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MEROTOPIC SPACES AND EXTENSIONS OF CLOSURE SPACES

K. C. CHATTOPADHYAY, OLAV NJÅSTAD AND W. J. THRON

1. Introduction. Proximity spaces and contiguity spaces, and more recently nearness spaces, have been studied not just because they provide various approaches to uniform structure. Possibly of greater importance is that they can be used as a means of introducing compactifications and more general extensions of the topological spaces on which they are defined. Riesz [**20**] was probably the first to recognize this connection. Since then the idea was used by Freudenthal [**9**], Alexandroff [**1**], Smirnov [**21**], Leader [**17**] and Ivanov and Ivanova [**13**, **14**, **15**] among others.

Recently Reed [**19**] using work of Bentley [**2**, **4**] and Herrlich [**11**, **12**] studied the 1 – 1 correspondence between the class $\mathscr{N}_{L0}$ of all cluster generated nearness spaces and the class $\mathscr{E}_T$ of all principal T_1-extensions of a given T_1-space. She succeeded in showing that the mapping induces a 1 – 1 correspondence between the contingual nearness spaces in $\mathscr{N}_{L0}$ and the compactifications in $\mathscr{E}_T$. The proximal nearness spaces are mapped onto the linkage compactifications (she called them clan complete extensions). The Efremovič proximal nearness spaces correspond to the T_2-compactifications.

As was pointed out by Čech [**6**], closure spaces, rather than topological spaces, provide the natural substructure for proximity relations. Similarly, general merotopic spaces induce closure spaces rather than topological spaces. The question thus arose whether results analogous to those of Bentley, Herrlich and Reed would also be valid for extensions of closure spaces. Extensions of closure spaces were recently studied in [**7**]. This article will be referred to as CT and will be extensively used in the sequel.

Even though the role of principal extensions is not as convincing in the theory of extensions of closure spaces (they have only rather involved extremal properties, for example) as it is for topological spaces, we are nevertheless able to use principal extensions to establish a 1 – 1 correspondence between the class $\mathscr{M}_{RI}$ of cluster generated Riesz merotopic spaces on a given T_1-closure space and the class $\mathscr{E}$ of principal

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T_1-extensions of the space. Since compact closure spaces are not in general conjointly compact and since contigual merotopic spaces in $\mathcal{M}_{RI}$ correspond to conjoint compactifications in $\mathcal{E}$, a new class of merotopic spaces needs to be introduced. These are the concrete weakly contigual merotopic spaces. They correspond to the compactifications. Finally, as in Reed [**19**], the proximal merotopic spaces correspond to the linkage compactifications.

We shall be concerned with sets X and Y, their elements shall be denoted by lower case roman letters. Roman capitals shall denote subsets of X or Y, that is elements of $\mathcal{P}X$ or $\mathcal{P}Y$. Elements of $\mathcal{P}^2 X$ shall be script capitals. For elements of $\mathcal{P}^3 X$ we use lower case greek letters and in special cases X^* or X'', and for elements of $\mathcal{P}^4 X$ we use capital greek letters.

2. Closure spaces. A *closure operator* c on X is a mapping from $\mathcal{P}X$ into $\mathcal{P}X$ satisfying

$$c\,\phi = \phi,\ cA \supset A,\ c(A \cup B) = cA \cup cB.$$

A pair (X, c) where c is a closure operator on X is called a *closure space*. We note that all topological spaces are closure spaces but that the closure operator defining a topological space also satisfies

$$ccA \subset cA,$$

which need not be true in a closure space.

A *grill* $\mathcal{G}$ on X is a collection of subsets satisfying

$$B \supset A \in \mathcal{G} \Rightarrow B \in \mathcal{G},\ A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G} \text{ or } B \in \mathcal{G},\ \phi \notin \mathcal{G}.$$

Grills were introduced by Choquet [**8**]. They were extensively studied in [**22**]. Here it suffices to observe that every grill is the union of ultrafilters and that every union of ultrafilters is a grill. By $\Gamma(X)$ we shall mean the set of all grills on X, and by $\Omega(X)$ we shall mean the set of all ultrafilters on X. A $1 - 1$ correspondence can be established between the set of all filters on X and $\Gamma(X)$ by the mapping sec, defined by

$$\begin{aligned}\sec \mathcal{F} &= \{B \subset X : B \cap F \neq \emptyset \text{ for all } F \in F\}\\ &= \cup\, \{\mathcal{U} \in \Omega\,(X) : \mathcal{U} \supset \mathcal{F}\}.\end{aligned}$$

A subset $\mathcal{G}^\dagger$ of $\Omega(X)$ is defined for every $\mathcal{G} \in \Gamma(X)$ as follows:

$$\mathcal{G}^\dagger = \{\mathcal{U} \in \Omega\,(X) : \mathcal{U} \subset \mathcal{G}\}.$$

For a given closure operator c on X and every point $x \in X$ we define the *adherence grill* of x to be

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$$\mathscr{G}(c, x) = \{A : x \in cA\}.$$

Note that $\mathscr{G}(c, x) \in \Gamma(X)$. Moreover, knowledge of $\{\mathscr{G}(c, x) : x \in X\}$ completely determines c, for we have $cA = \{x : A \in \mathscr{G}(c, x)\}$.

A point x is said to be an *adherence point* for a grill $\mathscr{G}$ if $\mathscr{G} \subset \mathscr{G}(c, x)$. The filter $\mathscr{F}(c, x) = \sec \mathscr{G}(c, x)$ is called the *neighbourhood filter* of x. Following Herrlich [**12**] we shall say that the grill $\mathscr{G}$ *converges* to the point X if $\mathscr{F}(c, x) \subset \mathscr{G}$. (An equivalent condition for convergence of $\mathscr{G}$ to x is $G^{\dagger} \cap \mathscr{G}(c, x)^{\dagger} \neq \phi$.)

