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TOPOLOGICAL RINGS OF QUOTIENTS

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We investigate here the notion of a topological ring of quotients of a topological ring with respect to an arbitrary Gabriel (idempotent) filter of right ideals. We describe the topological ring of quotients first as a subring of the algebraic ring of quotients, and then show it is a topological bicommutator of a topological injective R-module. Unlike R. L. Johnson in [**6**] and F. Eckstein in [**2**] we do not always make the ring an open subring of its ring of quotients. This would exclude examples such as $C(X)$, the ring of continuous real-valued functions on a compact space, and its ring of quotients as described in Fine, Gillman and Lambek [**3**].

Let R be a ring with 1 and $\mathscr{D}$ a Gabriel filter of right ideals for R. Let M be a right R-module, $Q_{\mathscr{D}}(M)$ its quotient module with respect to $\mathscr{D}$, $T_{\mathscr{D}}(M)$ the torsion submodule, and $F_{\mathscr{D}}(M)$ be $M/T_{\mathscr{D}}(M)$. We shall omit the subscript $\mathscr{D}$ if we are only dealing with one Gabriel filter. We now consider an operator Γ_M which assigns to subsets of M, subsets of $Q(M)$. We require the following properties to hold, where $X_1, X_2 \subseteq M$, $X_3 \subseteq R$ and

$$M \xrightarrow{f} N$$

is an R-homomorphism:

(1) $X_1 + T(M)/T(M) \subseteq \Gamma_M(X_1)$,
(2) $X_1 \subseteq X_2 \Rightarrow \Gamma_M(X_1) \subseteq \Gamma_M(X_2)$,
(3) $\Gamma_{Q(M)} \circ \Gamma_M = \Gamma_M$,
(4) $\Gamma_M(X_1)\Gamma_R(X_3) \subseteq \Gamma_M(X_1X_3)$,
(5) $\Gamma_M(X_1) + \Gamma_M(X_2) \subseteq \Gamma_M(X_1 + X_2)$, if $0 \in X_1$ or X_2,
(6) $Q(f)(\Gamma_M(X_1)) \subseteq \Gamma_N(f(X_1))$.

Properties (1), (2) and (3) say that if M is torsionfree divisible, then Γ_M is a closure operator. (4), (5) and (6) just express compatibility with the R-module structure. To put a topology on the quotient ring, we shall take as neighborhoods of 0 the images under Γ of neighborhoods of 0 in R, but first we give some examples of possible choices of Γ.

Example 1. Let $\Gamma_M(X)$ be the image of X under the canonical map $M \to M/T(M)$. This corresponds to making $R/T(R)$ an open subring of its quotient ring.

Example 2. If $X \subseteq Q(M)$ let

$$X^+ = \{q \in Q(M): \text{there exists } D \in \mathscr{D} \text{ so that for all } d \in D,\ qd \in Xd\}.$$

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If $Z \subseteq M$, let

$$\Gamma_M(Z) = \cap\{X \supseteq Z + T(M)/T(M) : X \subseteq Q(M) \text{ and } X = X^+\}.$$

We verify that Γ_M satisfies property (4), the others are checked similarly. First we show that $X_1^+X_3^+ \subseteq (X_1X_3)^+$ whenever $X_1 \subseteq Q(M)$ and $X_3 \subseteq Q(R)$. Take $q_i \in X_i^+$, and let $D_i \in \mathscr{D}$, such that $q_id \in X_id$ all $d \in D_i$, $i = 1, 3$. Let $D_3' = D_3 \cap q_3^{-1}(D_1)$. Now if $d \in D_3'$, then $(q_1q_3)d = q_1(q_3d) = x_1(q_3d) = x_1(x_3d) = (x_1x_3)d$ for some $x_i \in X_i$, $i = 1, 3$. Thus $q_1q_3 \in (X_1X_3)^+$. Now let $X_1 \subseteq M$, $X_3 \subseteq R$, $\bar{X}_1 = X_1 + T(M)/T(M)$, $\bar{X}_3 = X_3 + T(R)/T(R)$, and $\bar{X}_4 = \bar{X}_1\bar{X}_3$. We then define for each ordinal α

