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A DAM WITH GENERAL RELEASE RULE

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Abstract

A dam is considered with independently and identically distributed inputs occurring in a renewal process, and in particular a Poisson process, with a general release rate $r(\cdot)$ depending on the content. This is related to a $GI/G/1$ queue with service times dependent on the waiting time. Some results are obtained for the limiting content distribution when it exists; these are more complete for some special release rates, such as $r(x) = \mu x^{\alpha}$ and $r(x) = a + \mu x$, and particular input size distributions.

1. Introduction

The waiting time $X(t)$ at time t in a single server queue $GI/G/1$, or the content of an equivalent dam, has been extensively studied [12]; there are more complete results for a Poisson arrival process ($M/G/1$). More general release rules than unit rate per unit time have also been considered [4], [7], [10], [11], [14], [15]; we consider an instantaneous release rate $r(X(t))$, which is a function of the content $X(t)$, at time t, such that $r(0-) = 0$, and $r(x)$ is continuous and positive on $(0, \infty)$. For illustrative and numerical examples we shall consider the two special cases (i) $r(x) = \mu x^{\alpha}$, $0 < \mu < \infty$, $0 \leqq \alpha < \infty$, so that the instantaneous release rate is proportional to the αth power of the content, and (ii) $r(x) = a + \mu x$, $0 \leqq a < \infty$, $0 < \mu < \infty$ [10]. Case (i) with $\alpha = 0$ and $\mu = 1$ is $GI/G/1$, and with $\alpha = 1$ there is an exponential decay [8], [9], [14]; a variety of other values, such as $\alpha = 1/2$ for a parallel sided sink, might be appropriate in particular circumstances. If (ii) $r(x) = 1 + \mu x$, then the second factor gives a way of providing faster service for large waiting times or content, and it also guarantees ergodicity.

We consider a stochastic process $X(t)$, $0 \leqq t < \infty$, called the content of a dam of capacity $K \leqq \infty$, defined on $[0, K)$. Inputs, at $t_1 < t_2 < \cdots$ ($t_1 > 0 = t_0$) occur in a renewal process with $\tau_i = t_{i+1} - t_i$, $i = 0, 1, 2, \cdots$ being indepen-

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dently and identically distributed (i.i.d.) random variables with $P(\tau_i \leqq x) = A(x)$, $0 \leqq x < \infty$, $E(\tau_i) = 1/\lambda$, $0 < \lambda < \infty$, and with Laplace transform (LT) $a^*(\theta)$. The inputs S_n, $n = 1, 2, \cdots$, are i.i.d. random variables with $P(S_n \leqq x) = G(x)$, $0 \leqq x < \infty$, $G(0) = 0$, $E(S_n) = \beta < \infty$, and LT $g^*(\theta)$. When $A(x) = 1 - \exp(-\lambda x)$, we have a compound Poisson input process.

The content $X(t)$ of a dam with infinite capacity and unit release rate is equivalent to the virtual waiting time in a single server queue. For a general release rate this is no longer the case; the time a customer would wait for service depends on his own service time and possible subsequent arrivals. However, we may take $X(t)$ as a workload process and it is similar to the potential waiting process defined by Rubinovitch [13]. Let

$$D(x) = \int_{y=0}^{x} \frac{1}{r(y)}\, dy \qquad 0 < x < \infty \tag{1.1}$$

whenever the right-hand side is finite. Put $W(t) = D(X(t))$, $0 \leqq t < \infty$. Then $W(t)$ is a process with slope-1 except when inputs or overflow occur or when $W(t) = 0$; $W(t) \geqq 0$ for all $t \geqq 0$ and $W(t) = 0$ if and only if $X(t) = 0$ [2]. The process may be interpreted as the virtual waiting time in a modified $GI/G/1$ queueing system with service times depending on waiting times; it is thus an example of the important class of state dependent queueing systems.

We let $X(t_n -) = X_n$, $W(t_n -) = W_n$, and let $S_n^*(W_n)$ be the size of the nth input in the transformed process, i.e., "the service time of the nth customer" in the sense that it would take time $W_n + S_n^*(W_n)$ before the server became idle if no arrivals occurred in $(t_n, t_n + W_n + S_n^*(W_n))$. As $W_n = D(X_n)$ and $W_n + S_n^*(W_n) = D(X_n + S_n)$ we find

$$S_n^*(W_n) = D(S_n + D^{-1}(W_n)) - W_n \tag{1.2}$$

$$P\{S_n^*(W_n) \leqq x \mid W_n = w\} = G(D^{-1}(x + w) - D^{-1}(w)), \qquad 0 < x < \infty, \quad 0 \leqq w < \infty,$$

where D^{-1} is the (unique) inverse function of $D(x)$, such that $D^{-1}(D(x)) = x$; $S_n^*(W_n)$ is an increasing function of $D(x)$ and a decreasing function of W_n. For $K = \infty$ we have

$$X_{n+1} = D^{-1}[\{D(X_n + S_n) - \tau_n\}^+], \tag{1.3}$$

and

$$W_{n+1} = \{W_n + S_n^*(W_n) - \tau_n\}^+, \tag{1.4}$$

which is Lindley's [12] form. Any results obtained for the original $X(\cdot)$ process may be interpreted in terms of the transformed $W(\cdot)$ process.

