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A LINEAR COMPLEMENTARITY PROBLEM INVOLVING A SUBGRADIENT

J. PARIDA, A. SEN AND A. KUMAR

A linear complementarity problem, involving a given square matrix and vector, is generalised by including an element of the subdifferential of a convex function. The existence of a solution to this nonlinear complementarity problem is shown, under various conditions on the matrix. An application to convex nonlinear nondifferentiable programs is presented.

1. INTRODUCTION

For given $M \in \mathbf{R}^{n\times n}$ and $r \in \mathbf{R}^n$, the problem of finding an $x \in \mathbf{R}^n$ such that

$$x \geqslant 0, \qquad Mx + r \geqslant 0, \qquad \langle x, Mx + r\rangle = 0 \tag{1.1}$$

is called the linear complementarity problem. The existence of solutions for (1.1) has been investigated by many authors (see the references in [**1**]).

We consider the following extension of (1.1). Given a lower semicontinuous positively homogeneous finite convex function $h\colon \mathbf{R}^n \to \mathbf{R}$, find $x \in \mathbf{R}^n$ and $y \in \partial h(x)$ such that

$$x \geqslant 0, \qquad Mx + y + r \geqslant 0, \qquad \langle x, Mx + y + r\rangle = 0. \tag{1.2}$$

It may be observed that the problem of finding a stationary point of Kuhn-Tucker type of a nondifferentiable programming problem in which the objective function is the sum of a support function and a quadratic function, and the constraints are linear, becomes a linear complementarity problem of the form (1.2).

A lower semicontinuous positively homogeneous finite convex function is the support function of a certain closed convex set. In particular (Corollary 13.2.1 of [**9**]) such an h is representable as

$$h(x) = \max\{x^T v \mid v \in C\} \tag{1.3}$$

where

$$C = \{v \in \mathbf{R}^n \colon v^T x \leqslant h(x) \text{ for all } x\}.$$

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Note that since $h(x)$ is finite for every $x \in C$, C is compact (Theorems 8.4 and 13.2.2 of [**9**]). Moreover (Corollary 23.5.3 of [**9**]), one has the subdifferential formula

$$\partial h(x) = \{y \in C \colon x^T y = h(x)\}.$$

The representation of a lower semicontinuous positively homogeneous finite convex function as a support function is illustrated [**10**] in the following two cases:

(i) Let B be a symmetric positive semidefinite matrix. Then $\left(x^T B x\right)^{1/2} = h(x)$, where $C = \{Bw \colon w^T B w \leqslant 1\}$.

(ii) Let p and q be conjugate exponents; that is, $p^{-1} + q^{-1} = 1$, $1 < p < \infty$ and $1 < q < \infty$. Let E be a $k \times n$ matrix and let $\|z\|_p = \left(\sum_{i=1}^{k} |z_i|^p\right)^{1/p}$. Then $\|Ex\|_p = h(x)$, where $C = \{E^T z \colon \|z\|_q \leqslant 1\}$.

We give some generalised sets of conditions involving M, under each of which there exists a solution to (1.2). Several known classes of matrices M which are relevant to linear complementarity problem (1.1) are seen to satisfy these conditions.

2. The main results

In what follows, we denote by d and e any n-vector with all components positive and the n-vector with all components unity, respectively.

The following lemma on the variational inequality, which is a special case of Lemma 2.1 of [**8**], will be the basic tool for establishing our main results.

Lemma 1. *Let $S \subseteq \mathbb{R}^n$ be a compact convex set, $r \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, and let h be as defined by (1.3). Then there exists an $\overline{x} \in S$, and a $\overline{y} \in \partial h(\overline{x})$ such that*

$$\langle x - \overline{x}, M\overline{x} + \overline{y} + r\rangle \geqslant 0 \text{ for all } x \in S.$$

Theorem 1. *If the system*

$$(2.1) \qquad \begin{aligned} M_i u + t d_i &= 0 \qquad \text{if } u_i > 0 \\ M_i u + t d_i &\geqslant 0 \qquad \text{if } u_i = 0 \\ t = -u^T M u &\geqslant 0 \qquad 0 \neq u \geqslant 0 \end{aligned}$$

is inconsistent, then (1.2) has a solution.