We shall also use the following definitions. A closure space (X, c) is called a *T_0-space* if $\mathscr{G}(c, x) = \mathscr{G}(c, y)$ implies $x = y$. The space is called a *T_1-space* if $c\{x\} = \{x\}$ for every $x \in X$. These separation axioms are direct generalizations of the corresponding axioms for topological spaces, but their implications are not similar in all respects.

For spatial structures, defined in terms of closure operators, using grills and adherence grills instead of filters and neighbourhood filters is more convenient.

A mapping $\varphi : (X, c) \rightarrow (Y, k)$ is *continuous* if $\varphi(cA) \subset k\, \varphi(A)$ for all $A \subset X$.

A closure space (X, c) is called *compact* if every grill on X converges to a point of X. This is equivalent to the statement: $\{\mathscr{G}(c, x)^{\dagger} : x \in X\}$ covers $\Omega(X)$, that is

$$\cup\{\mathscr{G}(c, x)^{\dagger} : x \in X\} = \Omega\,(X).$$

It will become apparent that two stronger concepts of compactness are also needed. We call a grill $\mathscr{G}$ *conjoint* if for every finite subcollection $\{A_1, \dots A_n\}$ we have

$$\cap\{cA_k : k = 1, \dots, n\} \neq \phi.$$

The grill will be called *linked* if for any two elements A, B of $\mathscr{G}$ we have

$$cA \cap cB \neq \phi\,.$$

A space (X, c) is called *conjointly compact* (*linkage compact*) if every conjoint grill (linked grill) has an adherence point.

THEOREM 1. *The space (X, c) is conjointly compact if and only if $\cap\{c{:}A \in \mathscr{A}\} \neq \phi$ for every family $\mathscr{A}$ for which the family* $\{\mathrm{c}{:}A \in \mathscr{A}\}$ *has the finite intersection property.*

Proof. Clearly every space satisfying the second condition satisfies the first. That conjoint compactness implies the second property follows from the fact that every family $\mathscr{A}$ satisfying the finite intersection property for

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its closure is contained in a maximal family having this property. This maximal family is a grill.

It follows from Theorem 1 that in topological spaces conjoint compactness and compactness are identical. In closure spaces, however, the two concepts are in general distinct. This is shown in Example 7.1.

THEOREM 2. *Every linkage compact space is conjointly compact. Every conjointly compact space is compact.*

Proof. Since every conjoint grill is a linked grill the first statement follows. For the second assertion note that every ultrafilter $\mathcal{U}$ is a conjoint grill and hence $\mathcal{U} \subset \mathcal{G}(c, x)$ for some $x \in X$. It follows that $\{\mathcal{G}(c, x)^{\dagger} : x \in X\}$ covers $\Omega(X)$.

3. Extensions of closure spaces. We restrict ourselves to the bare outlines of the facts about extensions. For more detailed results we refer to CT. Principal extensions will be considered in the next section.

The triple $(\psi, (Y, k))$, where $\psi : X \to Y$ is a $1 - 1$ mapping and (Y, k) is a closure space, will be called an *extension of the closure space* (X, c) if

(1) $\psi(cA) = k\,\psi(A) \cap \psi(X)$ for all $A \subset X$,

and

(2) $k\,\psi(X) = Y$.

Since ψ is $1 - 1$, (1) insures that ψ is a homeomorphism from (X, c) to $(\psi(X), k')$, where $k'B = kB \cap \psi(X)$ for all $B \subset \psi(X)$. From (2) it follows that $\psi(X)$ is dense in (Y, k).

The *trace of the extension E on X at* $y \in Y$ *is*

$$\tau(y) = \tau(E, y) = \{A \subset X : y \in k\,\psi(A)\}.$$

The family

$$X^* = X^*(E) = \{\tau(E, y) : y \in Y\}$$

is called the *trace system* of the extension E. Since there is no danger of confusion we drop the prefix "dual" which was used in CT. Observe that for all $y \in Y$, $\tau(y) \in \Gamma(X)$ and that $\tau(\psi(X)) = \mathcal{G}(c, x)$ for all $x \in X$.

Two extensions $E_1 = (\psi_1, (Y_1, k_1))$ and $E_2 = (\psi_2, (Y_2, k_2))$ are called *equivalent* if there exists a homeomorphism χ such that $\chi \circ \psi_1 = \psi_2$ on X. The extension E_1 is said to be *greater* than the extension E_2 if there exists a continuous mapping θ from (Y_1, k_1) onto (Y_2, k_2) such that $\theta \circ \psi_1 = \psi_2$ on X. For $\tau : Y \to X^*$ to be a $1 - 1$ mapping it is necessary that (X, c) be a

T_0-space. However, even if (Y, k) is a T_1-space τ may not be $1 - 1$. There is an error on p. 1281 of CT. It is not necessarily true that $(\varphi, (X^*, k_0))$ is the smallest of all extensions of a T_0-space, since τ need not be $1 - 1$.

Nevertheless it will be convenient to consider only extensions $(\varphi, (X^*, h_r))$. Here $\varphi: X \rightarrow X^*$ is defined by $\varphi(x) = \mathscr{G}(c, x)$, A^* is given by

$$A^* = \{\mathscr{H} \in X^*: A \in H\},$$

and

$$h_r\,\alpha = [\varphi^{-1}(\alpha)]^* \cup r[\alpha \sim \varphi^* X)] \quad \text{for all } \alpha \subset X^*.$$

The mapping $r:\mathscr{P}[X^* - \varphi(X)] \rightarrow \mathscr{P}\, X^*$ is required to satisfy

$$r\phi = \phi,\ r\beta \supset \beta,\ r(\beta_1 \cup \beta_2) = r\beta_1 \cup r\beta_2.$$

We shall call an extension a T_0-extension, a compact extension, etc., if (Y, k) or (X^*, h_r) is a T_0-space, a compact space, etc.

