{b+d}$ *is not in* F_N. *All terms of* F_{N+1} *which are not already in* F_N *are of the form* $\frac{a+c}{b+d}$, *where* $\frac{a}{b}$, $\frac{c}{d}$ *are consecutive terms of* F_N.

Proofs of these results are given in Hardy and Wright [2, Ch. 3]. Fractions of the form $\frac{a+c}{b+c}$, with $\frac{a}{b}$ and $\frac{c}{d}$ consecutive terms of F_N, are called *mediants*.

An account of the continued fraction algorithm (giving the regular continued fraction expansion of a real number) is also given in Hardy and Wright [2, Chs. 10, 11], where proofs may be found for the following results:

To every real number α, *there corresponds a unique continued fraction* $[a_0; a_1, a_2, \ldots]$ (a_n *integral,* a_n $(n \geq 1)$ *positive*) *with value equal to* α. *This fraction is infinite if* α *is irrational and finite if* α *is rational. (In the latter case, the last integer* a_n *is greater than* 1 *if* n *is greater than* 0.) *If*

$$p_0 = a_0 , \quad q_0 = 1 ,$$

$$p_1 = a_1a_0 + a_1 , \quad q_1 = a_1 ,$$

and

$$\begin{matrix} p_k = a_kp_{k-1} + p_{k-2} \\ q_k = a_kq_{k-1} + q_{k-2} \end{matrix} \quad (k \geq 2) ,$$

then

$$\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k] \quad (k \geq 0) ,$$

$$q_kp_{k-1} - q_{k-1}p_k = (-1)^k \quad (k \geq 0) ,$$

and either

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$$\frac{p_n}{q_n} = \alpha = [a_0; a_1, \ldots, a_n] \text{ for some } n ,$$

or

$$\lim \frac{p_n}{q_n} = \alpha .$$

Finally,

$$(-1)^k (q_k \alpha - p_k) \geq 0 \quad (k \geq 0) .$$

The integers a_k occurring in the algorithm are called *partial quotients* and the fractions $\frac{p_k}{q_k}$ the *convergents* to α.

Theorem 2 implies that the q_k are strictly increasing for $k \geq 1$, and that $\frac{p_{k-1}}{q_{k-1}}$ and $\frac{p_k}{q_k}$ are consecutive terms in F_{q_k} for $k \geq 1$. A description of the continued fraction algorithm in terms of iterated mediants of fractions in F_N is contained in Hurwitz [3] and is given in a different notation by Szekeres in [6]. Briefly, if $\frac{p_{k-2}}{q_{k-2}}$ and $\frac{p_{k-1}}{q_{k-1}}$ $(k \geq 2)$ are successive convergents to α, and if α lies strictly between them, then form the successive mediants ("intermediate fractions")

$$\frac{p_{k-1}+p_{k-2}}{q_{k-1}+q_{k-2}}, \frac{2p_{k-1}+p_{k-2}}{2q_{k-1}+q_{k-2}} = \frac{p_{k-1}+(p_{k-1}+p_{k-2})}{q_{k-1}+(q_{k-1}+q_{k-2})}, \ldots, \frac{rp_{k-1}+p_{k-2}}{rq_{k-1}+q_{k-2}}, \ldots .$$

If r_k is the greatest value of r such that α lies in the closed interval with endpoints $\frac{rp_{k-1}+p_{k-2}}{rq_{k-1}+q_{k-2}}$ and $\frac{p_{k-1}}{q_{k-1}}$, then $r_k = a_k$ and

$$\frac{p_k}{q_k} = \frac{r_k p_{k-1}+p_{k-2}}{r_k q_{k-1}+q_{k-2}} .$$

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2. Extension to a rational line in R^n

We let $(x_1, \ldots, x_n)$ denote the usual coordinate representation of a point X in R^n $(n \geq 2)$. X is a rational point if each x_i is rational. Every rational point X in R^n has its coordinates x_i uniquely expressible in the form $x_i = \frac{p_i}{q}$ with $q \geq 1$ and $p_1, \ldots, p_n$, q relatively prime integers, and when the x_i are expressed in this canonical form, we call $q = q(X)$ the *denominator* of the rational point X.