Proof: Consider the compact convex sets

$$S_\alpha = \{x \in \mathbb{R}^n \colon x \geqslant 0, \quad d^T x \leqslant \alpha\}$$

for real $0 < \alpha < \infty$. By Lemma 1, there exists an x^α and $y^\alpha \in \partial h(x^\alpha)$ such that

$$\langle x - x^\alpha, Mx^\alpha + y^\alpha + r\rangle \geqslant 0 \qquad \text{for all } x \in S_\alpha,$$

and applying the duality theory of linear programming, we get a scalar ξ^α such that

$$x^\alpha \geqslant 0, \qquad Mx^\alpha + y^\alpha + r + \xi^\alpha d \geqslant 0 \tag{2.2}$$

$$\langle x^\alpha, Mx^\alpha + y^\alpha + r + \xi^\alpha d\rangle = 0 \tag{2.3}$$

$$\xi^\alpha \geqslant 0, \qquad d^T x^\alpha \leqslant \alpha, \qquad (\alpha - d^T x^\alpha)\xi^\alpha = 0. \tag{2.4}$$

We distinguish two cases.

Case 1. $\xi^\alpha = 0$ for some $\alpha = \overline{\alpha}$, $0 < \overline{\alpha} < \infty$. It follows from (2.2) and (2.3) that (1.2) has a solution $(x^{\overline{\alpha}}, y^{\overline{\alpha}})$.

Case 2. $\xi^\alpha > 0$ for every $0 < \alpha < \infty$. By (2.4), we have $d^T x^\alpha = \alpha$ for all these α. Let $u^\alpha = x^\alpha/\alpha$. Then $u^\alpha \geqslant 0$ and $d^T u^\alpha = 1$. This shows that the set of points (u^α, y^α) lies in the compact set $\{x\colon x \geqslant 0,\ \ d^T x = 1\} \times C$, and hence, there is a convergent sequence of (u^α, y^α) with $\alpha \to \infty$. Let this sequence be one with $\alpha = \alpha_1, \alpha_2, \alpha_3, \dots$, or, briefly, with $\alpha \in \Gamma$, and let (u, y) be the limit of the sequence. Clearly, $u \geqslant 0$ and $d^T u = 1$, which implies $u \neq 0$. Further, from (2.3) and (2.2) respectively, we have

$$0 > -\alpha^{-1}\xi^\alpha = \langle u^\alpha, Mu^\alpha\rangle + \alpha^{-1}\langle u^\alpha, y^\alpha + r\rangle,$$
$$Mu^\alpha + \alpha^{-1}(y^\alpha + r) + (\alpha^{-1}\xi^\alpha)d \geqslant 0$$

for all $\alpha \in \Gamma$, which in the limit gives $0 \geqslant u^T M u = -t$ (say) and $Mu + td \geqslant 0$. Since $d^T u = 1$, we also have $\langle u, Mu + td\rangle = 0$. This shows that u is a solution to the system (2.1), contradicting the assumption of the theorem. Hence, $\xi^\alpha = 0$ for at least one α. ∎

The following corollary is a consequence of Theorem 1 and the definitions of the matrices involved. For the definitions, we refer to Eaves [**2**] and Karamardian [**3**].

COROLLARY 1. *There exists a solution to (1.2) for every $r \in \mathbb{R}^n$ if M is any of the following matrices: positive definite, strictly copositive, P-matrix, strictly semi-monotone and regular matrix (for a regular matrix, take $d = e$ in (2.1)).*

THEOREM 2. *If there is a $\overline{u} \geqslant 0$, and a scalar $\beta > d^T\overline{u}$ such that*

$$\min\{\langle x - \overline{u}, Mx + y + r\rangle \mid y \in \partial h(x)\} \geqslant 0 \tag{2.5}$$

for every $x \in \{x\colon x \geqslant 0,\ \ d^T x = \beta\}$, then (1.2) has a solution.

