IV

Introduction to effective field theory

The purpose of an effective field theory is to represent in a simple way the dynamical content of a theory in the low-energy limit. One uses only those light degrees of freedom that are active at low energy, and treats their interactions in a full field-theoretic framework. The effective field theory is often technically non-renormalizable, yet loop diagrams are included and renormalization of the physical parameters is readily accomplished.

Effective field theory is used in all aspects of the Standard Model and beyond, from *QED* to superstrings. Perhaps the best setting for learning about the topic is that of chiral symmetry. Besides being historically important in the development of effective field theory techniques, chiral symmetry is a rather subtle subject, which can be used to illustrate all aspects of the method, viz., the low-energy expansion, non-leading behavior, loops, renormalization and symmetry breaking. In addition, the results can be tested directly by experiment since the chiral effective field theory provides a framework for understanding the very low-energy limit of *QCD*.

In this chapter we introduce effective field theory by a study of the linear sigma model, and discuss the generalization of these techniques to other settings.

IV–1 Effective lagrangians and the sigma model

The linear sigma model, introduced in Sects. I–4, I–6, provides a 'user friendly' introduction to effective field theory because all the relevant manipulations can be explicitly demonstrated. The Goldstone boson fields, the pions, are present at all stages of the calculation. It also introduces many concepts which are relevant for the low enegy limit of *QCD*. However, low-energy *QCD* is far less transparent, involving a transference from the quark and gluon degrees of freedom of the original lagrangian to the pions of the physical spectrum. Nevertheless, the low-energy properties of the two theories have many similarities.

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The first topic that we need to describe is that of an 'effective lagrangian'. First, let us illustrate this concept by simply quoting the result to be derived below. Recall the sigma model of Eq. (I–4.14),

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma \\ - g\bar{\psi}\left(\sigma - i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5\right)\psi + \frac{1}{2}\mu^2\left(\sigma^2 + \boldsymbol{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2\right)^2. \tag{1.1}$$

This is a renormalizable field theory of pions, and from it one can calculate any desired pion amplitude. Alternatively, if one works at low-energy ($E \ll \mu$), then it turns out that all matrix elements of pions are contained in the rather different looking 'effective lagrangian'

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\,\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \qquad U = \exp i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/F, \tag{1.2}$$

where $F = v = \sqrt{\mu^2/\lambda}$ at tree level (cf. Eq. (I–6.9)). This effective lagrangian is to be used by expanding in powers of the pion field

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} + \frac{1}{6F^2}\left[\left(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}\right)^2 - \boldsymbol{\pi}^2\left(\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\right)\right] + \cdots, \tag{1.3}$$

and taking tree-level matrix elements. This procedure is a relatively simple way of encoding all the low-energy predictions of the theory. Moreover, with this effective lagrangian is the starting point of a full effective field theory treatment including loops, which we will develop in Sect. IV–3.

Representations of the sigma model

In order to embark on the path to the effective field theory approach, let us rewrite the sigma model lagrangian as

$$\mathcal{L} = \frac{1}{4}\,\text{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right) + \frac{\mu^2}{4}\,\text{Tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left[\text{Tr}\,\Sigma^\dagger\Sigma\right]^2 \\ + \bar{\psi}_L i\partial\!\!\!/\psi_L + \bar{\psi}_R i\partial\!\!\!/\psi_R - g\left(\bar{\psi}_L\Sigma\psi_R + \bar{\psi}_R\Sigma^\dagger\psi_L\right), \tag{1.4}$$

with $\Sigma = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}$. The model is invariant under the $SU(2)_L \times SU(2)_R$ transformations

$$\psi_L \to L\psi_L, \qquad \psi_R \to R\psi_R, \qquad \Sigma \to L\Sigma R^\dagger \tag{1.5}$$

for L, R in $SU(2)$. This is the linear representation.[1]

[1] A number of distinct 2×2 matrix notations, among them Σ, U, and M, are commonly employed in the literature for either the linear or the nonlinear cases. It is always best to check the definition being employed and to learn to be flexible.

After symmetry breaking and the redefinition of the σ field,

$$\sigma = v + \tilde{\sigma}, \qquad v = \sqrt{\frac{\mu^2}{\lambda}}, \tag{1.6}$$

the lagrangian reads[2]

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} - 2\mu^2\tilde{\sigma}^2\right) + \frac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} - \lambda v\tilde{\sigma}\left(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\right)\\ &- \frac{\lambda}{4}\left(\tilde{\sigma}^2 + \boldsymbol{\pi}^2\right)^2 + \bar{\psi}\left(i\partial\!\!\!/ - gv\right)\psi - g\bar{\psi}\left(\tilde{\sigma} - i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5\right)\psi,\end{aligned} \tag{1.7}$$

indicating massless pions and a nucleon of mass gv. All the interactions in the model are simple nonderivative polynomial couplings.

There are other ways to display the content of the sigma model besides the above linear representation. For example, instead of $\tilde{\sigma}$ and $\boldsymbol{\pi}$ one could define fields S and $\boldsymbol{\varphi}$,

$$S \equiv \sqrt{\left(\tilde{\sigma} + v\right)^2 + \boldsymbol{\pi}^2} - v = \tilde{\sigma} + \cdots, \qquad \boldsymbol{\varphi} \equiv \frac{v\boldsymbol{\pi}}{\sqrt{\left(\tilde{\sigma} + v\right)^2 + \boldsymbol{\pi}^2}} = \boldsymbol{\pi} + \cdots, \tag{1.8}$$

where one expands in inverse powers of v. For lack of a better name, we can call this the *square-root* representation. The lagrangian can be rewritten in terms of the variables S and $\boldsymbol{\varphi}$ as

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left[\left(\partial_\mu S\right)^2 - 2\mu^2 S^2\right] + \frac{1}{2}\left(\frac{v+S}{v}\right)^2\left[\left(\partial_\mu\boldsymbol{\varphi}\right)^2 + \frac{\left(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi}\right)^2}{v^2 - \boldsymbol{\varphi}^2}\right]\\ &- \lambda v S^3 - \frac{\lambda}{4}S^4 + \bar{\psi}i\partial\!\!\!/\psi - g\left(\frac{v+S}{v}\right)\bar{\psi}\left[\left(v^2 - \boldsymbol{\varphi}^2\right)^{1/2} - i\boldsymbol{\varphi}\cdot\boldsymbol{\tau}\gamma_5\right]\psi.\end{aligned} \tag{1.9}$$

Although this looks a bit forbidding, no longer having simple polynomial interactions, it is nothing more than a renaming of the fields. This form has several interesting features. The pion-like fields, still massless, no longer occur in the potential part of the lagrangian, but instead appear with derivative interactions. For vanishing S, this is called the *nonlinear sigma model*.

Another nonlinear form, the *exponential* parameterization, will prove to be of importance to us. Here the fields are written as

$$\Sigma = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi} = (v + S)U, \qquad U = \exp\left(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}'/v\right) \tag{1.10}$$

such that $\boldsymbol{\pi}' = \boldsymbol{\pi} + \cdots$. Using this form, we find

[2] Here, and in subsequent expressions for $\mathcal{L}$, we drop all additive constant terms.

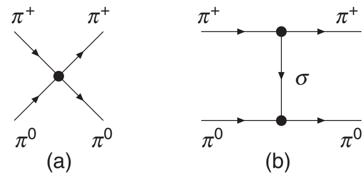

Fig. IV–1 Contributions to $\pi^+\pi^0$ elastic scattering.

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu S\right)^2 - 2\mu^2 S^2\right] + \frac{(v+S)^2}{4}\,\mathrm{Tr}\,\left(\partial_\mu U \partial^\mu U^\dagger\right)$$
$$- \lambda v S^3 - \frac{\lambda}{4} S^4 + \bar{\psi} i \not\partial \psi - g(v+S)\left(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L\right). \qquad (1.11)$$

The quantity U transforms under $SU(2)_L \times SU(2)_R$ in the same way as does Σ, i.e.,

$$U \to L U R^\dagger. \qquad (1.12)$$

This lagrangian is reasonably compact and also has only derivative couplings for pions.

Representation independence

We have introduced three sets of interactions with very different appearances. They are all nonlinearly related. In each of these forms the free-particle sector, found by looking at terms bilinear in the field variables, has the same masses and normalizations. To compare their dynamical content, let us calculate the scattering of the Goldstone bosons of the theory, specifically $\pi^+\pi^0 \to \pi^+\pi^0$. The diagrams that enter at tree level are displayed in Fig. IV–1. The relevant terms in the lagrangians and their tree-level scattering amplitudes are as follows.

(1) *Linear form*:

$$\mathcal{L}_I = -\frac{\lambda}{4}\left(\boldsymbol{\pi}^2\right)^2 - \lambda v \tilde{\sigma} \boldsymbol{\pi}^2,$$
$$i\mathcal{M}_{\pi^+\pi^0\to\pi^+\pi^0} = -2i\lambda + (-2i\lambda v)^2 \frac{i}{q^2 - m_\sigma^2}$$
$$= -2i\lambda\left[1 + \frac{2\lambda v^2}{q^2 - 2\lambda v^2}\right] = \frac{iq^2}{v^2} + \cdots, \qquad (1.13)$$

where $q = p'_+ - p_+ = p_0 - p'_0$ and the relation $m_\sigma^2 = 2\lambda v^2 = 2\mu^2$ has been used. The contributions of Figs. IV–1(a), 1(b) are seen to cancel at $q^2 = 0$. Thus, to leading order, the amplitude is momentum-dependent even though the interaction contains no derivatives. The vanishing of the amplitudes at zero momentum is universal in the limit of exact chiral symmetry.

(2) *Square-root representation*:

$$\mathcal{L}_I = \frac{1}{2}\frac{\left(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi}\right)^2}{\left(v^2-\boldsymbol{\varphi}^2\right)} + \frac{S}{v}\partial_\mu\boldsymbol{\varphi}\cdot\partial^\mu\boldsymbol{\varphi}. \tag{1.14}$$

For this case, the contribution of Fig. IV–1(b) involves four factors of momentum, two at each vertex, and so may be dropped at low-energy. For Fig. IV–1(a) we find

$$\mathcal{L}_I = \frac{1}{2v^2}\left(\varphi^0\partial_\mu\varphi^0+\varphi^+\partial_\mu\varphi^-+\varphi^-\partial_\mu\varphi^+\right)^2,$$
$$i\mathcal{M}_{\varphi^+\varphi^0\to\varphi^+\varphi^0} = \frac{i\left(p'_+-p_+\right)^2}{v^2} = \frac{i\,q^2}{v^2}+\cdots. \tag{1.15}$$

(3) *Exponential representation*:

$$\mathcal{L}_I = \frac{(v+S)^2}{4}\,\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right)+\cdots. \tag{1.16}$$

Again Fig. IV–1(b) has a higher-order ($\mathcal{O}(p^4)$) contribution, leaving only Fig. IV–1(a),

$$\mathcal{L}_I = \frac{1}{6v^2}\left[\left(\boldsymbol{\pi}'\cdot\partial_\mu\boldsymbol{\pi}'\right)^2-\boldsymbol{\pi}'^2\left(\partial_\mu\boldsymbol{\pi}'\cdot\partial^\mu\boldsymbol{\pi}'\right)\right],$$
$$i\mathcal{M}_{\pi^+\pi^0\to\pi^+\pi^0} = \frac{i\left(p'_+-p_+\right)^2}{v^2}+\cdots. \tag{1.17}$$

The lesson to be learned is that all three representations give the *same* answer despite very different forms and even different Feynman diagrams. A similar conclusion would follow for any other observable that one might wish to calculate.