4. Principal extensions. If (X, c) is a T_0-space we define the *principal extension of* (X, c) *with respect to the trace system* X^* to be $(\varphi, (X^*, g))$, where

$$g\,\alpha = [\varphi^{-1}(\alpha)]^* \cup \cap \{A^*: A^* \supset \alpha \sim \varphi(X)\}$$

for all $\alpha \subset X^*$. Hence $g = h_r$ with $r\beta = \eta\{A^*: A^* \supset \beta\}$. If, in particular, (X^*, g) is a topological space then it is easily seen that

$$g\,\alpha = \cap \{A^*: A^* \supset \alpha\} \text{ for all } \alpha \subset X^*$$

so that in this case $(\varphi, (X^*, g))$ is the classical principal (or strict) extension.

If one considers arbitrary T_0-extensions $E(\psi, (Y, k))$ of a T_0-space (X, c), then E can be shown to be equivalent to a principal extension if and only if

$$k\,B = \cap \{k\,\psi(A): k\,\psi(A) \supset B\} \quad \text{for all } B \subset Y - \psi(X).$$

The two properties of principal extensions which are used in this article are

I) $B \subset Y \sim \psi(X)$, $A \subset X$, $B \subset k\,\psi(A) \Rightarrow kB \subset k\,\psi(A)$.

II) For every $B \subset Y \sim \psi(X)$ there is a family $\mathscr{A}_B \subset \mathscr{P}X$ such that

$$kB = \cap \{k\,\psi(A): A \in \mathscr{A}_B\}.$$

The principal extension is the only T_0-extension with a fixed trace system having both properties. It is the smallest extension satisfying I. It is also the largest extension for which II holds.

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In what follows certain collections of grills in $\Gamma(X)$ play an important role. The family $\gamma = \{\mathscr{G}_i : i \in I\}$ shall be called a *finitely determined collection* (*binary collection*) if it satisfies the following two conditions:

(i) If for every finite subfamily (two element subfamily) $\mathscr{A}$ of a grill $\mathscr{G} \in \Gamma(X)$ there is an $i(\mathscr{A}) \in I$ such that $\mathscr{A} \subset \mathscr{G}_{i(A)} \in \gamma$, then $\mathscr{G}$ itself is contained in some $\mathscr{G}_j \in \gamma$.
(ii) $\gamma^\dagger = \{\mathscr{G}_i^\dagger : i \in I\}$ is a cover of $\Omega(X)$.

If c is a closure operator on X and $\gamma \subset \Gamma(X)$ we define a c-*collection* to be a family $\gamma = \{\mathscr{G}_i : i \in I\}$ which satisfies (ii) and

(i)′ If for every finite subfamily $\mathscr{A}$ of a grill $\mathscr{G}$ there is an $i(A) \in I$ such that $\mathscr{A} \subset \mathscr{G}_{i(A)} \in \gamma$, then $c(\mathscr{G}) = \{cA : A \in \mathscr{G}\}$ is contained in some $\mathscr{G}_j \in \gamma$.

By means of collections of these types we can characterize certain principal extensions as follows.

THEOREM 3. *A principal extension* $(\varphi, (X^*, g))$ *of a* T_0-*space* (X, c) *is compact* (*conjointly compact, linkage compact*) *if and only if* X^* *is a* c-*collection* (*a finitely determined collection, a binary collection*).

In the proof the following two observations play an important role.

LEMMA 1. *Let* $\prod \in \Gamma(X^*)$. *Then*

$$\mathscr{G}_X = \{A \subset X : \varphi(A) \in \textstyle\prod\}$$

and

$$\mathscr{G}_\sim = \{A \subset X : \text{There exists } \beta \in \textstyle\prod \cap \mathscr{P}[X^* - \varphi(X)] \text{ such that } A^* \supset \beta\}$$

are grills on X.

The proof is a straightforward verification.

Note that either of the two families may be empty, but that the null family is a grill.

LEMMA 2. *If* $\beta \subset X^* \sim \varphi(X)$ *and* $A^* \supset \beta$, *then* $A^* \supset g\beta$.

This follows immediately from the definition of g on $X^* - \varphi(X)$.

Proof of Theorem 3. Instead of proving all six parts of the theorem we only take up enough cases to illustrate the methods used.

We begin with: “If X^* is a finitely determined collection of grills on X then (X^*, g) is a conjointly compact space”.

Let $\prod$ be a conjoint grill on (X^*, g). Let $\mathscr{G}_X$ and $\mathscr{G}_\sim$ be as defined in Lemma 1. Since $\mathscr{G}_X$ and $\mathscr{G}_\sim$ are grills on X, so is $\mathscr{G}_X \cup \mathscr{G}_\sim$. Let $\mathscr{A}$ be a finite subfamily of $\mathscr{G}_X \cup \mathscr{G}_\sim$. Then $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$, where

$$\mathscr{A}_1 = \{A_{11}, A_{12}, \dots, A_{1m}\} \subset \mathscr{G}_X,$$
$$\mathscr{A}_2 = \{A_{21}, A_{22}, \dots, A_{2n}\} \subset \mathscr{G}_\sim .$$

We then have $\varphi(A_{1r}) \in \prod$, $r = 1, \dots m$, and for each A_{2s}, $s = 1, \dots, n$, there is a $\beta_s \in \prod \cap \mathscr{P}[X^* \sim \varphi(X)]$ such that $A^*_{2s} \supset \beta_s$.