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In the special case (i) $r(x) = \mu x^\alpha$ $(0 \leqq \alpha < 1)$ and (ii) $r(x) = a + \mu x$, we have respectively

(i)
$$D(x) = \frac{x^{1-\alpha}}{\mu(1-\alpha)}, \quad D^{-1}(x) = \{\mu(1-\alpha)x\}^{1/(1-\alpha)}$$
$$S_n^*(W_n) = \{W_n^{1/(1-\alpha)} + S_n(\mu(1-\alpha))^{\alpha-1}\}^{1-\alpha} - W_n,$$

(ii)
$$D(x) = \frac{1}{\mu}\ln\left(1+\frac{\mu x}{a}\right), \quad D^{-1}(x) = \frac{a}{\mu}(e^{\mu x}-1)$$
$$S_n^*(W_n) = \frac{1}{\mu}\ln\left\{1+\frac{\mu S_n}{a}e^{-\mu W_n}\right\}.$$

If the integral in (1.1) is infinite, e.g., $r(x) = \mu x^\alpha$, $\alpha \geqq 1$, then the dam can not empty in finite time from any positive value of the content. For any $\varepsilon > 0$ we put

$$D_\varepsilon(x) = \int_{y=\varepsilon}^{x} \frac{1}{r(y)}\,dy \qquad 0 < x < \infty \tag{1.5}$$

with $D(x) = \lim_{\varepsilon \downarrow 0} D_\varepsilon(x)$ whenever the limit exists and is finite; in this case $D_\varepsilon(x) = D(x) - D(\varepsilon)$. For $x < \varepsilon$ we have $D_\varepsilon(x) < 0$, but $D_\varepsilon(x)$ is still monotone in x. If we put $W_\varepsilon(t) = D_\varepsilon(X(t))$, then $W_\varepsilon(t)$ may be negative. However, at all relevant points in the argument below (also in [15]) $D_\varepsilon(x)$ actually occurs as a difference $D_\varepsilon(x) - D_\varepsilon(y)$ $(x > 0,\ y > 0)$, which eliminates the dependence on ε. Consequently results can be justified for the more general case, although we shall argue only for the case $D(x) < \infty$ and shall leave the generalization to the reader.

We wish to study the distribution function (d.f.) $F(x, t; x_0) = P\{X(t) \leqq x \mid X(0) = x_0\}$ and $H_n(x; x_0) = P\{X_n \leqq x \mid X(0) = x_0\}$ of the content at time t and before the nth input respectively, and the corresponding limiting d.f.'s $F(x) = \lim_{t\to\infty} F(x, t; x_0) = P(X \leqq x)$ and $H(x) = \lim_{n\to\infty} H_n(x; x_0)$, whenever they exist. By renewal theoretic arguments it follows [7] that $F(x)$ and $H(x)$ form proper d.f.'s whenever $K < \infty$ or $\lim_{x\to\infty} r(x) > \lambda\beta$. Further from (1.3) we have

$$H_{n+1}(x; x_0) = \int_{y=0}^{\infty} dH_n(y; x_0)P(y, x) = -\int_{y=0}^{\infty} H_n(y; x_0)d_yP(y, x)$$

$$H(x) = -\int_{y=0}^{\infty} H(y)d_yP(y, x) \tag{1.6}$$

where

$$P(y, x) = P\{X_{n+1} \leqq x \mid X_n = y\}$$
$$= \int_{w=0}^{\infty} \{1 - A(D(y+w) - D(w))\}\,dG(w). \tag{1.7}$$

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For $GI/G/1$ we have $P(y,x) = P\{S_n - \tau_n \leqq x - y\}$ ([12]), p. 49), and the Wiener–Hopf equation (1.6) can be solved by known methods; in general (1.6) presents a much more complex problem.

Except for the special case $r(x) = \mu x$ we consider an exponential inter-input distribution; the input process is then compound Poisson (class 1 in [2]). Some general results are given in the next section, with more explicit and numerical examples for some special input size distributions in Sections 3 and 4.