The above analysis demonstrates a powerful field-theoretic theorem, proved first by R. Haag [Ha 58, CoWZ 69, CaCWZ 69], on representation independence. It states that if two fields are related nonlinearly, e.g., $\varphi = \chi F(\chi)$ with $F(0) = 1$, then the same experimental observables result if one calculates with the field φ using $\mathcal{L}(\varphi)$ or instead with χ using $\mathcal{L}(\chi F(\chi))$. The proof consists basically of demonstrating that (i) two S-matrices are equivalent if they have the same single particle singularities, and (ii) since $F(0) = 1$, φ and χ have the same free field behavior and single-particle singularities. This result can be made plausible if we think of the scattering in non-mathematical terms. If the free particles are isolated, they have the same mass and charge and experiment cannot tell the φ particle from the χ particle. At this level they are in fact the same particles, due to $F(0) = 1$. The scattering experiment is then performed by colliding the particles. The results cannot depend on whether a theorist has chosen to calculate the amplitude using the φ or the χ names. That is, the physics cannot depend on a labeling convention.

This result is quite useful as it lets us employ nonlinear representations in situations where they can simplify the calculation. The linear sigma model is a good example. We have seen that the amplitudes of this theory are momentum-dependent. Such behavior is obtained naturally when one uses the nonlinear representations, whereas for the linear representation more complicated calculations involving assorted cancelations of constant terms are required to produce the correct momentum dependence. In addition, the nonlinear representations allow one to display the low-energy results of the theory without explicitly including the massive $\tilde{\sigma}$ (or S) and ψ fields.

IV–2 Integrating out heavy fields

When one is studying physics at some energy scale E, one must explicitly take into account all the particles which can be produced at that energy. What is the effect of fields whose quanta are too heavy to be directly produced? They may still be felt through virtual effects. When using an effective low-energy theory, one does not include the heavy fields in the lagrangian, but their virtual effects are represented by various couplings between light fields. The process of removing heavy fields from the lagrangian is called *integrating out* the fields. Here, we shall explore this process.

The decoupling theorem

There is a general result in field theory, called the decoupling theorem, which describes how the heavy particles must enter into the low-energy theory [ApC 75, OvS 80]. The theorem states that *if the remaining low-energy theory is renormalizable, then all effects of the heavy-particle appear either as a renormalization of the coupling constants in the theory or else are suppressed by powers of the heavy-particle mass*. We shall not display the formal proof. However, the result is in accord with physical expectations. If the heavy particle's mass becomes infinite, one would indeed expect the influence of the particle to disappear. Any shift in the coupling constants is not directly observable because the values of these couplings are always determined from experiment. Inverse powers of heavy-particle mass arise from propagators involving virtual exchange of the heavy particle.

In the Standard Model, the most obvious example of this is the role played in low-energy physics by the $W^{\pm}$ and Z gauge bosons. For example, while $W^{\pm}$-loops can contribute to the renormalization of the electric charge, the effect cannot be isolated at low energies. Also, the residual form of $W^{\pm}$-exchange amplitudes is that of a local product of two weak currents (Fermi interaction) with coupling strength G_F. Its effect is suppressed because $G_F \propto M_W^{-2}$.

However, in the Standard Model there is an example where the heavy-particle effects do *not* decouple. For a heavy top quark, there are many loop diagrams which do not vanish as $m_t \to \infty$, but instead behave as m_t^2 or $\ln(m_t^2)$. This can occur because the electroweak theory with the t quark removed violates the $SU(2)_L$ symmetry, as the full $\binom{t}{b}$ doublet is no longer present. Without the constraint of weak-isospin symmetry, the theory is not renormalizable and new divergences can occur in flavor-changing processes. These would-be divergences are cut off in the real theory by the mass m_t. Note that at the same time as $m_t \to \infty$, the top quark Yukawa coupling also goes to infinity, and hence induces strong coupling, which can also lead to a violation of decoupling.

In the sigma model, *all* the low-energy couplings of the pions are proportional to powers of $1/v^2 \propto 1/m_\sigma^2$, the simplest example being Eq. (1.9). Hence the effective renormalizable theory is in fact a free field theory, without interactions. The interactions have been suppressed by powers of heavy-particle masses. We shall use the energy expansion of the next section to organize the expansion in powers of the inverse heavy mass.

Integrating out heavy fields at tree level

The name of this procedure comes from the path-integral formalism, where the process of integrating out a heavy field H and leaving behind light fields ℓ_i is defined in terms of an effective action $W_{\text{eff}}[\ell_i]$,

$$Z[\ell_i] = e^{iW_{\text{eff}}[\ell_i]} \equiv \int [dH]\, e^{i\int d^4x\, \mathcal{L}(H(x),\ell_i(x))}. \tag{2.1}$$

However, the procedure is equally familiar from perturbation theory, in which the effect of the path integral is represented by a sum of Feynman diagrams.

Let us proceed with a path-integral example. Consider a linear coupling of H to some combination of fields J, with the lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu H \partial^\mu H - m_H^2 H^2\right) + JH. \tag{2.2}$$

One way to integrate out H is to 'complete the square', i.e., we write

$$\begin{aligned}\int d^4x\, \mathcal{L}(H,J) &= \int d^4x \left[-\frac{1}{2}H\mathcal{D}H + JH\right] \\ &= -\frac{1}{2}\int d^4x \left[\left(H - \mathcal{D}^{-1}J\right)\mathcal{D}\left(H - \mathcal{D}^{-1}J\right) - J\mathcal{D}^{-1}J\right] \\ &= -\frac{1}{2}\int d^4x \left[H'\mathcal{D}H' - J\mathcal{D}^{-1}J\right],\end{aligned} \tag{2.3}$$

where we have used the shorthand notations,

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$$
\begin{aligned}
\mathcal{D} &= \Box + m_H^2, \\
\mathcal{D}^{-1} J &= -\int d^4y\ \Delta_F(x-y) J(y), \\
\left(\Box_x + m_H^2\right) \Delta_F(x-y) &= -\delta^4(x-y), \\
H'(x) &= H(x) + \int d^4y\ \Delta_F(x-y) J(y), \\
\int d^4x\ J\mathcal{D}^{-1}J &= -\int d^4x\, d^4y\ J(x)\Delta_F(x-y)J(y),
\end{aligned}
\tag{2.4}
$$

and have integrated by parts repeatedly. Since we integrate in the path integral over all values of the field at each point of spacetime, we may change variables $[dH] = [dH']$ so

$$
\begin{aligned}
Z[J] = e^{iW_{\text{eff}}[J]} &= \int [dH] e^{i\int d^4x\, \mathcal{L}(H,J)} \\
&= \int [dH'] e^{i\int d^4x \left[-\frac{1}{2}H'\mathcal{D}H' + \frac{1}{2}J\mathcal{D}^{-1}J\right]} \\
&= Z[0]\, e^{\frac{i}{2}\int d^4x\ J\mathcal{D}^{-1}J},
\end{aligned}
\tag{2.5}
$$

where

$$
Z[0] = \int [dH'] e^{i\int d^4x\ \left[-\frac{1}{2}H'\mathcal{D}H'\right]}. \tag{2.6}
$$

Here, $Z[0]$ is an overall constant that can be dropped from further consideration. From this result we obtain the effective action

$$
W_{\text{eff}}[J] = -\frac{1}{2}\int d^4x\, d^4y\ J(x)\, \Delta_F(x-y) J(y). \tag{2.7a}
$$

This action is nonlocal because it includes an integral over the propagator. However, the heavy-particle propagator is peaked at small distances, of order $1/m_H^2$. This allows us to obtain a local lagrangian by Taylor expanding $J(y)$ as

$$
J(y) = J(x) + (y-x)^\mu \left[\partial_\mu J(y)\right]_{y=x} + \cdots . \tag{2.7b}
$$

Keeping the leading term and using

$$
\int d^4y\, \Delta_F(x-y) = -\frac{1}{m_H^2}, \tag{2.8}
$$

we obtain

$$
W_{\text{eff}}[J] = \int d^4x\, \frac{1}{2m_H^2} J(x)\, J(x) + \cdots , \tag{2.9}
$$

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where the ellipses denote terms suppressed by additional powers of m_H. Outside of the path-integral context, this result is familiar from W-exchange in the weak interactions.

Matching the sigma model at tree level

We can apply this procedure to the lagrangian for the sigma model, where the scalar field S is heavy with respect to the Goldstone bosons. Thus, considering the theory in the low-energy limit, we may integrate out the field S. Referring to Eq. (1.11) and neglecting the S^2 interactions, it is clear that we should make the identifications $H \rightarrow S$ and $J \rightarrow v\,\mathrm{Tr}\,(\partial_\mu U \partial^\mu U^\dagger)/2$. The effective lagrangian then takes the form

$$\mathcal{L}_{\mathrm{eff}} = \frac{v^2}{4}\,\mathrm{Tr}\,\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{v^2}{8m_S^2}\left[\mathrm{Tr}\,\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2 + \cdots , \tag{2.10}$$

where the second term in Eq. (2.10) is the result of integrating out the S-field and gives rise to the diagram of Fig. IV–1(b). Additional tree-level diagrams are implied by the sigma model when one includes the S^3 and S^4 interactions. Since these carry more derivatives, the above result is the correct tree-level answer with up to four derivatives.

This calculation is an illustration of the concept of ‘matching’, here applied at tree level. We match the effective field theory to the full theory in order to reproduce the correct matrix elements. From the starting point of Eq. (1.11), we expect that there will be a low-energy effective lagrangian, which is written as an expansion in powers of $\mathrm{Tr}\,(\partial_\mu U \partial^\mu U^\dagger)$, with coefficients that are initially unknown. In the matching procedure, we choose the coefficients to be those appropriate for the full theory.

In calculating transitions of pions, this is then used by expanding the U matrix in terms of the pion fields and taking matrix elements. At the lowest energies, only the lagrangian with two derivatives is required, justifying the result quoted in Eq. (1.2).[3] Interested readers may verify that the two terms in Eq. (2.10) reproduce the first two terms in the $\pi^+\pi^0$ scattering amplitude previously obtained in Eq. (1.13). However, we have gained a great deal by using the effective lagrangian framework, because now *all* matrix elements of pions can be calculated simply to this order in the energy by simply expanding the effective lagrangian and reading off the answer.

[3] We will show that this term is not modified by loop effects, aside from the renomalization of the parameter v.

IV–3 Loops and renormalization

The treatment above has left us with a nonlinear effective lagrangian of the form that is called 'non-renormalizable'. It is also incomplete because loop diagrams have not yet been considered. One might worry that because the effective lagrangian is non-renormalizeable, loops would cause trouble. However, that is not the case. Indeed, this situation helps demonstrate the 'effectiveness' of effective field theory – we will see that the important loop processes are reproduced in a simpler manner using the effective field theory.

Continuing our treatment of the linear sigma model, let us display the precise formal correspondence between the full theory and the effective theory. If we are only considering matrix elements involving the light pions, we can write the path integral defining the theory[4] as

$$Z[\mathbf{j}] = N \int [d\boldsymbol{\pi}(x)]\,[d\sigma(x)] \exp\left[i \int d^4x \; \left(\mathcal{L}\left[\boldsymbol{\pi}(x), \sigma(x)\right] + \mathbf{j}(x)\cdot\boldsymbol{\pi}(x)\right)\right]. \tag{3.1}$$

When working at low energies, we can then integrate out the heavy field σ to produce the effective theory

$$Z[\mathbf{j}] = N \int [d\boldsymbol{\pi}(x)] \exp\left[i \int d^4x \; \left(\mathcal{L}_{\text{eff}}\left[\boldsymbol{\pi}(x)\right] + \mathbf{j}(x)\cdot\boldsymbol{\pi}(x)\right)\right]. \tag{3.2}$$

Because the σ field is heavy, its influence will not propagate far and the resulting effective lagrangian will be local. However, this correspondence emphasizes the fact that one is still left with a full field theory. It is not only at tree level that the effective lagrangian must be applied. Loop processes must also be considered, as is the case in any field theory. The original theory involves both σ and π loops, while the effective theory has only the π loop diagrams. We will demonstrate how to match the effective theory to the full theory through an explicit calculation.