A line l in R^n contains either no rational points, one rational point, or two (and so an infinity of) rational points. l is called a rational line if it contains two distinct rational points.

Suppose now that l is a fixed rational line in R^n. The rational points on l can be determined explicitly in terms of any system of linear equations with rational coefficients used to define l. It suffices for our purpose to establish

THEOREM 1. *If X is a rational point on l, then $q(X)$ is divisible by a fixed positive integer depending only on l.*

Proof. Pick any rational point $B = \left(\frac{b_1}{d}, \ldots, \frac{b_n}{d}\right)$ on l of minimal denominator $q(B) = d$. The translation $y_j = x_j - \frac{b_j}{d}$ $(j = 1, \ldots, n)$ moves the origin to B, and l becomes a line through the origin which contains other rational points (since the set of rational points on l is preserved by the translation). Hence by homogeneity l contains points whose y-coordinates are integers, and the set of such points forms a lattice on l. Let $T = (t_1, \ldots, t_n)$ be a primitive point of this lattice, so that $t_1, \ldots, t_n$ are relatively prime integers. The correspondence

$$y_j = \frac{x}{d} t_j \quad (j = 1, \ldots, n)$$

between l and R^1 is a bijection which preserves rational points, as does the correspondence

$$x_j = \frac{b_j+xt_j}{d} \quad (j = 1, \ldots, n) . \tag{2.1}$$

Under (2.1) we see that a rational number x with denominator q corresponds to a rational point X on l with denominator dq, and conversely. This establishes the result.

The relation (2.1) enables us to order points on l by using the natural ordering of their images on R^1. For each positive integer N, we now define the Farey section F_N on l to be the ordered set of all rational points X on l whose denominators $q(X)$ satisfy $q(X) \le Nd$. Then we have proved

THEOREM 2. *F_N is the image of F_N under the mapping* (2.1). *X, X' are consecutive points of F_N if and only if the corresponding numbers x, x' are consecutive terms of F_N. This is so if and only if*

$$|q(X)p_j'-q(X')p_j| = d|t_j| \quad (j = 1, \ldots, n)$$

and

$$q(X) + q(X') > Nd .$$

When X and X' are consecutive points of some F_N, we shall write $X \oplus X'$ for their mediant, that is for the point on l corresponding under (2.1) to the mediant of x and x' on R. If $r > 1$ is an integer, $rX \oplus X'$ will denote the iterated mediant $X \oplus \left((r-1)X \oplus X'\right)$.

We now construct on a given rational line l in R^n an analogue of the continued fraction algorithm on R. Having first determined the minimal denominator d and selected a point B on l with $q(B) = d$, we then determine the integers t_j uniquely by specifying that the first non-zero integer in the sequence $t_1, \ldots, t_n$ be positive. Inserting these values into (2.1), we define the point B_k for each integer k as the image of k under (2.1). Thus if $B_k = \left(\frac{b_{k1}}{d}, \ldots, \frac{b_{kn}}{d}\right)$,

$$b_{kj} = b_j + kt_j \quad (j = 1, \ldots, n) .$$

Let A be a given point of l. We define a (possibly finite) sequence of points A_m $(m \geq 0)$ on l and a corresponding sequence a_m of integers as follows:

(i) if $A = B_k$ for some k, then $A_o = B_k = A$ and $a_o = k$. Otherwise, A_o is the unique B_k for which A lies between B_k and B_{k+1}, and $a_o = k$;

(ii) if $A_o = A$, the process stops. If A lies strictly between $B_k \oplus B_{k+1}$ and B_{k+1}, put $a_1 = 1$ and $A_1 = B_{k+1}$. Otherwise let $a_1 \geq 2$ be the largest integer r such that A lies between B_k and $(r-1)B_k \oplus B_{k+1}$, and put A_1 equal to $(a_1-1)B_k \oplus B_{k+1}$;

(iii) if A_{m-2}, A_{m-1} $(m \geq 2)$ have been defined, and $A \neq A_{m-1}$, then let a_m be the largest integer r such that A lies between A_{m-1} and $rA_{m-1} \oplus A_{m-2}$, and put $A_m = a_m A_{m-1} \oplus A_{m-2}$.