2. Compound Poisson input process

In the case of an exponentially distributed time between inputs we can formally obtain an integro-differential equation [7] for $F(x,t)$, which for $K = \infty$ has been solved in some special cases, such as for $r(x) = 1 (M/G/1)$ [12] and in terms of LT's for $r(x) = \mu x$ [8], [9]. For the limiting content distribution we have

$$r(x)F'(x) = \lambda F(x) - \lambda \int_{y=0}^{x} F(x-y)\, dG(y) \qquad 0 < x < K, \tag{2.1}$$

provided $K < \infty$ or $\lim_{x\to\infty} r(x) > \lambda\beta$. For the transformed process $W = D(X)$ with $L(w) = P\{W \leqq w\}$ we have

$$L'(w) = \lambda L(w) - \lambda \int_{u=0}^{w} L(u)\, dG(D^{-1}(w) - D^{-1}(u)), \qquad 0 < w < D(K). \tag{2.2}$$

For the remainder of this section we suppose $K = \infty$. We define $\psi(\theta) = \int_0^\infty \exp(-\theta x)\, dF(x)$ $(0 \leqq \theta < \infty)$ as the LT of the limiting content; from (2.1)

$$\int_{x=0-}^{\infty} r(x) e^{-\theta x} dF(x) = r(0)F(0) + \psi(\theta)\zeta(\theta) \tag{2.3}$$

$$\zeta(\theta) = \rho\{1 - g^*(\theta)\}/\theta. \tag{2.4}$$

The LT $\psi(\theta)$ is known in the special cases $r(x) = 1$ [12] and $r(x) = \mu x$ [8]. In the combined case $r(x) = \alpha + \mu x$ (2.3) gives

$$\psi'(\theta) + (\zeta(\theta) - \gamma)\psi(\theta) = -\gamma F(0)$$

where $\gamma = a/\mu$, $\rho = \lambda/\mu$. Using $\psi(0) = 1$ we obtain

$$\psi(\theta) = e^{-J(\theta)} \left\{1 - \gamma F(0) \int_{y=0}^{\theta} e^{J(y)} dy\right\}$$

$$J(\theta) = -\gamma\theta + \int_{y=0}^{\theta} \zeta(y)\, dy.$$

As a consequence of $E(S) = \beta < \infty$ we find $-J(\theta) \to +\infty$ as $\theta \to \infty$, so that

$$F(0) = \left\{ \gamma \int_{y=0}^{\infty} e^{J(y)} dy \right\}^{-1}.$$

The procedure can be used for more general input processes [3]. When $g^*(\theta) = \nu/(\nu + \theta)$ this gives (3.6) at $x = 0$, while for $g^*(\theta) = \exp(-\theta\beta)$ $F(0)$ can be evaluated numerically. Moments may be obtained by

$$E(X) = \rho\beta - \gamma(1 - F(0)) \tag{2.5}$$

$$V(X) = \frac{\rho E(S^2)}{2} + (\rho\beta - \gamma) E(X).$$

From (2.1) it follows that

$$E(r(X)) = \lambda\beta + r(0+)F(0), \tag{2.6}$$

which gives for $r(x) = \mu x^\alpha$ $(\alpha > 0)$ that

$$E(X^\alpha) = \rho\beta$$

$$E(X^{\alpha+1}) = \rho\beta E(X) + \frac{\rho}{2} E(S^2),$$

etc., which involves finding the moments of integer order. For $\alpha = 0$ (and $\rho\beta < 1$) and $\alpha = 1$ all moments can be obtained in this way by recurrence. Further if $r(x) = \mu x$ all moments can be found for inputs occurring in a renewal process; if X^* is the content just before an input occurs, then (see [6])

$$E(X^{*r}) = \frac{a^*(r\mu)}{1 - a^*(r\mu)} \sum_{j=0}^{r-1} \binom{r}{j} E(X^{*j}) E(S^{r-j})$$

$$E(X^r) = \frac{\rho\{1 - a^*(r\mu)\}}{r} \sum_{j=0}^{r} \binom{r}{j} E(X^{*j}) E(S^{r-j}).$$

In the case of $r(x) = \mu x^\alpha$ $(0 < \alpha < 1)$ we can use a fractional LT ([5], Section 4.7) to obtain

$$\lim_{c \to \infty} \int_{t=\theta}^{c} \frac{(1-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \psi'(t) dt = -\zeta(\theta)\psi(\theta).$$

By differentiation we obtain formally

$$\rho\beta E(X) + \frac{\rho}{2} E(S^2) = \int_{t=0}^{\infty} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \psi''(t) dt,$$

which gives known results as $\alpha \to 0$ or 1.