In order to accomplish the renormalization and matching procedure for the effective theory we will need a lagrangian similar to the tree level form, but with initially unknown coefficients that will be chosen later, i.e.,

$$\begin{aligned}\mathcal{L}_{\text{eff}} = {} & \frac{v^2}{4}\,\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) \\ & + \ell_1 [\,\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)]^2 + \ell_2\, \text{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right)\, \text{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right).\end{aligned} \tag{3.3}$$

This is the most general form consistent with the symmetry $U \to L U R^\dagger$, containing up to four derivatives. The first portion of this lagrangian, when expanded in terms of the pion field, yields the usual pion propagator as well as the lowest-order result for the $\pi\pi$ scattering amplitudes.

[4] Recall that $\sigma = S$ in some previous formulas.

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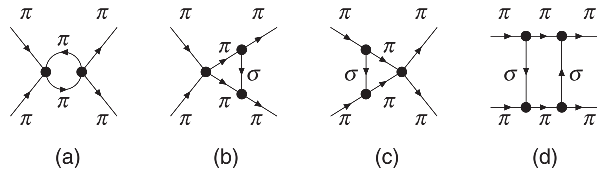

Fig. IV–2 A subset of one-loop diagrams contributing to $\pi^+\pi^0$ elastic scattering.

Let us again consider the process $\pi^+ + \pi^0 \to \pi^+ + \pi^0$, this time to one loop. The full linear sigma model is renormalizeable and will yield finite predictions in terms of the (renormalized) parameters of the theory. The effective theory has been constructed to have the same vertices at the lowest energies, but will have quite different high-energy properties because it is missing the extra high-energy degree of freedom. There will be new divergences present in perturbation theory. However, the low-energy effects will be similar in both calculations.

For example, consider the set of diagrams depicted in Fig. IV–2. In the full theory, all of these diagrams exist, and our previous result of Eq. (1.13) can be used to write the combined amplitudes as

$$i\mathcal{M}_{\rm full} = \int \frac{d^4k}{(2\pi)^4}\left[-2i\lambda + (-2i\lambda v)^2 \frac{i}{(k+p_+)^2 - m_\sigma^2}\right]\frac{i}{(k+p_+ + p_0)^2}\,\frac{i}{k^2} \times \left[-2i\lambda + (-2i\lambda v)^2 \frac{i}{(k+p'_+)^2 - m_\sigma^2}\right]. \tag{3.4}$$

The result is a sum of bubble, triangle, and box diagrams. The box in particular is a very complicated function of the kinematic invariants, involving di-logarithms ['tHV 79, DeNS 91, ElZ 08]. The divergence from the bubble diagram goes into the renormalization of the $\lambda\varphi^4$ coupling of the original lagrangian. For the effective theory, in contrast, one uses only pions and considers only the bubble diagram. The low-energy limit of the vertex is employed. Again, drawing from our results of Eq. (1.13), also visible by taking the leading approximation for the vertices in Eq. (3.4), one finds

$$i\mathcal{M}_{\rm eff} = \int \frac{d^4k}{(2\pi)^4}\,\frac{i(k+p_+)^2}{v^2}\,\frac{i}{(k+p_+ + p_0)^2}\,\frac{i}{k^2}\,\frac{i(k+p'_+)^2}{v^2}. \tag{3.5}$$

This diagram has a different divergence than the full theory. It is also much simpler kinematically, and its dimensional regularized form is easily evaluated as

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$$i\mathcal{M}_{\text{eff}} = \frac{i}{96\pi^2 v^4} s(s-u)\left[\frac{2}{4-d} - \gamma + \ln 4\pi - \ln\frac{-s-i\epsilon}{\mu^2}\right] + \frac{i}{288\pi^2 v^4}[2s^2 - 5su], \tag{3.6}$$

using the usual variables $s = (p_+ + p_0)^2,\ t = (p_+ - p'_+)^2,\ u = (p_0 - p'_+)^2$.

There are various interesting features of this result. Note that the whole amplitude is of order (energy)^4, while the original scattering vertex of Eq. (1.13) was of order (energy)^2. Technically, this follows simply from noting that the loop has factors of $1/v^4$ and that in dimensional regularization the only other dimensional factors are the external energies. On a more profound level it is an example of the energy expansion of the effective theory – loops produce results that are suppressed by higher powers of the momenta at low-energy. Because of this kinematic dependence, one can also readily see that the divergence cannot be absorbed into the renormalization of the original $\mathcal{O}(E^2)$ effective lagrangian. In fact we know that this divergence is spurious. It was generated because the effective theory had the wrong high-energy behavior compared to the full theory. This is to be expected in an effective theory – it does not pretend to know the content of the theory at all energies. However, the divergence will disappear in the matching of the two theories through the renormalization of a term in the $\mathcal{O}(E^4)$ lagrangian – this will be demonstrated below.

Even more interesting from the physics point of view is that the $s(s-u)\ln -s$ behavior is exactly what is found by taking the low-energy limit of the complicated result from the full theory and expanding it to this order in the momenta. This occurs because the $\ln -s$ factor comes from the low-energy regions of the loop momenta, of order $k \sim s$, so that the logarithm represents long-distance propagation.[5] Indeed, the imaginary part of the amplitude arising from $\ln(-s-i\epsilon) = \ln(s) - i\pi$ (for $s > 0$) comes from the on-shell intermediate state of two pions. This logarithm could never be represented by a local effective lagrangian and is a distinctive feature of long-distance (low-energy) quantum loops. These features match in the two calculations because when the loop momenta are small the effective field theory approximation for the vertex is valid. Overall, the effective field theory has an incorrect high-energy behavior but does capture the correct low-energy dynamics.

The comparison of the full theory and the effective theory can be carried out directly for this reaction. The dimensionally regularized result for the full theory is given in [MaM 08], but is too complicated to be reproduced here. However the

[5] Short distance pieces from higher values of k would be analytic functions able to be Taylor expanded around $s = 0$.

expansion of the full theory at low-energy in terms of renormalized parameters is relatively simple [GaL 84]

$$\begin{aligned}\mathcal{M}_{\text{full}} &= \frac{t}{v^2} + \left[\frac{1}{m_\sigma^2 v^2} - \frac{11}{96\pi^2 v^4}\right] t^2 \\ &\quad - \frac{1}{144\pi^2 v^4}[s(s-u) + u(u-s)] \\ &\quad - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{m_\sigma^2} + s(s-u)\ln\frac{-s}{m_\sigma^2} + u(u-s)\ln\frac{-u}{m_\sigma^2}\right]. \end{aligned} \tag{3.7}$$

The effective theory result [Le 72, GaL 84] has a very similar form but does not know about the existence of the σ,

$$\begin{aligned}\mathcal{M}_{\text{eff}} &= \frac{t}{v^2} + \left[8\ell_1^r + 2\ell_2^r + \frac{5}{192\pi^2}\right]\frac{t^2}{v^4} \\ &\quad + \left[2\ell_2^r + \frac{7}{576\pi^2}\right][s(s-u) + u(u-s)]/v^4 \\ &\quad - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{\mu^2} + s(s-u)\ln\frac{-s}{\mu^2} + u(u-s)\ln\frac{-u}{\mu^2}\right], \end{aligned} \tag{3.8}$$

where we have defined[6]

$$\begin{aligned}\ell_1^r &= \ell_1 + \frac{1}{384\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right] \\ \ell_2^r &= \ell_2 + \frac{1}{192\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]. \end{aligned} \tag{3.9}$$

At this stage we can match the two theories, providing identical scattering amplitudes to this order, through the choice

$$\begin{aligned}\ell_1^r &= \frac{v^2}{8m_\sigma^2} + \frac{1}{192\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right] \\ \ell_2^r &= \frac{1}{384\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right]. \end{aligned} \tag{3.10}$$

The reader is invited to compare this result with the tree-level matching, Eq. (2.10). We have not only obtained a more precise matching, we also have generated important kinematic dependence, particularly the logarithms, in the scattering amplitude.

We have seen that the predictions of the full theory can be reproduced even when using only the light degrees of freedom, as long as one chooses the coefficient of the effective lagrangian appropriately. This holds for *all* observables. Once

[6] Readers who compare with [GaL 84] should be aware that our normalization of the ℓ_i coefficients differs by a factor of four.

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the matching is done, other processes can be calculated using the effective theory without the need to match again for each process.[7] The total effect of the heavy particle, both tree diagrams and loops, has been reduced to a few numbers in the lagrangian which we have deduced from matching conditions to a given order in an expansion in the energy.

In this example we match to a known calculable theory. In other realizations of effective field theory, the full theory may be unknown (for example, in the case of gravity [Do 94]) or very difficult to calculate (as we will discuss for *QCD*). In cases where direct matching is not possible, the renormalized coefficients in the lagrangian could be determined through measurement. Measuring the value of the coefficients in one reaction would allow them to be used by the effective theory in other processes.

IV–4 General features of effective field theory

After this explicit example, let us think more generally about effective field theories. In quantum mechanics and quantum field theory, we face what appears to be an impossible situation. Intermediate states in perturbation theory and in loop diagrams include all energies, even beyond those which have been probed experimentally. Yet we expect more new particles and new interactions to be present eventually at higher energies. How can we then reliably perform *any* calculation without knowing the particles and interactions at all energies which enter in our calculations?

The answer essentially comes from the uncertainty principle. Effects from high energy appear local when viewed at low energy. This means that they are equivalent to terms in a local lagrangian. Most often the coefficient of a particular term in a lagrangian – a mass or a coupling constant – is something that we have to measure. So the effects of physics from high energy is contained in the parameters that we measure at low energy.

Effective field theory embraces this fact and uses it to perform calculations at low energy. In theories where the high-energy limit is known, such as our linear sigma model example above, the coefficients of the effective lagrangian can be determined by matching. In theories where the high-energy physics is not known, we still know that its effect is local, so that we parameterize it by the most general local lagrangian.

The decoupling theorem tells us that the high-energy effect appears in renormalized couplings or in terms suppressed by powers of the heavy scale. In this sense, all of our theories can be viewed as effective field theories. The class of renormalizable

[7] As part of our treatment of *QCD*, we show the universality of the renormalization in Appendix B–2.

field theories is a subset of effective field theories in which the power-suppressed lagrangians have not yet been needed.

Effective lagrangians and symmetries

What would happen if, instead of having a straightforward known theory like the linear sigma model, we were dealing with an unknown or unsolvable theory with the same $SU(2)_L \times SU(2)_R$ chiral symmetry? In this case there would exist some set of pion interactions which, although not explicitly known, would be greatly restricted by the $SU(2)$ chiral symmetry. Once again we could choose to describe the pion fields in terms of the exponential parameterization U, with a symmetry transformation

$$U \to LUR^\dagger \tag{4.1}$$

for L, R in $SU(2)$. Not having an explicit prescription, we would proceed to write out the most general effective lagrangian consistent with the chiral symmetry. In view of the infinite number of possible terms contained in such a description, this would appear to be a daunting process. However, the energy expansion allows it to be manageable.