PROOF: Consider the set $S_\beta = \{x\colon x \geqslant 0,\ \ d^T x \leqslant \beta\}$. Clearly, S_β is compact and convex. Now, applying Lemma 1, and then proceeding as in the proof of Theorem

1, we get vectors $\overline{x} \in \mathbf{R}^n$, $\overline{y} \in \partial h(\overline{x})$, and a scalar $\overline{\xi}$ such that

$$\text{(2.6)} \qquad \overline{x} \geqslant 0, \qquad M\overline{x} + \overline{y} + r + \overline{\xi} d \geqslant 0$$

$$\text{(2.7)} \qquad \langle \overline{x}, M\overline{x} + \overline{y} + r + \overline{\xi} d \rangle = 0$$

$$\text{(2.8)} \qquad \overline{\xi} \geqslant 0, \qquad d^T\overline{x} \leqslant \beta, \qquad (\beta - d^T\overline{x})\overline{\xi} = 0.$$

If $\overline{\xi} = 0$, then $(\overline{x}, \overline{y})$ solves (1.2). Assume that $\overline{\xi} > 0$, and by (2.8), we have $d^T\overline{x} = \beta$. Consequently, from (2.5)–(2.7), it follows that

$$0 \leqslant \langle \overline{x} - \overline{u}, M\overline{x} + \overline{y} + r \rangle \leqslant (d^T\overline{u} - \beta)\overline{\xi} < 0,$$

a contradiction. Therefore, we conclude that $\overline{\xi} = 0$. ∎

As a corollary of Theorem 2, we get the following result for a positive semidefinite matrix M, which can also be obtained by specialising to the present case the result of McLinden [**4**] for monotone multifunctions in a general setting.

COROLLARY 2. *If M is positive semidefinite and there exists a $\overline{u} \geqslant 0$, and a $\overline{v} \in \partial h(\overline{u})$ such that $M\overline{u} + \overline{v} + r > 0$, then (1.2) has a solution.*

PROOF: Set $d = M\overline{u} + \overline{v} + r$, and then choose a scalar $\beta > d^T\overline{u}$. Now, for any $x \geqslant 0$ wiht $d^Tx = \beta$, it follows from the positive semidefiniteness of M and the definition of a subgradient that

$$\begin{aligned}\langle x - \overline{u}, Mx + y + r \rangle &\geqslant \langle x - \overline{u}, M\overline{u} + \overline{v} + r \rangle \\ &= d^T(x - \overline{u}) = \beta - d^T\overline{u} > 0\end{aligned}$$

for all $y \in \partial h(x)$. Thus, the conditions of Theorem 2 are satisfied. ∎

The next corollary gives an existence result for the class of copositive matrices, which includes as a subclass the class of copositive plus matrices [**2**, p. 621].

COROLLARY 3. *If M is a copositive matrix and $r^Tx + h(x) \geqslant 0$ for every $x \geqslant 0$ with $e^Tx = 1$, then (1.2) has a solution.*

PROOF: The result follows immediately from Theorem 2 by setting $\overline{u} = 0$, $\beta = 1$ and $d = e$. ∎

3. An application

In a number of mathematical programming problems studied in detail, such as those in [**5**,6,7,10], the objective function is the sum of a lower semicontinuous positively homogeneous finite convex function and a differentiable convex function, while the constraint functions are differentiable. Below we consider a special case in which the objective function is the sum of a support function and a quadratic function. Though the objective function is not differentiable, the simple form of the subdifferential of a support function is helpful in framing a stationary point problem of Kuhn-Tucker type for this problem.

Let $D \in \mathbf{R}^{n\times n}$ be a symmetric positive semidefinite matrix, $A \in \mathbf{R}^{m\times n}$, $c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$ and let h be defined by (1.3). The problem then is as follows:

$$\begin{aligned}(\mathrm{P}) : \text{Minimise } Q(x) &= \frac{1}{2}x^T Dx + c^T x + h(x) \\ \text{subject to } Ax - b &\geqslant 0, \qquad x \geqslant 0.\end{aligned}$$