3. Exponentially distributed inputs

We assume that the size of an input has a negative exponential ditribution with mean $\beta = 1/\nu$, $0 < \nu < \infty$, so $G(x) = 1 - \exp(-\nu x)$, $0 \leqq x < \infty$. From (2.1)

$$r(x)F'(x) = \lambda F(x) - \nu\lambda e^{-\nu x}\int_{y=0}^{x} F(y)e^{\nu y}dy, \qquad 0 < x < K, \tag{3.1}$$

and hence by differentiation

$$F''(x) + \left(\nu + \frac{r'(x)}{r(x)} - \frac{\lambda}{r(x)}\right)F'(x) = 0, \qquad 0 < x < K.$$

Thus

$$\begin{aligned} F'(x) &= cr(x)^{-1}e^{Q(x)} \\ Q(x) &= -\nu x + \lambda D(x), \end{aligned} \qquad 0 < x < K \tag{3.2}$$

and using $F(K) = 1$ and (2.6) we find

$$F(x) = \frac{e^{Q(x)} + \nu\int_{y=0}^{x} e^{Q(y)}dy}{e^{Q(K)} + \nu\int_{y=0}^{K} e^{Q(y)}dy} \qquad 0 \leqq x \leqq K, \tag{3.3}$$

which has been obtained by McNeil [10], p. 253, using a limiting result of another problem. From (3.3) or (2.3) the equivalent result for $L(z)$ can easily be found.

More explicit results can be found for special cases of the release rate $r(x)$; for convenience we let $K = \infty$. When $r(x) = 1$ (and $\lambda/\nu < 1$) $F(x) = 1 - (\lambda/\nu)\exp(\nu - \lambda)x$, and when $r(x) = \mu x$ (3.3) is (truncated) gamma [8]. If $r(x) = \mu\sqrt{x}$, i.e., $\alpha = 1/2$, and if $\phi(x)$ and $\Phi(x)$ are the density function and distribution function of a standard normal random variable, and $\gamma = \rho\sqrt{(2/\nu)}$, it follows that $F(0) = \phi(\gamma)/(\phi(\gamma) + \gamma\Phi(\gamma))$, and $Y = \sqrt{(2\nu X)} - \gamma = \sqrt{\nu W} - \gamma$ is a truncated standard normal random variable on $(-\gamma, \infty)$ with density function $\gamma F(0)\phi(y)/\phi(\gamma)$ with a jump of size $F(0)$ at $-\gamma$. Further $E(\sqrt{X}) = \rho\beta$ (2.6) and

$$E(X) = \frac{\gamma F(0)}{2\nu\phi(\gamma)}\{2\gamma\phi(\gamma) + \gamma^2\Phi(\gamma) + 1 - \tfrac{1}{2}Q(\gamma^2; 3)\}$$

$$\begin{aligned} E(X^2) = \frac{\gamma F(0)}{4\nu^2\phi(\gamma)}\{&1 + 6\gamma^2 + (8\gamma + 8\gamma^3)\phi(\gamma) + \gamma^4\Phi(\gamma) - 3\gamma^2 Q(\gamma^2; 3) \\ &- \tfrac{1}{2}Q(\gamma^2; 5)\}, \end{aligned} \tag{3.4}$$

where $Q(x^2; k)$ is the tail of the gamma function with index $k/2$ ([1], p. 978).

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If $r(x) = \mu x^2$, i.e., $\alpha = 2$, then $F(0+) = 0$, and with $\sigma = \rho\nu = \lambda\nu/\mu$ we obtain

$$F'(x) = \frac{\sqrt{\rho}e^{-\nu x - \rho/x}}{2x^2\sqrt{\nu}K_1(2\sqrt{\rho\nu})} \qquad 0 < x < \infty, \tag{3.5}$$

where $K_j(x)$ is the modified Bessel function of the second kind and of order j ([1], p. 417). The constant c^{-1} in (3.2) is given by the solution of $a''(\sigma) = a(\sigma)/\sigma$, where

$$a(\sigma) = \int_{y=0}^{\infty} e^{-(y+\sigma/y)}dy = 2\sqrt{\sigma}K_1(2\sqrt{\sigma}).$$

In this case $E(X^2) = \rho\beta$ (2.6) and

$$E(X) = c\int_{x=0}^{\infty} \frac{1}{x} e^{-(\nu x+\rho/x)}dx = \frac{-\rho a'(\sigma)}{a(\sigma)}$$

$$= \sqrt{\frac{\rho}{\nu}}\frac{K_0(2\sqrt{\sigma})}{K_1(2\sqrt{\sigma})}.$$

TABLE 1

The mean $E(X)$, standard deviation $\sqrt{V(X)}$ and probability $F(0+)$ of emptiness for exponentially distributed (ν) inputs with $r(x) = \mu x^\alpha$, $\alpha = 0, \frac{1}{2}, 1, 2$, $\rho = \lambda/\mu = 0.5$, 1, 2, 4, and $r(x) = 1 + x\rho$.