If the coordinates of A_m are $\left(\frac{p_{m1}}{q_m}, \ldots, \frac{p_{mn}}{q_m}\right)$, where $q_m = q(A_m)$, then an easy calculation shows that for $j = 1, \ldots, n$,

$$\text{(2.2)} \qquad \begin{aligned} p_{oj} &= b_{oj} + a_o t_j , \quad q_o = d , \\ p_{1j} &= a_1 p_{oj} + t_j , \quad q_1 = a_1 d , \end{aligned}$$

and for $m \geq 2$,

$$\text{(2.3)} \qquad p_{mj} = a_m p_{m-1,j} + p_{m-2,j} , \quad q_m = a_m q_{m-1} + q_{m-2} .$$

Thus with each point A on l is associated a sequence $\{A_m\}$ of points of l and a sequence $\{a_m\}$ of integers. Conversely, given l, B, and the t_j, a given sequence $\{a_o, a_1, \ldots\}$ of integers a_m

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satisfying $a_m \geq 1$ for $m \geq 1$ clearly determines a corresponding sequence of points A_m on l. Let α, α_m respectively correspond to points A, A_m under the mapping (2.1).

THEOREM 3. *(i) Given a point A on l, let $\{A_m\}$ and $\{a_m\}$ be the sequences constructed above. Then the integers a_m are precisely the digits in the continued fraction expansion of α :*

$$\alpha = [a_0; a_1, \ldots] ,$$

and

$$\alpha_m = [a_0; a_1, \ldots, a_m] \quad (m \geq 0) .$$

(ii) Given a sequence $\{a_0, a_1, \ldots\}$ of integers a_m satisfying $a_m \geq 1$ for $m \geq 1$, the corresponding points A_m on l converge to that point A for which $\alpha = [a_0; a_1, \ldots]$.

The proof consists simply of interpreting the construction of the points A_m in terms of operations on the corresponding real numbers α_m, and using the properties of the continued fraction algorithm quoted in §1.

The representation of points A on a rational line l via sequences $\{a_m\}$ will be called the generalised continued fraction algorithm for l, and we write the expansion of A in the form

$$A = [a_0; a_1, a_2, \ldots] .$$

The preceding discussion shows that this algorithm requires two choices to be made - a point of minimal denominator on l must be selected as B_0, and a direction along l is chosen by specifying a choice of signs for the integers $t_1, \ldots, t_n$. It is clear that a new choice for B_0 alters the first digit a_0 in the expansion of A, but leaves the others unchanged. Choosing opposite signs for the set $t_1, \ldots, t_n$ produces the following easily verified alterations:

(a) *an expansion of the form* $[k; 1, a_2, \ldots]$ *is changed to*

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$[-(k+1);\ a_2,\ a_3,\ \ldots]$,

(b) an expansion of the form $[k;\ a_1,\ a_2,\ \ldots]$ $(a_1 \geq 2)$ is changed to $[-(k+1);\ 1,\ a_1-1,\ a_2,\ \ldots]$.

Geometrically, a change of origin leaves the sequence of points $\{A_m\}$ on l unaltered, while a change of direction inserts or removes one point initially and relabels the others.

3. Properties of the algorithm

Suppose now that we have selected a base point $B = \left(\frac{b_1}{d},\ \ldots,\ \frac{b_n}{d}\right)$ and a direction on the rational line l , so that $t_1,\ \ldots,\ t_n$ are known. If $X = \left(x_1,\ \ldots,\ x_n\right)$ is a point of l , let x be the unique real number determined from (2.1), and let

$$x = [a_0;\ a_1,\ \ldots]$$

be the regular continued fraction expansion of x . If ξ_m is the m-th convergent to x , the points X_m corresponding to the ξ_m under (2.1) will be called the convergents to X on l . The coordinates $\left(\frac{p_{m1}}{q_m},\ \ldots,\ \frac{p_{mn}}{q_m}\right)$ of the X_m can be calculated using (2.2) and (2.3).