$$X_i^0 = X_i,$$

$$X_i^\alpha = (X_i^\beta)^+ \quad \text{if } \alpha = \beta + 1,$$

$$X_i^\alpha = \bigcup_{\beta<\alpha} X_i^\beta \quad \text{if } \alpha \text{ is a limit ordinal,}$$

for $i = 1, 3, 4$. For a sufficiently large γ we have $X_i^\gamma = \Gamma_M(X_i)$, $i = 1, 3, 4$. We show by induction that $X_1^\alpha X_3^\alpha \subseteq X_4^\alpha$. If $\alpha = \beta + 1$, then

$$X_1^{\beta+1}X_2^{\beta+1} = (X_1^\beta)^+(X_2^\beta)^+ \subseteq (X_1^\beta X_2^\beta)^+ \subseteq (X_4^\beta)^+ = X_4^{\beta+1}$$

using the inductive assumption $X_1^\beta X_2^\beta \subseteq X_4^\beta$ and the easily checked fact that $^+$ is monotone. If α is a limit ordinal, and if $q_i \in X_i^\alpha$ $(i = 1, 3)$, then $q_i \in X_i^\beta$, for some $\beta < \alpha$, and so $q_1q_3 \in X_1^\beta X_3^\beta \subseteq X_4^\beta \subseteq X_4^\beta$.

Example 3. Let R be a commutative ring, $X \subseteq M$, where M is any R-module. We define

$$\Gamma_M(X) = \{\sum (x_i + T(R)/T(R))q_i : x_i \in X, q_i \in Q(R)\}.$$

It is easily verified that properties (1), (2), . . . , (6) hold. Commutativity is necessary for (3).

We now assume that R is any topological ring, $\mathscr{D}$ any Gabriel filter of right ideals, and Γ an operator satisfying the conditions (1), (2), . . . , (6). We shall usually omit the subscript for Γ. Let $\mathscr{V}$ be the neighborhood filter at 0 of R. Let $\mathscr{W} = \{\Gamma_R(V) : V \in \mathscr{V}\}$. Let

$$Q^* = \{q \in Q(R) : \text{for all } W \in \mathscr{W}, \text{ there exists } V \in \mathscr{V}, qV \subseteq W, Vq \subseteq W\}.$$

We may later write $Q_{\mathscr{D}}^*(R) = Q^*$.

PROPOSITION 1. *Q^* is a topological ring, with $\mathscr{W}$ a base for the neighborhood filter at* 0.

Proof. We first note that every element of $\mathscr{W}$ is contained in Q^* by (1) and (5). To show that Q^* is a topological group we need to show that if $V \in \mathscr{V}$, then

(a) there is a $U \in \mathscr{V}$, $\Gamma(U) + \Gamma(U) \subseteq \Gamma(V)$;

(b) there is a $T \in \mathscr{V}$, $\Gamma(T) \subseteq -\Gamma(T)$.

For (a), take $U \in \mathscr{V}$ such that $U + U \subseteq V$, and apply (4). To see (b), we have $(\Gamma(V))(-1) \subseteq \Gamma(V)\Gamma(-1) = -\Gamma(V)$ by (1) and (4). Then letting

$T = -V$, $\Gamma(T) \subseteq -\Gamma(-T) = -\Gamma(V)$. It remains to check continuity of multiplication. If $q \in Q^*$, $W \in \mathscr{W}$, $W = \Gamma(V)$, $V \in \mathscr{V}$, take $U, T \in \mathscr{V}$ such that $U + U \subseteq V$ and $qT \subseteq \Gamma(U)$, $Tq \subseteq \Gamma(U)$, and $T\ T \subseteq U$. Thus $(q + \Gamma(T))\ \Gamma(T) \subseteq q\Gamma(T) + \Gamma(T)\Gamma(T) \subseteq \Gamma(qT) + \Gamma(TT) \subseteq \Gamma(U) + \Gamma(U) \subseteq \Gamma(V)$.

We shall call Q^* the topological quotient ring of R with respect to $\mathscr{D}$ and Γ.