It is not difficult to generate candidate interactions which are invariant under chiral $SU(2)$ transformations. For the purpose of illustration, we list the following two-derivative, four-derivative, and six-derivative terms in the exponential parameterization,

$$\begin{aligned} &\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right), \quad \mathrm{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \cdot \mathrm{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right), \\ &\quad \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) \cdot \mathrm{Tr}\left(\partial_\nu U \Box \partial^\nu U^\dagger\right) . \end{aligned} \tag{4.2}$$

There can be no derivative-free terms in a list such as this because $\mathrm{Tr}\left(U U^\dagger\right) = 2$ is a constant. It is clear that one can generate innumerable similar terms with arbitrary numbers of derivatives. The general lagrangian can be organized by the dimensionality of the operators,

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \cdots \\ &= \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \ell_1 [\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)]^2 \\ &\quad + \ell_2 \mathrm{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \cdot \mathrm{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right) + \cdots . \end{aligned} \tag{4.3}$$

The important point is that, at sufficiently low energies, the matrix elements of most of these terms are very small since each derivative becomes a factor of the momentum q when matrix elements are taken. It follows from dimensional analysis that the coefficient of an operator with n derivatives behaves as $1/M^{n-4}$, where

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M is a mass scale which depends on the specific theory. Therefore, the effect of an n-derivative vertex is of order E^n/M^{n-4}, and, at an energy small compared to M, large-n terms have a very small effect. At the lowest energy, only a single lagrangian, the one in Eq. (3.3) with two derivatives, is required. We shall call this an '$\mathcal{O}(E^2)$' contribution in subsequent discussions. The most important corrections to this involve four derivatives, and are therefore '$\mathcal{O}(E^4)$'. In practice then, the infinity of possible contributions is reduced to only a small number. The coefficients of these terms are not generally known, and must thus be determined phenomenologically. However, once fixed by experiment (or by matching to the full theory if possible) they can be used to allow predictions to be made for a variety of reactions.

Power counting and loops

It would appear that loop diagrams could upset the dimensional counting described above. This might happen in the calculation of a given loop diagram if, for example, two of the momentum factors from an $\mathcal{O}(E^4)$ lagrangian are involved in the loop and are thus proportional to the loop momentum. Integrating over the loop momentum apparently leaves only two factors of the 'low' energy variable. It would therefore seem that for certain loop diagrams, an $\mathcal{O}(E^4)$ lagrangian could behave as if it were $\mathcal{O}(E^2)$. If this happened, it would be a disaster because arbitrarily high order lagrangians would contribute at $\mathcal{O}(E^2)$ when loops were calculated. As we shall show, this does not occur. In fact, the reverse happens. When $\mathcal{O}(E^2)$ lagrangians are used in loops, they contribute to $\mathcal{O}(E^4)$ or higher.

Before we give the formal proof of this result, let us note that we saw this effect in the linear sigma model calculation above. We started by using the order E^2 lagrangian in the loop diagram and the result was the renormalization of a lagrangian at order E^4. It is also straightforward to demonstrate why this occurs. Consider a pion loop diagram, as in Fig. IV–2. From the explicit form displayed in Eq. (3.5), we see that

$$\mathcal{M}^{(\text{loop})}_{\pi^+\pi^0\to\pi^+\pi^0} \equiv \frac{1}{v^4} I(p_+, p_0, p'_+), \tag{4.4}$$

where I is the loop integral with the factor v^{-4} extracted. Counting powers of energy factors is most easily done in dimensional regularization. The loop integral contains no dimensional factors other than p_+, p_0, and p'_+. Since, in four dimensions it has the overall energy unit E^4, it must therefore be expressible as fourth order in momentum. Despite the loop integration, the end result is expressed only in terms of the external momenta. These momenta are small, and hence all the energy factors involved in power counting are taken at low-energy. In dimensional regularization, there can also be a dependence on the arbitrary scale μ,

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$$\int d^4\ell \rightarrow \mu^{4-d} \int d^d\ell, \tag{4.5}$$

but in the limit $d \rightarrow 4$ this occurs only in dimensionless logarithms such as $\ln(E^2/\mu^2)$. Thus, the order of momentum can be found by counting the factors of $1/v^2$ which occur for every vertex from the lowest-order lagrangians. Each factor of $1/v^2$ must be accompanied by momenta in the numerator in order to produce a dimensionless amplitude. Each vertex in a diagram contributes powers of $1/v^2$, and higher-order loop diagrams require more vertices. Thus, every time a loop is formed, the overall momentum power of the amplitude must *increase* rather than decrease.

We have also seen that any divergences present can be handled in the usual way, by renormalizations of the parameters in the theory. Again, the uncertainty principle comes into play – the divergences come from the extreme high-energy part of the calculation and thus they must look like some term in a local lagrangain. If the original effective lagrangian which we have written down is indeed the most general one consistent with the given symmetry, then it must have enough parameters of the right form to encompass any divergences which occur. In particular, our power-counting argument tells us that when $\mathcal{L}_2$ is used in one-loop diagrams, the divergences are of order E^4 and should be capable of being absorbed into the parameters of that order. Since the parameters are generally unknown and are to be determined phenomenologically, the only difference this makes is to cast physical results in terms of the renormalized parameters instead of the bare ones.

Weinberg's power-counting theorem

To prove this result [We 79b], consider some diagram with a total of N_V vertices. Then letting N_n be the number of vertices arising from the subset of effective lagrangians which contain n derivatives (e.g. N_4 is the number of vertices coming from four-derivative lagrangians), we have $N_V = \Sigma_n N_n$. The overall energy dimensionality of the coupling constants is thus M^{N_C} with

$$N_C = \sum_n N_n(4-n), \tag{4.6}$$

where M is a mass scale entering into the coefficients of the effective lagrangian (e.g., the quantity v in the sigma model). Each pion field comes with a factor of $1/v$, so that associated with N_E external pions and N_I internal pion lines is an energy factor $(1/M)^{2N_I+N_E}$. (Recall that two pions must be contracted to form an internal line.) However, the number of internal lines can be eliminated in terms of the number of vertices and loops (N_L),

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$$N_I = N_L + N_V - 1 = N_L + \sum_n N_n - 1. \tag{4.7}$$

Any remaining dimensional factors must be made up of powers of the energy E times a dimensionless factor of E/μ where μ is the scale employed for renormalizing the coupling constants. (When using dimensional regularization, these factors of E/μ enter only in logarithms.) Thus the overall matrix element is composed of energy factors

$$\begin{aligned}\mathcal{M} &\sim (M)^{\sum_n N_n(n-4)} \frac{1}{M^{N_E+2N_L+2\sum_n N_n-2}} E^D F(E/\mu) \\ &\sim (\text{mass or energy})^{4-N_E},\end{aligned} \tag{4.8}$$

where the second line is the overall dimension of an amplitude with N_E external bosons. The renormalization scale μ can be chosen of the order of E so no large factors are present in $F(E/\mu)$. Overall the energy dimension is then

$$D = 2 + \sum_n N_n(n-2) + 2N_L. \tag{4.9}$$

A diagram containing N_L loops contributes at a power E^{2N_L} higher than the tree diagrams. This theorem is of great practical consequence. At low energy, it allows one to work with only small numbers of loops. In particular, at $\mathcal{O}(E^4)$ only one-loop diagrams generated from $\mathcal{L}_2$ need to be considered.

The end result is a very simple rule for counting the order of the energy expansion. The lowest-order (E^2) behavior is given by the two-derivative lagrangians treated at tree level. There are two sources at the next order (E^4): (i) the $\mathcal{O}(E^2)$ one-loop amplitudes, and (ii) the tree-level $\mathcal{O}(E^4)$ amplitudes. When the coefficients of the E^4 lagrangians are renormalized, finite predictions result. Other effective field theories will have power-counting rules analogous to this one appropriate for chiral theories.

The limits of an effective field theory

The effective field theory of the linear sigma model is valid for energies well below the mass of the scalar particle in the theory, the σ or S. Once there is enough energy to directly excite the S particle, it is clear that the effective theory is inadequate. This energy scale is visible even within the effective theory itself. Scattering matrix elements are an expansion in the energy, with a schematic form

$$\mathcal{M} \sim \frac{q^2}{v^2}\left[1 + \frac{q^2}{m_\sigma^2} + \cdots\right] \tag{4.10}$$

and the scale of the energy dependence is determined largely by the scalar mass. As the energy increases the corrections to the lowest-order result grow and eventually

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all terms in the energy expansion become equally important and the effective theory breaks down. Thus, the effective theory reveals its own limits.

In more general effective field theories, there is always a separation of the heavy degrees of freedom, which are integrated out from the theory, and the light degrees of freedom, which are treated dynamically. In many instances, the natural separation scale is set by a particle's mass, as in the linear sigma model. We will see that in the case of *QCD*, the meson resonances such as the $\rho(770)$ do not appear explicitly in the low-energy effective theory. Therefore, these have been integrated out and help define the limits of the effective field theory. In other cases, we could integrate some of the high-momentum modes of certain fields, while still keeping the low-momentum modes of these same fields as active dynamical participants in the low-energy theory. This is done for the effective hamiltonian for weak decays, where we integrate out the high-energy modes of the gluonic fields. In these cases, the scale that we have used to separate high and low energy defines the limit of validity of the effective field theory.

Let us also address a rather subtle point concerning the energy scale of the effective theory. While we regularly use this idea of an energy scale defining the limit of validity of the effective theory, there are times that we do not apply this separation fully. In loop diagrams, if we wanted to only include loop effects below a certain energy scale, we would need to employ a cut-off in the loop integral. This is often inconvenient and if done carelessly could upset some of the symmetries of the theory. Moreover, the presence of an additional dimensional factor in loop diagrams would upset some of the power-counting arguments described above. Most often, practical calculations are performed using dimensional regularization. This regulator has no knowledge of the energy scale of the theory and thus loop diagrams will in general include effects from energies where the effective theory is not valid. However, again the uncertainty principle comes to our rescue. Even if these spurious high-energy contributions are not correct, we know that their effect is equivalent to a local term in the effective lagrangian. Any mistakes made in the loop can be corrected by modifying the coefficients of the terms in the effective lagrangian. Careful application of the procedures for matching or measuring the parameters will return the the same physical predictions independent of the choice of regularization scheme.

IV–5 Symmetry breaking

Effective lagrangians can be used not only in the limit of exact symmetry but also to analyze the effect of small symmetry breaking. Let us first return to the sigma model for an illustration of the method, and then consider the general technique.

The $SU(2)_L \times SU(2)_R$ symmetry of the sigma model is explicitly broken if the potential $V(\sigma, \boldsymbol{\pi})$ is made slightly asymmetric, e.g., by the addition of the term

$$\mathcal{L}_{\text{breaking}} = a\sigma = \frac{a}{4}\,\text{Tr}\,\left(\Sigma + \Sigma^{+}\right) \tag{5.1}$$

to the basic lagrangian of Eq. (1.4). To first order in the quantity a, this shifts the minimum of the potential to

$$v = \sqrt{\frac{\mu^2}{\lambda}} + \frac{a}{2\mu^2}, \tag{5.2}$$

and produces a pion mass

$$m_\pi^2 = \frac{a}{v}. \tag{5.3}$$

Although the latter result can be found by using the linear representation and expanding the fields about their vacuum expectation values, it is easier to use the exponential representation,

$$\begin{aligned}\mathcal{L}_{\text{breaking}} &= \frac{a}{4}(v+S)\,\text{Tr}\,(U+U^\dagger) = \frac{a}{4}(v+S)\,\text{Tr}\left(2 - \left(\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{v}\right)^2 + \cdots\right) \\ &= a(v+S) - \frac{a}{2v}\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \cdots = a(v+S) - \frac{m_\pi^2}{2}\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \cdots .\end{aligned} \tag{5.4}$$

The chiral $SU(2)$ symmetry is seen to be slightly broken, but the vectorial $SU(2)$ isospin symmetry remains exact.