From Lemma 2 it follows that $g\beta_s \subset A^*_{2s}$. An immediate consequence of the definition of g is $g\,\varphi(A) = A^*$. Since $\prod$ is assumed to be a conjoint grill on (X^*, g),

$$\phi \neq \cap\{g\,\varphi\,(A_{1r}):r = 1, \dots, m\} \cap \cap\,\{g\,\beta_s:s = 1, \dots, n\}$$
$$\subset \cap\,\{A^*_{1r}:r = 1, \dots, m\} \cap \cap\,\{A^*_{2s}:s = 1, \dots n\}.$$

Hence there exists an $\mathscr{H} \in X^*$ in this intersection. Clearly $\mathscr{A} \subset \mathscr{H}$. Since $\mathscr{A}$ is an arbitrary finite subfamily of $\mathscr{G}_X \cup \mathscr{G}_\sim$ and since X^* is assumed to be a finitely determined collection it now follows that there exists a $\mathscr{K} \in X^*$ such that $\mathscr{G}_X \cup \mathscr{G}_\sim \subset \mathscr{K}$.

Now let $\alpha \in \prod$ be such that $\varphi^{-1}(\alpha) = A \neq \phi$. Then

$$A \in G_X \subset \mathscr{K} \quad \text{and} \quad \mathscr{K} \in A^* = g\,\varphi(A) = g\,\alpha,$$

that is $\alpha \in \mathscr{G}(g, K)$. If $\beta \in \prod \cap \mathscr{P}[X^* - \varphi(X)]$, then let D be such that $D^* \supset \beta$. Then $D \in \mathscr{G}_\sim \subset \mathscr{K}$ and hence $\mathscr{K} \in D^*$. Consequently

$$\mathscr{K} \in g\,\beta = \cap\{D^*:D \supset \beta\} \quad \text{and} \quad \beta \in \mathscr{G}(g, \mathscr{K}).$$

Thus, finally, $\prod \subset \mathscr{G}(g, \mathscr{K})$, and it follows that (X, g) is conjointly compact.

We next prove: "If (X^*, g) is linkage compact then X^* is a binary collection".

Let $\mathscr{G} \in \Gamma(X)$ be such that $\{A, B\} \subset \mathscr{G}$ implies that

$$\{A, B\} \subset \mathscr{H}\,(A, B) \in X^*.$$

Set

$$\varphi(G) = \{\alpha \subset X^*:\alpha \supset \varphi(A) \text{ for some } A \in \mathscr{G}\}.$$

Then $\varphi(G)$ is a grill on X^*. Let α_1, α_2 be in $\varphi(G)$. Then there exist $A_1, A_2 \in \mathscr{G}$ such that $\alpha_i \supset \varphi(A_i)$, $i = 1, 2$. Hence

$$\mathscr{H}(A_1, A_2) \in A^*_1 \cap A^*_2 \subset g\alpha_1 \cap g\alpha_2,$$

and it follows that $\varphi(G)$ is a linked grill on (X^*, g). Since we assume that (X^*, g) is linkage compact there exists $\mathscr{H} \in X^*$ such that $\varphi(G) \subset \mathscr{G}(g,$

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H). Let $A \in \mathscr{G}$. Then $\varphi(A) \subset \varphi(G) \subset \mathscr{G}(g, \mathscr{H})$. Thus $\mathscr{H} \in g\,\varphi(A) = A^*$, which means $A \in \mathscr{H}$. Hence $\mathscr{G} \subset \mathscr{H}$ and X^* satisfies condition (i) of a binary collection.

Since the space (X^*, g) is linkage compact it is also compact (Theorem 2). It follows from CT p. 1285 that $\{\mathscr{H}^\dagger : \mathscr{H} \in X^*\}$ covers $\Omega(X)$, and hence condition (ii) is satisfied.

The last statement that will be proved is: "If (X^*, g) is compact, then X^* is a c-collection".

As in the argument above it follows that X^* satisfies condition (ii).

Let $\mathscr{G} \in \Gamma(X)$ be such that for every finite subfamily $\mathscr{A}$ of $\mathscr{G}$ there exists an $\mathscr{H}_s \in X^*$ such that $\mathscr{A} \subset \mathscr{H}_s$. Set

$$\sum = \{g\,\alpha : \alpha \supset \varphi(A) \text{ for some } A \in \mathscr{G}\}.$$

If $g\alpha \in \sum$ then $g\alpha \supset A^*$ for some $A \in \mathscr{G}$. Hence $\sum$ has the finite intersection property so that there exists an ultrafilter $\Phi \in \Omega(X^*)$ with $\sum \subset \Phi$. Since (X^*, g) is assumed to be compact there exists an $\mathscr{H} \in X^*$ so that $\sum \subset \Phi \subset \mathscr{G}(g, \mathscr{H})$. This is equivalent to the statement: $\mathscr{H} \in g\,g\varphi(A)$ for every $A \in \mathscr{G}$.