Properties of the ordinary continued fraction algorithm can now be easily carried over. For example, Borel's theorem becomes:-

THEOREM 4. *If X is not a rational point of l , then at least one of every three consecutive convergents X_m to X satisfies*

$$\left|x_j - \frac{p_{mj}}{q_m}\right| < \frac{d|t_j|}{\sqrt{5}q_m^2} \quad (j = 1,\ \ldots,\ n) .$$

Choosing X as the point on l corresponding to $x = \frac{\sqrt{5}-1}{2}$ shows that Theorem 4 is best possible.

Similarly, periodicity of the generalised continued fraction expansion of X is a necessary and sufficient condition that the

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coordinates of X lie in the same quadratic field (and that at least one coordinate is irrational).

The fact that the convergents ξ_m to x give all the best approximations to x implies that the convergents X_m to X give all the best approximations to X among points on the line l , in the sense that if $Y = \left(\frac{p_1}{q}, \ldots, \frac{p_n}{q}\right)$ is a rational point on l with $q(Y) = q \le q_m = q(X_m)$, then

$$\max_j |qx_j-p_j| > \max_j |q_m x_j-p_{mj}| .$$

It follows from a simple general result of the author (Mack [4]) that the X_m necessarily give all best approximations to X with denominators greater than some constant depending only on l .

The condition that a set of n real numbers $\theta_1, \ldots, \theta_n$ be the coordinates of a point P lying on a rational line l in R^n is equivalent to the numbers $\theta_1, \ldots, \theta_n$ being rationally dependent on 1 and at most one of the θ_j . If $\theta_1, \ldots, \theta_n$ are all rational, then there are an infinity of rational lines l passing through P , and there is a generalised algorithm for each line. Those lines l for which $d\max|t_j|$ is minimal are determined, and the algorithm for one of these lines yields good rational approximations to P . (It is possible to select a line with $d = 1$, but then the line with $\max|t_j|$ minimal need be neither the line joining P to the origin, nor the line joining P to the nearest point with integer coordinates.) When one of the θ_j is irrational, the rational line l is uniquely determined and the generalised algorithm for l can be applied to the point P .

We close with a simple example of the algorithm. The point $X = \left(\frac{5-3\sqrt{2}}{4}, \frac{\sqrt{2}-1}{2}\right)$ lies on the rational line $2x_1 + 3x_2 = 1$ in R^2 , for which $d = 1$. The lattice of integer points is given by

$$x_1 = -1 + 3n , \quad x_2 = 1 - 2n \quad (n \in Z) ,$$

so we may take as base point $B = (-1, 1)$, while $t_1 = 3$, $t_2 = -2$. The number x corresponding to X is

$$x = \frac{\frac{5-3\sqrt{2}}{4}+1}{3} = \frac{\frac{\sqrt{2}-1}{2}-1}{-2} = \frac{3-\sqrt{2}}{4} .$$

The continued fraction expansion of x is $[0; 2, 1, 1, \overline{10, 1, 1, 1}]$ (the bar denotes the periodic part) and the first few convergents are

$$\xi_0 = 0 , \quad \xi_1 = \frac{1}{2} , \quad \xi_2 = \frac{1}{3} , \quad \xi_3 = \frac{2}{5} , \quad \xi_4 = \frac{21}{53} ,$$

giving as convergents X_m to X the points

$$X_0 = (-1, 1), \ X_1 = \left(\frac{1}{2}, 0\right), \ X_2 = \left(0, \frac{1}{3}\right), \ X_3 = \left(\frac{1}{5}, \frac{1}{5}\right), \ X_4 = \left(\frac{10}{53}, \frac{11}{53}\right) .$$

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