As we have seen, the $\mathcal{O}(E^2)$ lagrangian is obtained by setting $S = 0$,

$$\mathcal{L}_2 = \frac{v^2}{4}\,\text{Tr}\,\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{m_\pi^2}{4} v^2\,\text{Tr}\,\left(U + U^\dagger\right). \tag{5.5}$$

Higher-order terms will contain products like

$$\left[m_\pi^2\,\text{Tr}\,\left(U+U^\dagger\right)\right]^2, \quad m_\pi^2\,\text{Tr}\,\left(U+U^\dagger\right)\cdot\,\text{Tr}\,\left(\partial_\mu U \partial^\mu U^\dagger\right), \;\ldots, \tag{5.6}$$

and can be obtained by integrating out the field S as was done in Sect. IV–2. It is important to realize that the symmetry-breaking sector also has a low-energy expansion, with each factor of m_π^2 being equivalent to two derivatives. If m_π^2 is small, *the expansion is a dual expansion in both the energy and the mass*.

If we encounter a theory more general than the sigma model, the effect of a small pion mass can be similarly expressed in low orders by,

$$\begin{aligned}\mathcal{L}_{\text{breaking}} &= a_1 m_\pi^2\,\text{Tr}\,\left(U+U^\dagger\right) + a_2\left[m_\pi^2\,\text{Tr}\,\left(U+U^\dagger\right)\right]^2 \\ &+ a_3 m_\pi^2\,\text{Tr}\,\left(U+U^\dagger\right)\,\text{Tr}\,\left(\partial_\mu U\partial^\mu U^\dagger\right) + a_4 m_\pi^2\,\text{Tr}\,\left[(U+U^\dagger)\partial_\mu U\partial^\mu U^\dagger\right],\end{aligned} \tag{5.7}$$

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with coefficients that are generally not known. An important consideration is the symmetry-transformation property of the perturbation. The symmetry-breaking term of Eq. (5.1) is not invariant under separate left-handed and right-handed transformations but only under those with $L = R$. All the terms in Eq. (5.7) have this property.

Other symmetry breakings can be analyzed in a manner analogous to the treatment just given of the mass term. One identifies the symmetry-transformation property of the perturbing effect and writes the most general effective lagrangian with that property. Most often the perturbation is treated to only first order, but higher-order behavior can also be studied.

IV–6 Matrix elements of currents

There is an elegant technique which allows one, at a minimal increase in complexity, to calculate matrix elements of currents from a chiral effective lagrangian [GaL 84, 85*a*]. The idea is to add to the lagrangian terms containing external sources coupled to the currents in question. Construction of the effective lagrangian, including source terms, then allows the current matrix elements to be easily identified. We shall explain this technique here, and use it extensively in our discussion of *QCD* in subsequent chapters.

First, consider how current matrix elements are identified in a path-integral framework. We have seen in Chap. III (see also App. A) that by adding a source coupled to the desired current, matrix elements can be obtained from differentiation of the path integral, e.g., Eqs. (III–2.2), (III–2.4). For example, we can modify three-flavor *QCD* by adding sources to obtain

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \bar{\psi}i\not{D}\psi - \bar{\psi}\gamma_\mu\frac{1+\gamma_5}{2}\ell^\mu\psi - \bar{\psi}\gamma_\mu\frac{1-\gamma_5}{2}r^\mu\psi \\ &- \bar{\psi}_L(s+ip)\psi_R - \bar{\psi}_R(s-ip)\psi_L,\end{aligned} \tag{6.1}$$

where ℓ_μ, r_μ, s, p are 3×3 matrix source functions expressible as

$$\ell_\mu = \ell^0_\mu + \ell^a_\mu\lambda^a, \quad r_\mu = r^0_\mu + r^a_\mu\lambda^a, \quad s = s^0 + s^a\lambda^a, \quad p = p^0 + p^a\lambda^a, \tag{6.2}$$

with $a = 1, \ldots, 8$. The lagrangian in Eq. (6.1) reduces to the usual *QCD* lagrangian in the limit $\ell_\mu = r_\mu = p = 0$, $s = \mathbf{m}$, where $\mathbf{m}$ is the 3×3 quark mass matrix. The electromagnetic coupling can be obtained with the choice $\ell_\mu = r_\mu = eQA_\mu$, where A_μ is the photon field and Q is the electric charge operator defined in units of e. Various currents can be read off from the lagrangian, such as the left-handed current

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$$J^k_{L\mu}(x) = -\frac{\partial \mathcal{L}}{\partial \ell^\mu_k(x)} = \bar{\psi}(x)\gamma_\mu \frac{1+\gamma_5}{2}\lambda^k \psi(x) \tag{6.3}$$

or the scalar density

$$\bar{\psi}(x)\psi(x) = -\frac{\partial \mathcal{L}}{\partial s^0(x)}. \tag{6.4}$$

Moreover, matrix elements of these currents can be formed from the path integral by taking functional derivatives. The simplest example is

$$\langle 0 \left| \bar{\psi}(x)\psi(x) \right| 0\rangle = i\,\frac{\delta \ln Z}{\delta s^0(x)}\bigg|_{\substack{\ell=r=p=0 \\ s=m}}, \tag{6.5}$$

while other examples appear in Sect. III–2.

Matrix elements and the effective action

A low-energy effective action for the Goldstone bosons of *QCD* will be a functional of the external sources. One way to define the connection of the effective action with *QCD* is to consider the effect of the sources,

$$e^{iW(\ell_\mu, r_\mu, s, p)} = \int [d\psi]\left[d\overline{\psi}\right]\left[dA^a_\mu\right]\, e^{i\int d^4x\ \mathcal{L}_{QCD}(\psi,\overline{\psi},A^a_\mu,\ell_\mu,r_\mu,s,p)}. \tag{6.6}$$

At low-energy, all heavy degrees of freedom can be integrated out and absorbed into coefficients in the effective action W. However, the Goldstone bosons propagate at low-energy, and they must be explicitly taken into account. One then writes a representation of the form

$$e^{iW(\ell_\mu, r_\mu, s, p)} = \int [dU]\, e^{i\int d^4x\ \mathcal{L}_{\rm eff}(U,\ell_\mu,r_\mu,s,p)}, \tag{6.7}$$

where as usual U contains the Goldstone fields. This form then allows inclusion of all low-energy effects while maintaining the symmetries of *QCD*.

The lagrangian of Eq. (6.1) has an exact *local* chiral $SU(3)$ invariance if we have the external fields transform in the same way as gauge fields. In particular, the transformations

$$\begin{aligned} \psi_L &\to L(x)\psi_L, \qquad \psi_R \to R(x)\psi_R, \\ \ell_\mu &\to L(x)\ell_\mu L^\dagger(x) + i\partial_\mu L(x)L^\dagger(x), \\ r_\mu &\to R(x)r_\mu R^\dagger(x) + i\partial_\mu R(x)R^\dagger(x), \\ (s+ip) &\to L(x)(s+ip)R^\dagger(x) \end{aligned} \tag{6.8}$$

provide an invariance for any $L(x)$, $R(x)$ in $SU(3)$.

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In constructing the effective action, these invariances must be included. This is easy to do if ℓ_μ and r_μ enter in the same way as gauge fields. In particular, upon defining a covariant derivative

$$D_\mu U = \partial_\mu U + i\ell_\mu U - iUr_\mu, \tag{6.9}$$

and field-strength tensors

$$\begin{aligned} L_{\mu\nu} &= \partial_\mu \ell_\nu - \partial_\nu \ell_\mu + i[\ell_\mu, \ell_\nu], \\ R_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu + i[r_\mu, r_\nu], \end{aligned} \tag{6.10}$$

we obtain the following covariant responses to local transformations:

$$\begin{array}{llll} U & \to L(x)UR^\dagger(x), & D_\mu U \to L(x)D_\mu U R^\dagger(x), \\ L_{\mu\nu} & \to L(x)L_{\mu\nu}L^\dagger(x), & R_{\mu\nu} \to R(x)R_{\mu\nu}R^\dagger(x). \end{array} \tag{6.11}$$

The effective action is then expressed in terms of these quantities. At order E^2, there are only two terms in the effective lagrangian,

$$\mathcal{L}_2 = \frac{F_\pi^2}{4} \operatorname{Tr}\left(D_\mu U D^\mu U^\dagger\right) + \frac{F_\pi^2}{4} \operatorname{Tr}\left(\chi U^\dagger + U\chi^\dagger\right), \tag{6.12}$$

where

$$\chi \equiv 2B_0(s + ip) \tag{6.13}$$

and B_0 is a constant with the dimension of mass. In the limit $\ell_\mu = r_\mu = p = 0$, $s = m$, this is the same effective lagrangian with which we have been dealing in the $SU(2)$ examples, with the identification $m_\pi^2 = (m_u + m_d)B_0$. Note that this usage requires B_0 to be positive.

Having constructed the effective action, we can obtain a number of interesting matrix elements. For example, use of Eq. (6.5) provides the identification of the vacuum scalar-density matrix element as

$$\langle 0 \left| \bar{\psi}_i \psi_j \right| 0 \rangle = -F_\pi^2 B_0 \delta_{ij} \tag{6.14}$$

to this order in the effective lagrangian. Similarly, use of Eq. (6.3) reveals the left-handed current to be

$$L_\mu^k = -i\frac{F_\pi^2}{2} \operatorname{Tr}\left(\lambda^k U \partial_\mu U^\dagger\right). \tag{6.15}$$

One other advantage of the source method is to allow the use of the equations of motion. The standard Noether procedure for identifying currents does not work if the equations of motion are employed in the lagrangian. To become convinced of this, one can consider the following exercise. We examine the response of the two trial lagrangians,

$$\mathcal{L}_1 = \varphi^* \Box \varphi, \quad \mathcal{L}_2 = -m^2 \varphi^* \varphi \tag{6.16}$$

to a phase transformation $\varphi \to e^{i\alpha}\varphi$. The first contributes to the Noether current while the second does not. However, these two forms are identical on-shell if φ satisfies the Klein–Gordon equation. In an effective lagrangian which is meant to be used always on-shell it is often convenient to drop terms which vanish by virtue of the equations of motion. The use of source fields as described above avoids this problem.

IV–7 Effective field theory of regions of a single field

In our presentation earlier in this chapter, the construction of an effective field theory was described by the integrating out of heavy particles, while leaving the light particles as dynamical degrees of freedom. However, often one can make an effective field theory from a single particle. In this case, certain energy regions of the field are treated as heavy and others are light, and one retains the light regions in the effective field theory. Indeed, sometimes there are multiple regions that are 'light' in some sense, and one splits the original single field into multiple fields. This section provides some of the background for such decompositions.

The simplest example of the division of a single field into 'heavy' and 'light' is in the nonrelativistic reduction. When the energy is small, the antiparticle degrees of freedom are heavy and can be removed from the theory, leaving a nonrelativistic particle description. For example, if one redefines a four-component Dirac field ψ into upper and lower two-component fields, ψ_u and ψ_ℓ by factoring out the leading energy dependence at low-energy via

$$\psi(x,t) = e^{-imt}\begin{pmatrix}\psi_u(x,t)\\ \psi_\ell(x,t)\end{pmatrix}, \tag{7.1}$$

ψ_u will behave as a nonrelativistic field and ψ_ℓ will account for the two heavy degrees of freedom. The free Dirac lagrangian shows this separation,

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ &= \psi_u^* i\partial_t \psi_u + \psi_\ell^*[i\partial_t + 2m]\psi_\ell + \psi_u^* i\sigma\cdot\nabla\psi_\ell + \psi_\ell^* i\sigma\cdot\nabla\psi_u .\end{aligned} \tag{7.2}$$

While no approximation has yet been made by this redefinition, the nonrelativistic limit is taken by assuming that the residual energy dependence is small compared to the mass (i.e., one neglects ∂_t compared to $2m$). One can then integrate out the lower component through its equation of motion,

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$$(i\partial_t + 2m)\psi_\ell \approx 2m\ \psi_\ell = i\sigma \cdot \nabla\psi_u, \tag{7.3}$$

leaving the upper component as the active non-relativistic degree of freedom.

$$\mathcal{L} = \psi_u^* i\partial_t \psi_u - \frac{\left(\nabla\psi_u^*\right) \cdot \nabla\psi_u}{2m}. \tag{7.4}$$

With inclusion of the interactions, this can lead to a full nontrivial effective field theory. A well-developed example of this is the Non-Relativistic *QCD* (*NRQCD*) effective field theory [CaL 86]. We will also return to this procedure in more generality in the discussion of Heavy Quark Effective Theory (*HQET*) in Chap. XIII.