ρ	ν	α	$E(X)$	$\sqrt{V(X)}$	$F(0+)$
0.5	1	0	1	1.732	0.500
		$\frac{1}{2}$	0.567	0.960	0.367
		1	0.500	0.707	0
		2	0.537	0.452	0
		$1 + x/2$	0.302	0.639	0.604
1	1	$\frac{1}{2}$	1.449	1.588	0.103
		1	1	1	0
		2	0.814	0.580	0
		$1 + x$	0.500	0.866	0.500
2	1	1/2	4.499	2.919	0.0028
		1	2	1.414	0
		2	1.212	0.728	0
		$1 + 2x$	0.800	1.166	0.400
4	1	$\frac{1}{2}$	16.500	5.701	0.000
		1	4	2	0
		2	1.788	0.896	0
		$1 + 4x$	1.243	1.567	0.311
1	2	0	0.500	1.225	0.500
		$\frac{1}{2}$	0.444	0.610	0.223
		1	0.500	0.500	0
		2	0.606	0.364	0
		$1 + x$	0.167	0.373	0.667

When $r(x) = a + \mu x$ we find

$$F(x) = \frac{\Gamma\left(\frac{\nu a}{\mu} + \nu x ; \rho + 1\right) - \Gamma\left(\frac{\nu a}{\mu} ; \rho + 1\right) + \left(1 + \frac{\mu x}{a}\right)^{\rho} e^{-\nu x} \gamma\left(\frac{\nu a}{\mu} ; \rho + 1\right)}{1 - \Gamma\left(\frac{\nu a}{\mu} ; \rho + 1\right)} \quad (3.6)$$

where $\gamma(x;\rho)$ and $\Gamma(x;\rho)$ are respectively the density function and the distribution function of a gamma distribution with index ρ ($\Gamma(x;\rho) = 1 - Q(2x;2\rho)$ ([1], p. 978)).

For $r(x) = \mu x^{\alpha}$, $\alpha = 0, \frac{1}{2}, 1, 2$, and $r(x) = 1 + \rho x$ and a selection of values of ρ and ν the probability of emptiness $F(0+)$, the mean $E(X)$ and the standard deviation $\sqrt{V(X)}$ of the content are shown in Table 1.

4. Bounded inputs

We consider inputs which have zero mass on $[0, \eta)$, $0 < \eta < \infty$, with mean β; many input distributions would satisfy such a mild restriction. Further this would give an approximation when counting only inputs of magnitude at least η occurring in a stable input process as defined in [2]. In this case (2.2) can be solved iteratively over $[0, \eta), [\eta, 2\eta), \cdots$ to obtain

$$F(x) = \frac{e^{\lambda D(x)} \sum_{j=0}^{[x/\eta]} (-1)^j \xi_j(x)}{e^{\lambda D(K)} \sum_{j=0}^{[K/\eta]} (-1)^j \xi_j(K)} \qquad 0 \leqq x \leqq K \quad (4.1)$$

where $\xi_0(x) = 1$, $0 \leqq x < \infty$ and

$$\xi_j(x) = \lambda \int_{w=j\eta}^{x} \frac{e^{-\lambda D(w)}}{r(w)} dw \int_{y=(j-1)\eta}^{w-\eta} e^{\lambda D(y)} \xi_{j-1}(y) dG(w-y) \qquad j\eta \leqq x < \infty.$$

If the inputs are constant, i.e., $G(x) = 1$ for $x \geqq \eta = \beta$, then

$$\xi_j(x) = \lambda \int_{y=j\beta}^{x} \frac{e^{-\lambda D(y) + \lambda D(y-\beta)}}{r(y)} \xi_{j-1}(y-\beta) dy \qquad j\beta \leqq x < \infty. \quad (4.2)$$

For $r(x) = \mu x^{\alpha}$, $\alpha = 0, 0.25, 0.50, 0.75, 1, 1.50, 2$, $\beta = 1$, $\rho = 0.50, 1, 2, 4$, and x an integer $\leqq 7$ (4.2) has been evaluated [14], [15].

If the interinput is Erlang with index k $(= 2, 3, \cdots)$, then the iterative procedure used above carries over.

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