To evaluate $g\,g\,\varphi(A)$ we proceed as follows:

$$\begin{aligned} g\,g\,\varphi(A) &= g[g\,\varphi(A) \cap \varphi(X)] \cup g[\varphi(A)^\sim \varphi(X)] \\ &= g\,\varphi(cA) \cup g[g\,\varphi(A)^\sim \varphi(X)] \\ &= g\,\varphi(cA) \cup \{\cap [g\,\varphi(B) : g\,\varphi(B) \supset g\,\varphi(A)^\sim \varphi(X)]\}. \end{aligned}$$

Since the intersection is contained in $g\,\varphi(A)$ it then follows that

$$g\,g\,\varphi(A) = g\,\varphi(cA) = (cA)^*.$$

Thus $\mathscr{H} \in g\,g\,\varphi(A)$ is equivalent to $cA \in \mathscr{H}$, so that $c(\mathscr{G}) = \{cA : A \in \mathscr{G}\} \subset \mathscr{H}$. This establishes that X^* satisfies condition (i)′.

5. Merotopic spaces. A *merotopic space* (X, ν) is a set X together with a collection $\nu \subset \mathscr{P}^2 X$ such that the following conditions are satisfied:

$N_1 : \cap\{A : A \in A\} \neq \phi \Rightarrow \mathscr{A} \in \nu$,
$N_2 : \mathscr{B} < \mathscr{A} \in \nu \Rightarrow \mathscr{B} \in \nu$,
$N_3 : \mathscr{A} \vee \mathscr{B} \in \nu \Rightarrow \mathscr{A} \in \nu$ or $\mathscr{B} \in \nu$,
$N_4 : \mathscr{A} \in \nu \Rightarrow \phi \notin \mathscr{A}$.

Here

$\mathscr{A} < \mathscr{B}$ if and only if each set in $\mathscr{A}$ contains a set in $\mathscr{B}$,
$\mathscr{A} \vee \mathscr{B} = \{A \cup B : A \in \mathscr{A}, B \in \mathscr{B}\}$.

If (X, ν) is a merotopic space then we may sometimes use the expression

"$\mathscr{A}$ is a near collection" if $\mathscr{A} \in \nu$.

Each merotopic space has associated with it a closure operator c_ν on X defined as follows:

$$c_\nu A = \{x:\{\{x\}, A\} \in \nu\}.$$

The concept of a merotopic space is due to Katětov [**16**]. We have stated the axioms for a merotopic space in terms of near collections, which are due to Herrlich [**11**, **12**]. The nearness spaces of Herrlich involve also the axiom

$$N_5 : \{c_\nu A:A \in \mathscr{A}\} \in \nu \Rightarrow \mathscr{A} \in \nu .$$

(These spaces have also been termed *Lodato nearness spaces*, cf. [**10**, **19**].) Thus every nearness space is a merotopic space. By a *Riesz merotopic space* (Riesz space) we mean a merotopic space (X, ν) which satisfies

$$N_6 : \{c_\nu A:A \in \mathscr{A}\} \neq \phi \Rightarrow \mathscr{A} \in \nu .$$

The condition N_6 can be traced back to Riesz [**20**].

If (X, ν) is a merotopic space, then every maximal near collection is a grill. Even if (X, ν) is a nearness space, then not every near collection need be contained in a maximal near collection. If every near collection on a nearness space is contained in a maximal near collection, then (X, ν) is called *concrete*. (These spaces have been given an external characterization by Bentley [**3**].)

Again let (X, ν) be a merotopic space. A grill $\mathscr{G}$ in ν is called a *ν-clan*. A maximal member of ν is called a *ν-cluster*. It follows that every ν-cluster is a ν-clan. The following theorem which characterizes Riesz merotopic spaces is easily verified.

THEOREM 4. *A merotopic space (X, ν) is a Riesz space if and only if, for all $x \in X$, the adherence grills $\mathscr{G}(c_\nu, X)$ are ν-clusters.*

A collection $\gamma = \{\mathscr{G}_i:i \in I\} \subset \Gamma(X)$ will be called a *flat collection* if $\mathscr{G}_i \subset \mathscr{G}_j$ implies $i = j$, that is if all $\mathscr{G}_i$ are maximal in γ. A merotopic space (X, ν) has been called *cluster generated* if there exists a flat collection $\gamma = \{\mathscr{G}_i:i \in I\}$ such that

$$\mathscr{A} \in \nu \Leftrightarrow \text{there exists } i(\mathscr{A}) \in I \text{ such that } \mathscr{A} \subset \mathscr{G}_{i(\mathscr{A})}.$$

The family γ is then exactly the family of clusters on (X, ν). Since this definition is equivalent to that of concreteness, we shall use the latter term from now on. It will be convenient to set

$$X^\nu = \{\mathscr{H}:\mathscr{H} \text{ is a } \nu\text{-cluster}\}.$$

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Next, we call a grill $\mathscr{G}$ a ξ_ν-clan if every finite subfamily of $\mathscr{G}$ belongs to ν. A π_ν-clan is a grill $\mathscr{G}$ such that every two element subfamily of $\mathscr{G}$ belongs to ν.

Clearly, every ν-clan is a ξ_ν-clan and every ξ_ν-clan is a π_ν-clan. It is known [**10**] that every ξ_ν-clan is contained in a maximal ξ_ν-clan and that every π_ν-clan is contained in a maximal π_ν-clan. A family of sets (not necessarily a grill) is called ξ_ν (or π_ν) *compatible* if every finite (two-element) subset is in ν. It is true that every maximal ξ_ν-compatible family is a grill. The same need not be true for π_ν-compatible families [**22**].

A merotopic space (X, ν) is called *proximal* if

$\mathscr{A} \in \nu$ if and only if $\mathscr{A} \subset \mathscr{G}$ for some π_ν-clan $\mathscr{G}$.

The space is called *contigual* if

$\mathscr{A} \in \nu$ if and only if $\mathscr{A} \subset \mathscr{G}$ for some ξ_ν-clan $\mathscr{G}$.