A second common way of splitting up a single field is to integrate out the high momentum portions of a field. This logic is often called Wilsonian [Wi 69]. Imagine splitting the momenta in a problem into those above an energy scale Λ and those below this scale. By first performing the calculation of the high-energy portion, one is left with an effective field theory. The operators defining that theory will carry factors, the Wilson coefficients, that depend on the scale Λ. This means that one obtains a set of new operators $\mathcal{O}_n$ in the lagrangian

$$\mathcal{L} = \cdots + \sum_n \mathcal{C}_n(\Lambda)\mathcal{O}_n, \tag{7.5}$$

where $\mathcal{C}_n(\Lambda)$ are the Wilson coefficients and the series is infinite. The operators are local because they capture high-energy physics, and their matrix elements will depend on the separation scale, $\langle\mathcal{O}_n\rangle = \langle\mathcal{O}_n(\Lambda)\rangle$. One regularly uses the renormalization group to describe the running of the Wilson coefficients with changes of scale. The low-energy theory remains a full field theory and one must calculate the full quantum effects in the matrix elements of $\mathcal{O}_n$ up to the scale Λ. When the high-energy physics in $\mathcal{C}_n$ and the low-energy physics in the matrix elements of $\mathcal{O}_n$ are properly matched, in the end the separation scale Λ will disappear from the description. Nevertheless, this separation is often useful. For example, in *QCD* the high-energy behavior may be reliably calculated in perturbation theory, while the low-energy behavior may be best accomplished with lattice calculations. Examples appearing in this book include the Wilson coefficients of the non-leptonic weak hamiltonian, cf. Sect. VIII–3, and those used in *QCD* sum rules, cf. Sect. XI–5.

In practice, however, we most often do *not* use a Wilsonian separation scale Λ, but instead employ dimensional regularization. Dimensional regularized loop integrals do not carry information about any particular scale, and therefore extend over all energies. The extension to $d < 4$ damps the high-energy divergences in a scale-independent way. Nonetheless, this procedure works for logarithmically running Wilson coefficients. Aside from the momenta, the only scale in a dimensionally regularized integral is the $\mu^{2\epsilon}$ inserted in front of the loop integral. This ends up

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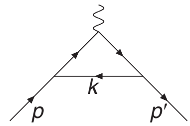

Fig. IV–3 The scalar vertex diagram analysed in the text.

appearing in the final answer as $\ln \mu^2$ when expanded close to $d = 4$. The fact that cut-off regularization and dimensional regularization have the equivalence

$$\ln \Lambda^2 \Leftrightarrow \frac{1}{\epsilon} + \ln \mu^2 \tag{7.6}$$

allows the scale μ to be a proxy for the separation scale Λ. However, the correspondence of μ with a Wilsonian separation scale does not hold for Wilson coefficients with power-law running [CiDG 00].

For a yet more subtle example, consider the interaction of a high-energy massless particle in the vertex diagram of Fig. IV–3. For the purposes of this example, let us consider these as scalars and the current vertex as $J = \varphi^2/2$. We can analyse the resultant scalar vertex integral,

$$I = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2} \frac{1}{k^2} \frac{1}{(p'+k)^2}, \tag{7.7}$$

in the limit where $p^2 \sim p'^2 \ll Q^2 = (p - p')^2$. The only scales in this problem are Q^2, which is treated as a large scale, and $p^2 \sim p'^2$, which is the small scale. The relative size is labeled $\lambda^2 \sim p^2/Q^2 \sim p'^2/Q^2$.

This integral can be analyzed by the *method of regions* [BeS 98, Sm 02].[8] In this technique, one identifies all the important momentum regions of the loop integral, and makes appropriate approximations within each region. A portion of the integral will have all the components of the loop momenta of order Q and higher. This will be called the *hard* region. A region labeled *soft* has all the components much smaller than Q. In addition, there will be regions where the momentum is of order Q in the direction of p or p'. In these *collinear* regions, some invariant products can be smaller than Q^2.

In order to quantify this one takes light-like reference four-vectors

$$n^\mu = (1, 0, 0, 1), \quad \bar{n}^\mu = (1, 0, 0, -1), \quad n^2 = \bar{n}^2 = 0, \quad n \cdot \bar{n} = 2. \tag{7.8}$$

For an arbitrary four-vector expressed using these and a transverse component,

$$V^\mu = n \cdot V \frac{\bar{n}^\mu}{2} + \bar{n} \cdot V \frac{n^\mu}{2} + V_\perp^\mu \equiv V_+ \frac{\bar{n}^\mu}{2} + V_- \frac{n^\mu}{2} + V_\perp^\mu, \tag{7.9}$$

the invariant product is

[8] This example and the treatment of it follows the lectures of [Be 10].

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$$V^2 = (n \cdot V)(\bar{n} \cdot V) + V_\perp^2 = V_+ V_- + V_\perp^2.$$

$$A_\mu B^\mu = \frac{1}{2}(A_+ B_- + A_- B_+) + A_\perp \cdot B_\perp \tag{7.10}$$

These are useful because we can choose a frame with p along n and with p' along $\bar{n}$, and we can refer to the n direction as 'right' and the $\bar{n}$ direction as 'left'. This allows us to classify the different regions. Of the original momenta, we have

$$\begin{aligned} &(V_+, V_-, V_\perp) \\ p \sim &(\lambda^2, \;\; 1, \;\; 0)\; Q \\ p' \sim &(1, \;\; \lambda^2, \;\; 0)\; Q. \\ Q \sim &(1, \;\; 1, \;\; 0)\; Q \end{aligned} \tag{7.11}$$

Q is a hard momentum because it takes a hard interaction to change an energetic right-moving particle into one moving left. Using this decomposition, one can identify the regions of the loop momentum

$$\begin{aligned} &(k_+, k_-, k_\perp) \\ k \sim &(1, \;\; 1, \;\; 1)\; Q \quad \text{hard} \\ k \sim &(\lambda^2, \;\; 1, \;\; \lambda)\; Q \quad \text{collinear R.} \\ k \sim &(1, \;\; \lambda^2, \;\; \lambda)\; Q \quad \text{collinear L} \\ k \sim &(\lambda^2, \lambda^2, \lambda^2)\; Q \quad \text{soft} \end{aligned} \tag{7.12}$$

In each region, one can drop small momentum components in terms of large ones. For example, when k is in the hard region, one can drop p^2, p'^2, $k_- p_+$, $k_+ p'_-$, which are all of order λ^2, in order to obtain[9]

$$\begin{aligned} I_{\text{hard}} &= \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)(k^2 + k_- p_+ + i\epsilon)(k^2 + k_+ p'_- + i\epsilon)} \\ &= \frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2} Q^2} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{-Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{-Q^2} - \frac{\pi^2}{6} \right]. \end{aligned} \tag{7.13}$$

Similarly, in the right collinear region, one can expand $(k + p')^2 = k_- p'_+ + \mathcal{O}(\lambda^2)$, such that

$$\begin{aligned} I_{\text{col}-R} &= \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\epsilon)((k + p)^2 + i\epsilon)(k_- p'_+ + i\epsilon)} \\ &= \frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2} Q^2} \left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{-p'^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{-p'^2} + \frac{\pi^2}{6} \right]. \end{aligned} \tag{7.14}$$

An observation that will be relevant for the eventual construction of an effective theory is that when the exchanged propagator carrying momentum k is in the right

[9] The integrals of this section are displayed in the useful appendix of [Sm 02]. See also [Sm 12].

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collinear region, the other propagator on the p side is also collinear, but the third propagator on the p' side is hard. A similar result is obviously found when k is in the left collinear region, obtained by replacing p by p'. Finally, in the soft region, one keeps only terms of order λ^2, finding

$$\begin{aligned} I_{\text{soft}} &= \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left(k^2 + i\epsilon\right)\left(k_- p_+ + p^2 + i\epsilon\right)\left(k_+ p'_- + p'^2 + i\epsilon\right)} \\ &= \frac{i\Gamma\,(1+\epsilon)}{(4\pi)^{d/2}\,Q^2} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2\,Q^2}{-p^2\,p'^2} + \frac{1}{2} \ln^2 \frac{\mu^2\,Q^2}{-p^2\,p'^2} + \frac{\pi^2}{6}\right]. \end{aligned} \tag{7.15}$$

If one tries to identify other regions besides these and makes the corresponding simplifications of the loop integral, one ends up with a scale-less integral which vanishes within dimensional regularization. For example, if one considers the region where k scales as $k \sim (\lambda^2, \lambda^2, \lambda)$[10], one would use $k^2 \sim k_\perp^2$ and keep terms of order λ^2 in each propagator

$$\begin{aligned} I' &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left(k_\perp^2 + i\epsilon\right)\left(k_- p_+ + p^2 + k_\perp^2 + i\epsilon\right)\left(k_+ p'_- + p'^2 + k_\perp^2 + i\epsilon\right)} \\ &= \frac{1}{p_+ p'_-} \int \frac{d^d k'}{(2\pi)^d} \frac{1}{\left(k_\perp^2 + i\epsilon\right)\left(k'_- + i\epsilon\right)\left(k'_+ + i\epsilon\right)} \\ &= 0, \end{aligned} \tag{7.16}$$

where in the second line we have defined shifted variables $k'_- = k_- + (p^2 + k_\perp^2)/p_+$ and $k'_+ = k_+ + (p'^2 + k_\perp^2)/p'_-$, with the result being an integral without any scale. Such integrals are set to zero within dimensional regularization.

The sum of the four subregions yields the correct total integral,

$$I = \frac{i}{16\pi^2 Q^2} \left[\ln \frac{Q^2}{p^2}\ \ln \frac{Q^2}{p'^2} + \frac{\pi^2}{3}\right], \tag{7.17}$$

up to terms suppressed by powers of λ. As expected, this result is finite, even though the integrals from the individual regions are not. The approximations that we made lead to infrared divergences in the hard integral, and ultraviolet divergences in the others. However, these cancel when added together.

The other interesting feature of this procedure is that we have not restricted the integration ranges when calculating the integrals for the different regions. The full integration range is used in each case. The reason that this does not amount to double counting within dimensional regularization is that if there is a single unique scale within the integral, as has been deliberately constructed in each region, the integral is determined by momenta around that scale. This is the key observation

[10] This region is referred to as the *Glauber* region. The treatment of the integral given in the text appears adequate for this example, although the understanding of the Glauber region is still evolving [BaLO 11].

that allows the method of regions to work. By constructing approximations that scale in unique fashions, one can isolate the physics of that region alone.[11] That this actually happens in these integrals can be seen from the above integral where the factors of Q^2, p^2, and $p^2 p'^2/Q^2$ all signal the dominant scale in the respective diagrams, showing that the effects come from different regions of the momentum integration.