Example A. Let R be $C(X)$, X compact Hausdorff, the $\mathscr{D}$ be the Utumi filter, the topology for R be the one induced by the sup norm, and Γ be as defined in Example 2. Then Q^* is the ring of real-valued functions which are continuous and bounded on a dense open subset of X, and its topology is that induced by the sup norm. We recall from **[3]** that $Q(R)$ is the ring of all real-valued functions continuous on a dense open subset of X, and an ideal D of R is in $\mathscr{D}$ if and only if the cozero set of D is dense. To prove our assertion about Q^*, if $\epsilon > 0$ let

$$W = \{g \in Q(R): |g(x)| \leqq \epsilon \text{ all } x \text{ in an open dense set } \mathscr{O} \subseteq X\}.$$

We claim that $\Gamma(W \cap R) = W$. Take $g \in (W \cap R)^+$. Thus

there exists $D \in \mathscr{D}$ such that for all $d \in D$, $gd \in (W \cap R)d$.

The cozero set of D is open and dense, and for $x \in \operatorname{coz} D$ choose $d \in D$, $d(x) \neq 0$. Then $g(x)d(x) = w(x)d(x) \leqq \epsilon d(x)$. Thus $|g(x)| \leqq \epsilon$ and we have $g \in W$. Conversely if $g \in W$, and $\mathscr{O}$ is the open dense set in the definition of W, let

$$D' = \{d \in R: d^{-1}(0) \text{ is a neighborhood of } X\backslash\mathscr{O}\}.$$

Since X is normal, $\operatorname{coz} D' = \mathscr{O}$, and thus $D' \in \mathscr{D}$. We define

$$w(x) = \begin{cases} 0 \text{ on } X\backslash\mathscr{O} \\ g(x) \text{ on the closure of } X\backslash d^{-1}(0) \\ \text{otherwise extend it continuously with values in } [-\epsilon, \epsilon]. \end{cases}$$

We have $gd = wd$, $w \in W \cap R)$, and therefore $g \in (W \cap R)^+$. We have shown that $(W \cap R)^+ = W$, but it is easily verified that $W^+ = W$, and thus $\Gamma(W \cap R) = W$. Every function which is bounded and continuous on a dense open set is in some W for a sufficiently large ϵ, and therefore in $\Gamma(W \cap R) \subseteq \Gamma(R) \subseteq Q^*$. Conversely if $g \in Q^*$, then there exists $\delta > 0$ such that

$$g \cdot \{f \in R: |f| \leqq \delta\} \subseteq W.$$

Thus $g \cdot \delta = w$, for some $w \in W$, and $g = w/\delta$, a function which is bounded (by ϵ/δ) on a dense open set. That the topology on Q^* is the sup norm, is clear from $\Gamma(W \cap R) = W$.

Example B. This example also uses Γ as in Example 2. We give here a general construction, which applies to any ring, and which when applied to a ring of algebraic integers gives the ring of Adele's (together with its topology) modulo

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the Archimedean part (see [1]). Let R be a ring, $\mathscr{D}$ the Utumi filter of right ideals. Let $\hat{R}$ be the Hausdorff completion. $\hat{R}$ will have a linear topology, i.e. a base for the neighborhood filter at 0 consisting of right ideals, so let $\mathscr{D}'$ be the smallest Gabriel filter containing it. We now form the topological ring $Q^*_{\mathscr{D}'}(\hat{R})$. To see what this is if R is a ring of algebraic integers, we first note that

$$\hat{R} = \prod_{P_i \in \operatorname{Spec} R} \hat{R}_{P_i}$$

by [3] where $\hat{R}_{P_i}$ denotes the P_i-adic completion of R. The topology on $\hat{R}$ is the product topology, and thus ideals in $\mathscr{D}'$ are of the form $\prod_i X_i$, where $X_i = \hat{R}_{P_i}$ for almost all i, and otherwise $X_i = \mathscr{M}_i^{n_i}$, $n_i \in \mathbf{N}$, $\mathscr{M}_i$ the maximal ideal in $\hat{R}_{P_i}$. Let $\mathscr{M}_i = a_i\hat{R}_{P_i}$. $Q_{\mathscr{D}'}(\hat{R})$ is then the local product of the $Q_i = a_i^{-1}\hat{R}_{P_i}$, with respect to the $\hat{R}_{P_i}$. $Q_{\mathscr{D}'}(\hat{R}) = Q_{\mathscr{D}'}{}^*(\hat{R})$, and it is clear that the topology of Q^* is the usual one.