It can easily be checked that if there exists $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^m$ and $y \in \partial h(x)$, satisfying

$$\begin{aligned} x \geqslant 0, &\qquad \lambda \geqslant 0 \\ Ax - b \geqslant 0, &\qquad Dx - A^T\lambda + y + c \geqslant 0 \\ \langle \lambda, Ax - b\rangle = 0, &\qquad \langle x, Dx - A^T\lambda + y + c\rangle = 0 \end{aligned} \tag{3.1}$$

then x is an optimal solution of (P). Now we define

$$\begin{aligned} M = \begin{bmatrix} D & -A^T \\ A & 0 \end{bmatrix} &\qquad r = \begin{bmatrix} c \\ -b \end{bmatrix} \\ h_0(x, \lambda) &= h(x) + 0 \end{aligned} \tag{3.2}$$

for each $(x, \lambda) \in \mathbf{R}^{n+m}$, and note that $(y, s) \in \partial h_0(x, \lambda)$ if and only if $y \in \partial h(x)$ and $s = 0$. Taking M, r and $h_0(x, \lambda)$ as above, the stationary point problem (3.1) can be projected into a complementarity problem of the form (1.2). Clearly, M in (3.2) is positive semidefinite. An application of Corollary 2 yields the following theorem.

THEOREM 3. *If there exist $\hat{x} \in \mathbf{R}^n$, $\hat{\lambda} \in \mathbf{R}^m$ and $\hat{y} \in \partial h(\hat{x})$ such that*

$$\begin{aligned} \hat{x} \geqslant 0, &\qquad \hat{\lambda} \geqslant 0 \\ A\hat{x} - b > 0, &\qquad D\hat{x} - A^T\hat{\lambda} + \hat{y} + c > 0 \end{aligned} \tag{3.3}$$

then (P) has an optimal solution.

4. Numerical examples

We give below some examples to illustrate the existence results of Sections 2 and 3.

Example 1. Let $h(x) = \left(x^T Bx\right)^{1/2}$,

$$M = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}, \qquad r = \begin{bmatrix} -\sqrt{3}/2 \\ 1 - \sqrt{3} \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here M is a regular matrix (see [**3**, p. 126]). By Corollary 1, (1.2) has a solution, and we see that $x = \left(1/2, \sqrt{3}/2\right)$, $y = \left(1/2, \sqrt{3}/2\right)$ is a solution.

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Example 2. Let $h(x)$ be as in Example 1, and let

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad r = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Here M is positive semidefinite, and, for $\overline{u} = (2,0)$, $\overline{v} = (1,0) \in \partial h(\overline{u})$, we have $M\overline{u} + \overline{v} + r > 0$. By Corollary 2, (1.2) has a solution, and we see that $x = (1,0)$, $y = (1,0)$ is a solution.

Example 3. Let $h(x) = \|Ex\|_2$,

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad r = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Here M is copositive, and, for $x \geqslant 0$ with $e^T x = 1$, $r^T x + h(x)$ has values between zero and $\left(1 + \sqrt{2}\right)$. Consequently, Corollary 3 ensures the existence of a solution, and we find that $x = (1,0)$, $y = \left(\sqrt{2}, 0\right)$ is a solution of (1.2).

Example 4. Let $h(x)$ be as in Example 1. Consider the problem: minimise $Q(x) = -x_1 - x_2 + h(x)$ over $x_1 > 0$, $0 \leqslant x_2 \leqslant 1$. It can easily be seen that -1 is the infimum of $Q(x)$ over the constraint set, but the problem has no optimal solution. Consequently Theorem 3 implies that the system (3.3) cannot be consistent. In fact, we need $\hat{\lambda} \geqslant 0$, $(\hat{y}_1, \hat{y}_2) \in \partial h(\hat{x})$ such that $\hat{y}_1 > 1$ and $\hat{\lambda} + \hat{y}_2 > 1$, which is not possible, since $\hat{y}_1^2 + \hat{y}_2^2 \leqslant 1$.

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Dr. J. Parida,
Department of Mathematics,
Regional Engg. College,
Rourkela 769008
India

Dr. A. Kumar
Lecturer in Mathematics
Vedvyas College
Rourkela - 769041
India

Dr. A. Sen
Lecturer in Mathematics
SKDAV Women's College
Rourkela
India