One can convert the analysis of the method of regions into an effective field theory whose applicability extends beyond this particular example. The initial field can be divided up into new effective fields for each of the important regions. The goal is to choose these fields and their interactions to yield the same results as the method of regions analysis outlined above. The hard-momentum region can be integrated out completely and replaced by effective operators of the light fields. These operators will come with Wilson coefficients to ensure the matching with the full calculation. However, the dynamical light fields need to come in three varieties for the different light-momentum regions. Thus, the original scalar field $\varphi(x)$ now comes in three varieties, $\varphi(x) = \varphi_{cR}(x) + \varphi_{cL}(x) + \varphi_s(x)$. The interactions of the light fields among themselves is relatively simple to construct. If the interaction vertex of the original theory was a simple φ^3 vertex, we expand that to include the possible interaction between the light fields,

$$-\mathcal{L} = \frac{g}{3}\varphi^3 \to \frac{g}{3}\varphi_{cR}^3 + \frac{g}{3}\varphi_{cL}^3 + \frac{g}{3}\varphi_s^3 + g\varphi_{cR}^2\varphi_s + g\varphi_{cL}^2\varphi_s. \tag{7.18}$$

Vertices not listed above, such as $\varphi_{cR}\varphi_s^2$, are ones which cannot occur due to momentum conservation (e.g., a collinear particle cannot split into two soft particles).

It is somewhat more subtle to choose the other effective operators and their Wilson coefficients. For the scalar example shown above, the 'current' carrying the momentum Q in the full theory is $J = \varphi^2/2$. Since it transfers this large momentum it can connect φ_{cR} to φ_{cL} such that we expect a vertex $J \sim \varphi_{cR}\varphi_{cL}$. However, in addition we need to recall that we have integrated out the hard scalars. This leads to additional vertices. For example, in the diagram of Fig. IV–4(a) the propagator is hard because it carries the momenta of both left-moving and right-moving fields, which couple to it at the lower vertex. When the other fields are light, this propagator shrinks to a point vertex as in Fig. IV–4(b). This, then, is a new contribution to the current operator, and we expect that the current has the form

$$J = \mathcal{C}_2\varphi_{cR}\varphi_{cL} + \mathcal{C}_3\varphi_{cR}^2\varphi_{cL} + \mathcal{C}_3'\varphi_{cR}\varphi_{cL}^2 + \cdots, \tag{7.19}$$

[11] In cases where regions are defined which have overlapping contributions there are also methods for cleanly separating the regions [MaS 07].

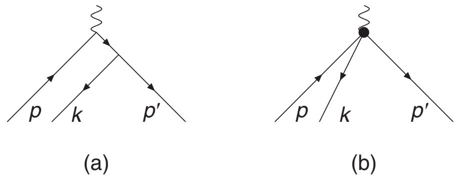

Fig. IV–4 (a) An interaction of collinear particles through a hard propagator; (b) the effective local vertex representing this interaction at low-energy.

where $\mathcal{C}_2$ and $\mathcal{C}_3$ are the Wilson coefficients. Calculation from the original theory shows that to this order

$$\mathcal{C}_2 = 1 + g^2 I_{\text{hard}}, \qquad \mathcal{C}_3 = \frac{2g}{k_- p'_+ - i\epsilon}, \qquad C'_3 = \frac{2g}{k_+ p_- - i\epsilon}, \tag{7.20}$$

where I_{hard} refers back to Eq. (7.13).

At this stage, we can reproduce the original vertex calculation using the effective theory by the calculation of the diagrams of Fig. IV–5. The diagrams of Fig. IV–5 (a),(b),(c) refer to the new vertices given in Eq. (7.20), while Fig. IV–5 (d) refers to the soft contribution of Eq. (7.15). By construction, one can see how all four of the regions of the original diagram are reproduced. We note how the hard propagators that occur when k is in one of the collinear regions have been accounted for by a new local vertex in the current operator, with the Wilson coefficient describing the effect of the hard propagator.

The reader may object that the construction of the effective theory was more trouble than evaluating the original diagram. However, once we have developed the effective theory, we can apply it in multiple new contexts. The example above is analogous to the Soft Collinear Effective Theory (*SCET*) of *QCD* [BaFPS 01]. Similar techniques are used in the various realizations of *NRQCD* [CaL 86, PiS 98, BrPSV 05]. Outside of the Standard Model, related methods are applied in the classical effective field theory of General Relativity [GoR 06], which has been used to systematize the classical treatment of gravitational radiation from binary systems [PoRR 11]. Further development of the method of regions and threshold expansions can be found in [BeS 98, Sm 02].

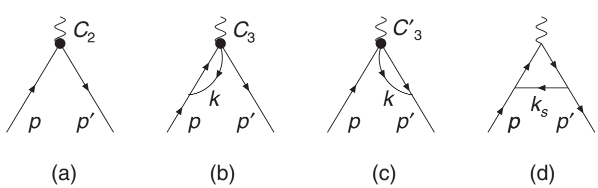

Fig. IV–5 The diagrams involving the light fields reconstructing the scalar vertex.

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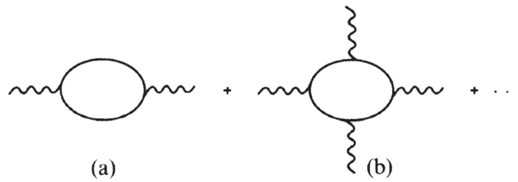

Fig. IV–6 Photon amplitudes containing a single fermion loop.

IV–8 Effective lagrangians in *QED*

We have explored in some detail the structure of effective field theory by using chiral symmetry as an example. However, this is not meant to imply that effective lagrangians are useful only in that one context. In fact, they can be applied to a wide variety of situations. Here, we apply the technique to *QED*.

Consider situations in which the photon's four-momentum is small compared to the electron mass. In such cases, the electron and other fermions cannot be produced directly, but instead influence the physics of photons only through virtual processes. The lowest-order diagrams, i.e., those which contain a single electron loop, with increasing numbers of external photon legs, are shown in Fig. IV–6. Note that the one-loop diagram containing three photons, or indeed any odd number of photons, vanishes by virtue of charge-conjugation invariance. This is true to all orders in the coupling e, and is refered to as *Furry's theorem*. Diagrams like those in Fig. IV–6 have effects at low-energy which are typically calculated in perturbation theory. The associated amplitudes have coefficients which scale as some power of the inverse electron mass. They can be generated by means of an effective lagrangian, as we shall now discuss.

Let us seek a description which eliminates the electron degrees of freedom. That is, we wish to write a lagrangian which involves only photons, but nevertheless includes effects like the ones in Fig. IV–6. The result must of course be gauge-invariant. The procedure may be defined by the path-integral relation

$$\int \left[dA_\mu\right] \exp\left[i\int d^4x\ \mathcal{L}_{\text{eff}}(A_\mu)\right] \equiv \frac{\int \left[dA_\mu\right]\left[d\psi\right]\left[d\bar{\psi}\right] \exp\left[i\int d^4x\ \mathcal{L}_{QED}(A_\mu,\psi,\bar{\psi})\right]}{\int \left[d\psi\right]\left[d\bar{\psi}\right] \exp\left[i\int d^4x\ \mathcal{L}_0(\psi,\bar{\psi})\right]}, \tag{8.1}$$

where $\mathcal{L}_{QED}$ is the full *QED* lagrangian, and $\mathcal{L}_0$ is the free fermion lagrangian. Thus $\mathcal{L}_{\text{eff}}$ has precisely the same matrix elements for photons as does the full *QED* theory. Specifically, it includes the virtual effects of electrons. The techniques described in App. A–5 enable us to formally express the content of Eq. (8.1) as [Sc 51]

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$$\int d^4x\, \mathcal{L}_{\text{eff}}(A_\mu) = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} - i\,\text{Tr}\,\ln\left[\frac{i\not{D} - m}{i\not{\partial} - m}\right]. \tag{8.2}$$

This form, although formally correct, does not readily lend itself to physical interpretation. However, we can determine various interesting effects directly from perturbation theory. For example, the vacuum polarization of Fig. IV–6(a) modifies the photon propagator, i.e., the two-point function. From Eqs. (II–1.26), (II–1.29), we determine the result for a photon of momentum q to be

$$i\hat{\Pi}_{\mu\nu}(q) = i\frac{\alpha}{15\pi}\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\frac{q^2}{m^2} + \cdots . \tag{8.3}$$

The essence of the effective lagrangian approach is to represent such information as the matrix element of a local lagrangian. In the present example, we find that the term in Eq. (8.3) corresponds to the interaction

$$\mathcal{L}_{\text{eff}} = \frac{\alpha}{60\pi m^2}F_{\mu\nu}\Box F^{\mu\nu}, \tag{8.4}$$

where $\Box \equiv \partial_\mu\partial^\mu$.

The calculation of Fig. IV–6(b) is a somewhat more difficult, but still straightforward, exercise in perturbation theory. We shall lead the reader through a calculation using path integrals in a problem at the end of this chapter. It too can be represented as a local lagrangian, and is usually named after Euler and Heisenberg [ItZ 80]. One finds the full result to one-loop order to be

$$\begin{aligned}\mathcal{L}_{\text{eff}}(A_\mu) = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{60\pi m^2}F_{\mu\nu}\Box F^{\mu\nu} \\ &+ \frac{\alpha^2}{90m^4}\left[\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{7}{4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right] + \cdots ,\end{aligned} \tag{8.5}$$

where $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Corrections to this effective lagrangian can be of two forms: (i) even at one loop there are additional terms of higher dimension

$$F_{\mu\nu}\frac{\Box^2}{m^4}F^{\mu\nu}, \quad F_{\mu\nu}\frac{\Box}{m^6}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \quad \frac{1}{m^8}\left(F_{\mu\nu}F^{\mu\nu}\right)^3, \ldots, \tag{8.6}$$

involving either more fields or more derivatives; or (ii) the coefficients of these operators can receive corrections of higher order in α through multi-loop diagrams. We see here an example of the energy expansion, which we have discussed at length earlier in this chapter. In this case it is an expansion in powers of q^2/m^2. The effective lagrangian of Eq. (8.5) can be used to compute aspects of low-energy

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photon physics such as the low-energy contribution of the vacuum polarization process or the matrix element for photon–photon scattering.

IV–9 Effective lagrangians as probes of New Physics

One of the most common and important uses of effective lagrangians is to parameterize how *new* physics at high energy may influence low-energy observables. The general procedure can be abstracted from our earlier discussion. Remember that one is trying to represent the low-energy effects from a 'heavy' sector of the theory. This is accomplished by employing an effective lagrangian

$$\mathcal{L}_{\rm eff} = \sum_n \mathcal{C}_n \, O_n, \tag{9.1}$$

where the $\{O_n\}$ are local operators having the symmetries of the theory and are constructed from fields that describe physics at low-energy. There need be no restriction to renormalizable combinations of fields. Most often the operators can be organized by dimension. The lagrangian itself has mass dimension 4, so that if an operator has dimension d_i the coefficient must have mass dimension

$$\mathcal{C}_n \sim M^{4-d_n}. \tag{9.2}$$

The mass scale M is associated with the heavy sector of the theory. It is clear that operators of high dimension will be suppressed by powers of the heavy mass. To leading order, this allows one to keep a small set of operators.

Some applications will involve phenomena for which the dynamics is well understood. If so, the coefficients of the effective lagrangian can be determined through direct calculation as in the preceding sections. Other examples occur in the theory of weak nonleptonic interactions (cf. Sect. VIII–3) and in the interactions of W-bosons (cf. Sect. XVI–3). Even more generally, effective lagrangians can also be used to describe the effects of new types of interactions. In these cases, dimensional analysis supplies an estimate for the magnitude of the energy scales of possible New Physics. We shall conclude this section by using effective lagrangians to characterize the size of possible violations of some of the symmetries of the Standard Model.

Given certain input parameters, the Standard Model is a closed, self-consistent description of physics up to at least the mass of the Z^0, and is described by the most general renormalizable lagrangian consistent with the underlying gauge symmetries. What would happen if there were new interactions having an intrinsic energy scale of several TeV or beyond? In general, such new theories would be expected to modify predictions of the Standard Model. The modifications would be described by non-renormalizable interactions, organized by dimension in an effective lagrangian description as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \cdots , \tag{9.3}$$

where $\mathcal{L}_n$ has mass dimension n and Λ is the energy scale of the new interaction.