It is called *weakly contigual* if

every $\mathscr{U} \in \Omega(X)$ belongs to ν

and

$\{c_\nu A : A \in \mathscr{A}\}$ belongs to ν whenever $\mathscr{A}$ is contained in a ξ_ν-clan $\mathscr{G}$.

From what was said above it follows that proximal and contigual merotopic spaces are concrete. The same is not in general true for weakly contigual merotopic spaces (see Example 7.3).

There is a one to one correspondence between (generalized) proximity spaces and proximal merotopic spaces. A proximal merotopic space is a space whose structure is the largest one compatible with the corresponding proximity space. This correspondence is functorial in nature and associates with each proximity space (X, π) the indiscrete proximal space (X_{ν_π}) on (X, π). For category theoretic details, see Bentley and Herrlich [**5**]. Similarly there is a one to one correspondence between contiguity spaces and contigual merotopic spaces.

We have chosen to study proximal merotopic spaces and contigual merotopic spaces in preference to proximity spaces and contiguity spaces because thereby we gain an advantage in the next section by having to define only a correspondence between merotopic spaces and principal extensions. Otherwise we would have to define correspondences separately for proximities and extensions and for contiguities and extensions.

We are now ready to prove

THEOREM 5. *A concrete Riesz space is proximal (contigual, weakly contigual) if and only if* $X^\nu = \{\mathcal{H}_i : i \in I\}$, *its family of clusters, is a binary collection (a finitely determined collection, a* c_ν*-collection).*

Proof. We note that X^ν contains all $\mathcal{G}(c_\nu, x)$, $x \in X$, when (X, ν) is a Riesz space. Further, for X^ν to be the collection of clusters on such a space it is necessary that for every $x \in X$ there is a unique $i(x) \in I$ such that $\{x\} \in \mathcal{H}_{i(x)}$. Clearly $\mathcal{H} = \mathcal{G}(c_\nu, x)$. We prove two of the six cases of the theorem.

(1) "If (X, ν) is a contigual space then X^ν is a finitely determined collection".

Let $\mathcal{G} \in \Gamma(X)$ be given and assume that for every finite subset $\mathcal{A}$ of $\mathcal{G}$ there is an $i(\mathcal{A})$ such that $\mathcal{A} \subset \mathcal{H}_{i(\mathcal{A})}$. Then $\mathcal{G}$ is a ξ_ν-clan and hence $\mathcal{G} \subset \mathcal{H}_i$ for some $i \in I$, since every ξ_ν-clan on a contigual space is contained in a ν-cluster. The collection X^ν therefore satisfies condition (i) of a finitely determined collection.

Next, every ultrafilter $\mathcal{U} \in \Omega(X)$ has the finite intersection property and therefore is a ξ_ν-clan. Hence $\mathcal{U} \subset \mathcal{H}_i$ for some i. Thus $\{\mathcal{H}_i^\dagger : i \in I\}$ covers $\Omega(X)$ and so X^ν is a finitely determined collection.

(2) "If X^ν is a c_ν-collection then (X, ν), determined by $\mathcal{A} \in \nu$ if and only if $\mathcal{A} \subset \mathcal{H}_i$ for some $i \in I$, is a concrete weakly contigual merotopic space".

Throughout the proof it is assumed that X^ν is indeed the set of clusters on (X, ν). Hence X^ν is a flat collection. Now let $\mathcal{G}$ be a ξ_ν-clan. Then every finite subset $\mathcal{A}$ of $\mathcal{G}$ is in ν. That is, there exists an $i(\mathcal{A})$ so that $\mathcal{A} \subset \mathcal{H}_{i(\mathcal{A})} \in X^\nu$. Since X^ν is assumed to be a c_ν-collection we have

$$c_\nu(\mathcal{G}) = \{c_\nu A : A \in \mathcal{G}\} \subset \mathcal{H}_j \quad \text{for some } \mathcal{H}_j \in X^\nu,$$

and hence $c_\nu(\mathcal{G}) \in \nu$. It is also true that every $\mathcal{U} \in \Omega(X)$ is in ν, since X^ν is assumed to satisfy condition (ii). Thus the space (X, ν) is weakly contigual. It is obviously concrete.

6. Correspondence between merotopic spaces and extensions. Let (X, c) be a given closure space and let (X, ν) be a merotopic space with $c_\nu = c$. We say that (X, ν) is a merotopic space on (X, c). We begin by considering when $(\varphi, (X^\nu, g))$, where X^ν and g are as defined in Sections 5 and 4, respectively, will be an extension of (X, c). First, one must have $\mathcal{G}(c, x) \in X^\nu$ for all $x \in X$. That is, all $\mathcal{G}(c, x)$ must be ν-clusters. This is the case if and only if (X, ν) is a Riesz space. Next, φ must be one to one. If (X, c) is not a T_1-space and (X, ν) is a Riesz space, then there exist $x \neq y$ with $\mathcal{G}(c, x) = \mathcal{G}(c, y)$. Hence we must start with a T_1-space (X, c).

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Assume this has been done. We are thus led to consider a mapping from the non-empty family $\mathcal{M}_{RI}$ of concrete Riesz merotopic spaces (X, ν) on (X, c) into the family $\mathcal{E}$ of principal T_1-extensions of (X, c) with trace system X^ν. For $\nu \in \mathcal{M}_{RI}$ set

$$E_\nu = (\varphi, (X^\nu, g)).$$

For $E \in \mathcal{E}$ set

$$\nu_E = \{\mathcal{A} \subset \mathcal{P}\, X : \mathcal{A} \subset \mathcal{H}_i \text{ for some } \mathcal{H}_i \in X^*(E)\}.$$

Then the following is true.