There is a single operator of dimension 5 which will be displayed in the neutrino chapter. At dimension 6, there are 80 distinct operators consistent with the gauge symmetries of the Standard Model [BuW 86]. These can generate a variety of effects which deviate from the Standard Model. For example, the operator

$$\mathcal{L}_6\left(c'\right) \equiv \frac{c'}{\Lambda^2}(\Phi^\dagger\Phi)\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu}, \tag{9.4}$$

containing the Higgs field Φ and the field tensor $\mathbf{W}_{\mu\nu}$ of $SU(2)$ gauge bosons, produces a deviation from unity in the rho-parameter,[12]

$$\rho \equiv \frac{M_W^2}{M_Z^2\cos^2\theta_{\text{w}}} = 1 - c'\frac{v^2}{\Lambda^2} + \cdots . \tag{9.5}$$

The current level of precision, $|\rho - 1| \leq 0.0029$ (at 2σ), requires $\Lambda > 4.5$ TeV for $c' = 1$. Yet another possibility concerns the violation of flavor symmetries in the Standard Model. The operator,

$$\mathcal{L}_6(c'') \equiv \frac{c''}{\Lambda^2}\bar{e}\gamma_\mu(1+\gamma_5)\mu\;\bar{s}\gamma^\mu(1+\gamma_5)d + \text{h.c.}, \tag{9.6}$$

conserves generational or family number, but violates the separate lepton-number symmetries. It leads to the transition $K_L \to e^-\mu^+$ such that

$$\frac{\Gamma_{K_L^0\to\mu^+e^-}}{\Gamma_{K^+\to\mu^+\nu_\mu}} = \left(\frac{c''}{V_{\text{us}}G_F\Lambda^2}\right)^2 . \tag{9.7}$$

The present bound, $\text{Br}_{K_L^0\to\mu e} < 4.7\times10^{-12}$ at 90% confidence level, requires $\Lambda > 1700$ TeV for $c'' \simeq 1$. In a similar manner, constraints on other physical processes imply bounds on their corresponding energy scales Λ, generally in the range $5 \to 5000$ TeV.

Dimension-six contact interactions also are searched for at the highest energies of the Large Hadron Collider (LHC). The effect of the contact interaction becomes relatively more pronounced at high energy when competing with background processes which fall off due to propagator effects. For example, an operator leading to $q\bar{q} \to \mu^+\mu^-$,

$$\mathcal{L}_6(g) \equiv \frac{g^2}{2\Lambda^2}\bar{q}_L\gamma_\nu q_L\bar{\mu}_L\gamma^\nu\mu_L, \tag{9.8}$$

[12] More precisely the comparison is with a form of the rho-parameter after radiative corrections.

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becomes increasingly visible over the Drell–Yan process at high energy. Early LHC results [Aa *et al*. (ATLAS collab.) 11] bound this interaction with $\Lambda > 4.5$ TeV at 95% confidence for $g^2/4\pi = 1$; such limits will clearly improve in the future. Interestingly, some operators are better bounded by low-energy precision experiments and others are better probed at high energy [Bh *et al*. 12], demonstrating the value of both lines of research.

Of course, if there is new physics in the TeV energy range, it need not generate all 80 possible effective interactions. The ones actually appearing would depend on the couplings and symmetries of the new theory. In addition, the coefficients of contributing operators could contain small coupling constants or mixing angles, diminishing their effects at low-energy. However, the effective lagrangian analysis indicates that the continued success of the Standard Model is quite nontrivial and places meaningful bounds on possible new dynamical structures occurring at TeV, and even higher, energy scales.

Problems

(1) ***U*(1) effective lagrangian**

Consider a theory with a complex scalar field φ with a $U(1)$ global symmetry $\varphi \to \varphi' = \exp(i\theta)\,\varphi$. The lagrangian will be

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \mu^2 \varphi^* \varphi - \lambda(\varphi^* \varphi)^2.$$

(a) Minimize the potential to find the ground state and write out the lagrangian in the basis

$$\varphi = \frac{1}{\sqrt{2}}(v + \varphi_1(x) + i\varphi_2(x))$$

Show that φ_2 is the Goldstone boson.

(b) Use this lagrangian to calculate the low-energy scattering of $\varphi_2 + \varphi_2 \to \varphi_2 + \varphi_2$. Show that despite the nonderivative interactions of the lagrangian, cancelations occur such that leading scattering amplitude starts at order p^4.

(c) Instead of the basis above, express the lagrangian using an exponential basis,

$$\varphi = \frac{1}{\sqrt{2}}(v + \Phi(x))e^{i\chi(x)/v}.$$

Show that in this basis a 'shift symmetry' $\chi \to \chi + c$ is manifest.

(d) Calculate the same scattering amplitude using this basis and show that the results agree. Note that the fact that the amplitude is of order p^4 is more readily apparent in this basis.

(2) **Path integrals and the Fermi effective lagrangian**
Consider the path integral $Z_W = \int \left[dW^+\right]\left[dW^-\right] \exp\left[i \int d^4x \mathcal{L}_W(x)\right]$, where $\mathcal{L}_W$ is the $W^\pm$-boson lagrangian $\mathcal{L}_W = \mathcal{L}_W^{(\text{free})} + \mathcal{L}_W^{(\text{int})}$, with

$$\mathcal{L}_W^{(\text{free})} = -\frac{1}{2} F_{\mu\nu}^+ F_-^{\mu\nu} + M_W^2 W_\mu^+ W_-^\mu, \quad \mathcal{L}_W^{(\text{int})} = -\frac{g_2}{\sqrt{8}} \left(W_\mu^+ J_{\text{ch}}^\mu + \text{h.c.}\right).$$

Integrating out the heavy $W^\pm$ fields in Z_W leads to an effective interaction between charged weak currents called the Fermi model.

(a) Show that, upon discarding a total derivative term, one can write the free field contribution in Z_W as

$$\int d^4x \; \mathcal{L}_W^{(\text{free})} = \int d^4x \; d^4y \; W_\mu^+ \alpha K^{\mu\nu}(x, y) W_\nu^-(y),$$

where $K_{\mu\nu}(x, y) \equiv \delta^{(4)}(x - y) \left[g_{\mu\nu}\left(\partial^2 + M_W^2\right) - \partial_\mu \partial_\nu\right]$.

(b) Further steps allow the path integral to be expressed as

$$Z_W = \exp\left[-i \frac{g_2^2}{8} \int d^4x \; d^4y \; J_{\text{ch}}^{\mu\dagger}(x) \Delta_{\mu\nu}(x, y) J_{\text{ch}}^\nu(y)\right],$$

where $\Delta_{\mu\nu}(x, y)$ is the Fourier transform of the $W^\pm$ propagator $\Delta_{\mu\nu}(k) = -\left(g_{\mu\nu} - k_\mu k_\nu / M_W^2\right)$. Upon expanding this form of Z_W in powers of M_W^{-2}, show that to lowest order,

$$\mathcal{L}_W^{(\text{eff})}(x) = -\frac{G_F}{\sqrt{2}} \; J_{\text{ch}}^{\mu\dagger}(x) J_\mu^{\text{ch}}(x) \qquad \text{(Fermi model)},$$

where the Fermi constant obeys $G_F/\sqrt{2} \equiv g_2^2/(8M_W^2)$.

(3) **The Euler–Heisenberg lagrangian: constant magnetic field**
Consider a charged scalar field φ interacting with a constant external magnetic field $\mathbf{B} = B\hat{\mathbf{k}}$. The corresponding Klein–Gordon equation is $(D^2 + m^2)\varphi(x) = 0$, where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, and the effective action is then given by

$$e^{iS_{\text{eff}}(B)} = \frac{\int [d\varphi(x)][d\varphi^*(x)] e^{i\int d^4x \; \varphi^*(x)(D^2+m^2)\varphi(x)}}{\int [d\varphi(x)][d\varphi^*(x)] e^{i\int d^4x \; \varphi^*(x)(\Box^2+m^2)\varphi(x)}}$$

$$= \det(\Box^2 + m^2)/\det(D^2 + m^2),$$

$$S_{\text{eff}}(B) = i \; \text{Tr} \; \ln \frac{D^2 + m^2}{\Box + m^2}.$$

The operation ‘Tr ln’ applied to a differential operator is not a trivial one and the purpose of this problem is to evaluate this quantity for the case at hand.

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(a) Demonstrate that

$$S_{\text{eff}}(B) = i \,\text{Tr} \int_0^\infty e^{-m^2 s} \left(e^{-\Box s} - e^{-D^2 s}\right).$$

(b) In order to evaluate the trace we require a complete set of solutions to the equations

$$D^2 \bar{\varphi}_n(x, y, z, t) = \lambda_n \bar{\varphi}_n(x, y, z, t),$$
$$\Box \varphi_n(x, y, z, t) = \kappa_n \varphi_n(x, y, z, t),$$

so that we may write

$$S_{\text{eff}}(B) = i \sum_n \int_0^\infty \frac{ds}{s}\, e^{-m^2 s} \left(e^{-\kappa_n s} - e^{-\lambda_n s}\right).$$

(c) With the gauge choice $A_\mu = (0, Bx\hat{\mathbf{j}}\,)$ show that the eigenstates are

$$\varphi(x, y, z, t) = e^{i(k_x x + k_y y + k_z z - k_t t)},$$
$$\bar{\varphi}(x, y, z, t) = e^{i(k_z z + k_y y - k_t t)\psi_n(x - k_y/eB)},$$

where $\psi_n(x)$ is an eigenstate of the harmonic-oscillator hamiltonian, and the eigenvalues are $\kappa_n = -k_t^2 + k_x^2 + k_y^2 + k_z^2$, $\lambda_n = -k_t^2 + k_z^2 + eB(2n+1)$.

(d) Rotate to euclidean space and evaluate the trace using box quantization. Taking a box with sides L_1, L_2, L_3 and a time interval T, we have

$$\kappa : \sum_n \to L_1 L_2 L_3 T \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4},$$
$$\lambda : \sum_n \to L_2 L_3 T \int_0^{eBL_1} dk_y \int_{-\infty}^{\infty} \frac{dk_0 dk_z}{(2\pi)^2} \sum_{n=0}^{\infty},$$

where the integration on k_y is over all values with $x' = x - k_y/eB$ positive.

(e) Evaluate the effective action

$$S_{\text{eff}}(B) = L_1 L_2 L_3 T \int_0^\infty \frac{ds}{s} \int_{-\infty}^{\infty} \frac{dk_0 dk_z}{(2\pi)^2} e^{-(m^2 + k_0^2 + k_z^2)s} \times \left[\frac{eB}{2\pi} \sum_{n=0}^{\infty} e^{-eB(2n+1)s} - \int_{-\infty}^{\infty} \frac{dk_x dk_y}{(2\pi)^2} e^{-(k_x^2 + k_y^2)s}\right]$$

and show that

$$S_{\text{eff}}(B) = L_1 L_2 L_3 T \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left(\frac{eBs}{\sinh eBs} - 1\right).$$

Expand this in powers of B, finding the (divergent) wavefunction renormalization and the B^4 piece of the Euler–Heisenberg lagrangian.

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(f) Show that the corresponding result of a constant *electric* field can be found by the substitution $B \to iE$ so that

$$S_{\text{eff}}(E) = L_1 L_2 L_3 T \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s} \left(\frac{eEs}{\sin eEs} - 1 \right).$$

(g) Demonstrate that, although $\text{Im}\, S_{\text{eff}}(B) = 0$, one nonetheless obtains

$$\text{Im}\, S_{\text{eff}}(E) = L_1 L_2 L_3 T \frac{e^2 E^2}{16\pi^3} \sum_{n=1}^{\infty} \frac{(-)^n}{n^2} e^{-n\pi m^2/eE},$$

and discuss the meaning of this result [Sc 51, Sc 54].

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