THEOREM 6. *Let (x, c) be a fixed T_1-space. Then the mappings*

$$E_\nu : \mathcal{M}_{RI} \to \mathcal{E}$$

and

$$\nu_E : \mathcal{E} \to \mathcal{M}_{RI}$$

are inverses of each other and hence provide bijections from $\mathcal{M}_{RI}$ to $\mathcal{E}$ and from $\mathcal{E}$ to $\mathcal{M}_{RI}$, respectively.

Proof. We have shown that if $\nu \in \mathcal{M}_{RI}$ then E_ν is indeed a principal extension of (X, c). We now show that (X^ν, g) is a T_1-space. It will be convenient to set, in analogy with the definition of A^*,

$$A^\nu = \{\mathcal{H} \in X^\nu : A \in \mathcal{H}\}.$$

Then

$$g\alpha = [\varphi^{-1}(\alpha)]^\nu \cup \cap \{A^\nu : A^\nu \supset \alpha^\sim \varphi(X)\}.$$

Since (X, ν) is a Riesz space, the only $\mathcal{H} \in X^\nu$ that contains $\{x\}$ is $\mathcal{G}(c_\nu, x)$. It follows that

$$g\{\mathcal{G}(c_\nu, x)\} = \{x\}^\nu = \{\mathcal{G}(c_\nu, x)\}.$$

For $\mathcal{H} \neq \mathcal{G}(c_\nu, x)$, $x \in X$, $\mathcal{H} \in X^\nu$ one obtains

$$g\{\mathcal{H}\} = \{\phi\}^\nu \cup \cap \{A^\nu : A^\nu \supset \{\mathcal{H}\}\}.$$

If $\mathcal{K} \in A^\nu$ for all $A^\nu \supset \{\mathcal{H}\}$ then $A \in \mathcal{K}$ for all $A \in \mathcal{H}$. This is possible only if $\mathcal{H} = \mathcal{K}$, since X^ν is a flat collection. Hence $g\{\mathcal{H}\} = \{\mathcal{H}\}$, and it follows that (X^ν, g) is a T_1-space.

Next let (X^ν, g) be a T_1-space. Then the merotopic space (X, ν_E), where $E = (\varphi, (X^\nu, g))$, is concrete. It is a Riesz space since the $\mathcal{G}(c, x)$ are the only elements of X^ν that contain $\{x\}$. For if $\{x\} \in \mathcal{H} \neq \mathcal{G}(c, x)$ then $\{\mathcal{H}\} \in [x]^\nu$. Hence $\nu_E \in \mathcal{M}_{RI}$.

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Next, $\{ \{x\}, A\} \in \nu$, that is $x \in c_\nu A$, if and only if $A \in \mathcal{G}(c, x)$, which is equivalent to $x \in cA$. It follows that $c_\nu = c$. The equalities

$$E_{\nu_E} = E \quad \text{and} \quad \nu_{E_\nu} = \nu$$

now follow by direct substitution.

Combining the results of Theorem 6 with those of Theorems 3 and 5 one obtains

THEOREM 7. *Let (X, c) be a given T_1-space. The mapping E_ν maps the proximal (contigual, weakly contigual and concrete) Riesz spaces on (X, c) one to one onto the linkage compact (conjointly compact, compact) principal T_1-extensions of (X, c).*

7. Examples. The first two examples establish the distinctness of the three types of compactness (and hence the distinctness of the three types of merotopic spaces) considered in this article. The last example shows that weakly contigual merotopic spaces need not be concrete.

Example 7.1. A compact T_1-closure space (X, c) which is not conjointly compact.

Set $\mathbf{N} = \{1, 2, 3, \ldots\}$ and define

$$X_n = \{ (m, n) : m \in \mathbf{N}\} \quad \text{for all } n \in \mathbf{N}.$$

Next let $w(p_1 \ldots p_q)$ be defined for every finite collection of distinct elements of $\mathbf{N}$ and let

$$w(p_1 \ldots p_q) = w(r_1 \ldots r_m) \text{ if and only if } \{p_1, \ldots, p_q\} = \{r_1, \ldots r_m\}.$$

Finally introduce two other points w_0 and w distinct from all $w(p_1 \ldots p_q)$. Set

$$X = Y \cup Z$$

where

$$Y = \cup\{X_n : n \in \mathbf{N}\}$$

and

$$Z = \{w(p_1 \ldots p_q) : \{p_1, \ldots, p_q\} \subset \mathbf{N}\} \cup \{w_0, w\},$$

and it is understood that $Y \cap Z = \phi$. The closure operator c is defined as

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follows: $cF = F$ for all finite subsets $F \subset X$. For infinite sets $C \subset Z$ one sets $cC = C \cup \{w\}$. For infinite sets $B \subset Y$ one sets

$$cB = B \cup \{w(p_1 \ldots p_q) : X_{p_i} \cap B \text{ is infinite for at least one } p_i, \\ i = 1, \ldots q\} \\ \cup \{w_0 : \text{if } B \text{ contains } (m, n) \text{ with arbitrarity large } n\}.$$

In general

$$cA = c(A \cap Y) \cup c(A \cap Z).$$

It is easily verified that (X, c) is a T_1-closure space.

Let $\mathcal{U}$ be an ultrafilter on X. If $\mathcal{U} = \mathcal{U}(x)$ then $\mathcal{U} \subset \mathcal{G}(c, x)$. If $\mathcal{U}$ is non-principal then either $Y \in \mathcal{U}$ or $Z \in \mathcal{U}$. In the first case we distinguish two subcases:

(a) There exists an $n \in \mathbf{N}$ such that $X_n \in \mathcal{U}$, then $\mathcal{U} \in \mathcal{G}(c, w(n))$.

(b) No $X_n \in \mathcal{U}$, then $w_0 \in c\,\mathcal{U}$ for all $U \in \mathcal{U}$ and hence $\mathcal{U} \subset \mathcal{G}(c, w_0)$.
If $Z \in \mathcal{U}$ then $\mathcal{U} \subset \mathcal{G}(c, w)$. It follows that $\{\mathcal{G}(c, x)^{\dagger} : x \in X\}$ covers $\Omega(X)$ and hence (X, c) is compact.

Now consider the grill

$$\mathcal{G} = \cup\{\mathcal{U} \in \Omega(X) : \mathcal{U} \text{ is non-principal and } \mathcal{U} \supset X_n \\ \text{for some } n \in \mathbf{N}\}.$$

Clearly

$$\cap\{cX_n : n \in \mathbf{N}\} = \phi$$

and

$$w(p_1 \ldots . p_q) \in \cap \{cX_{p_i} : i = 1, \ldots q\}.$$

Moreover for each $A \in \mathcal{G}$ there is an $n \in \mathbf{N}$ such that $A \cap X_n$ is infinite, so that

$$w(np_2 \ldots p_q) \in cA .$$

Hence $\mathcal{G}$ is a conjoint grill which has no adherence point.

Example 7.2. A compact T_1-topological space (X, k) which is not linkage compact. This is also an example of a conjointly compact closure space which is not linkage compact, since every compact topological space is conjointly compact.

Let

$$X = \{x : x \text{ is a real number}, -2 < x < 2\}.$$

Let d be the ordinary closure operator on X induced from the reals, and let (α, β) denote the real interval: $\alpha < x < \beta$. Then k is defined by specifying the adherence grills for all $x \in X$ as follows:

$$\mathscr{G}(k, x) = \mathscr{G}(d, x) \text{ for } x \notin \{-1, 0, 1\},$$

$$\mathscr{G}(k, -1) = \mathscr{G}(d, -1) \cup \{A : A \cap (-\epsilon, 0) \text{ is infinite for all } \epsilon > 0\}$$
$$\cup \{B : B \cap [(-2, -2 + \epsilon) \cup (2 - \epsilon, 2)] \text{ is infinite for all } \epsilon > 0\},$$

$$\mathscr{G}(k, 0) = \mathscr{G}(d, 0) \cup \{A : A \cap (-1 - \epsilon, -1 + \epsilon) \text{ is infinite for all } \epsilon > 0\}$$
$$\cup \{B : B \cap (-\epsilon, 1 + \epsilon) \text{ is infinite for all } \epsilon > 0\},$$

$$\mathscr{G}(k, 1) = \mathscr{G}(d, 1) \cup \{A : A \cap (0, \epsilon) \text{ is infinite for all } \epsilon > 0\},$$
$$\cup \{B : B \cap [(-2, -2 + \epsilon) \cup (2 - \epsilon, 2)] \text{ is infinite for all } \epsilon > 0\}.$$

Clearly, (X, k) is a T_1-space. The space is compact by the following argument. For any open cover of (X, k) the points $-1, 0, 1$ will be covered by three or fewer elements. What is not covered by these open sets will be the union of at most four closed intervals of the form $a_j \leqq x \leqq b_j, j = 1, 2, 3, 4$.

Here $-2 < a_1 < b_1 < -1 < a_2 < b_2 < 0 < a_3 < b_3 < 1 < a_4 < b_4 < 2$. The cover of each of these intervals can be replaced by a finite subcover.

Now consider the grill

$$\mathscr{G} = \{A : A \in \mathscr{G}(k, -1) \cup \mathscr{G}(k, 1), A \text{ is infinite}\}.$$

It is easy to verify that for every $A \in \mathscr{G}$, the closure kA contains at least two of the three points $-1, 0, 1$. Hence $\mathscr{G}$ is a linked grill. However, one can find $A_1, A_2, A_3 \in \mathscr{G}$ such that $kA_1 \cap kA_2 \cap kA_3 = \phi$. Hence $\mathscr{G}$ has no adherence point, and so (X, k) is not linkage compact.

Example 7.3. A weakly contigual merotopic space which is not concrete. Let (X, c) be the space of Example 7.1. Define

$$\mathscr{A} \in \nu \Leftrightarrow \cap \{cA : A \in \mathscr{A}\} \neq \phi .$$

Then the space (X, ν) is weakly contigual and concrete. Its clusters are exactly the grills $\mathscr{G}(c, x)$, $x \in X$. Now introduce

$$\mathscr{G}_m = \cup \{\mathscr{U} \in \Omega(X) : X_m \in \mathscr{U}, \mathscr{U} \text{ non-principal}\}$$
$$\mathscr{G}^{(n)} = \cup \{\mathscr{G}_m : m \neq 2^n, 2^{n+1}, \ldots . \}$$

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and

$$\gamma = \{\mathscr{G}(c, x):x \in X\} \cup \{\mathscr{G}^{(n)}:n \in \mathbf{N}\}.$$

A merotopic space (X, μ) is determined by

$$\mathscr{A} \in \mu \Leftrightarrow \mathscr{A} \subset \mathscr{G} \quad \text{for some } \mathscr{G} \in \gamma.$$

Then (X, μ) is not concrete. However, every finite family $\mathscr{A} \in \mu$ satisfies $\mathscr{A} \in \nu$ and hence (X, μ) is weakly contigual.

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Burdwan University, Burdwan,
West Bengal, India;
University of Trondheim-NTH,
Trondheim, Norway;
University of Colorado, Boulder,
Boulder, Colorado

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