Example C. If R commutative, Γ is as in Example 3, and if we localize at a prime P, then $Q^* = R_P$ and its topology is the M-adic topology where $M = PR_P$.

Example D. If R is any topological ring, $\mathscr{D}$ the Utumi filter, and Γ is as in Example 1, then $Q^* = C(R)$, where $C(R)$ is defined by R. L. Johnson in [6].

We now wish to show that Q^* can be obtained as a topological bicommutator. If M is a topological R-module, E the ring of continuous endomorphisms $\widetilde{\text{End}}\,(M_R, M_R)$, and $S = \widetilde{\text{End}}\,({}_EM, {}_EM)$, we shall call S, endowed with the topology of pointwise convergence, the topological bicommutator of M. It comes equipped with the continuous canonical ring homomorphism $R \to S$.

Given $\mathscr{D}$ a Gabriel filter for R, and Γ an operator satisfying (1), (2), . . . , (6) we let

$$I = \prod_{j \in J} \{E(R/K_j): \quad K_j \leqq_r R, R/K_j \text{ torsionfree}\}.$$

Let $i_0 = (1 + K_j)_{j \in J} \in I$. A base for neighborhoods of 0 in I, will be

$$\mathscr{U} = \{\Gamma_I(i_0 V): V \in \mathscr{V}\}$$

where $\mathscr{V}$ is the neighborhood filter of 0 in R. We let

$$I^* = \{i \in I: \text{for all } U \in \mathscr{U} \text{ there exists } V \in \mathscr{V} \text{ with } iV \subseteq U\}.$$

It is clear that I^* is a topological R-module.

A topological R-module E is said to be a topological injective (see [5]) if for M' an open submodule of a topological module M, any continuous map $M' \xrightarrow{f} E$ admits a continuous extension to M.

LEMMA. *$I_R{}^*$ is a topological injective.*

Proof. Let

$$M' \xrightarrow{f} I^*$$

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as in the definition. We know there is a map

$$M \xrightarrow{\bar{f}} I$$

extending f. If $m \in M$ and $U \in \mathscr{U}$, there is a $V \in \mathscr{V}$ such that $mV \subseteq f^{-1}(U)$. Thus $\bar{f}(m)V \subseteq U$, so $\bar{f}(M) \subseteq I^*$. Continuity is clear since M' is open and $\bar{f}|_{M'}$ is continuous.

THEOREM. *Q^* is the topological bicommutator of I^*.*

Proof. First we show that $Ei_0 = I^*$. Take $i \in I^*$. Define $h: i_0R \to I^*: i_0r \mapsto ir$. We can extend h to $\bar{h}: I^* \to I$. If $x \in I^*$, and $U \in \mathscr{U}$ we know there is a $V \in \mathscr{V}$ with $iV \subseteq U$ and a $W \in \mathscr{V}$ such that $xW \subseteq \Gamma(i_0V)$. We have $\bar{h}(x)W = \bar{h}(xW) \subseteq \bar{h}(\Gamma(i_0V)) \subseteq \Gamma(\bar{h}(i_0V)) = \Gamma(iV) \subseteq \Gamma(U) = U$. Thus $\bar{h}(I^*) \subseteq I^*$. $\bar{h}$ is continuous, for if $U \in \mathscr{U}$, take $V \in \mathscr{V}$ such that $iV \subseteq U$, and then $\bar{h}(\Gamma(i_0V)) \subseteq U$.

Since $Ei_0 = *$, we have a monomorphism $S \to I^*: s \mapsto i_0s$. We wish to show that

$$R \xrightarrow{\kappa} S$$

is essential, or equivalently $i_0R \subseteq i_0S$ is essential. Suppose for $i = i_0s \in i_0S$, $iR \cap i_0R = 0$. Then we define $e: iR + i_0R \to I^*$ by $e(i) = i$ and $e(i_0) = 0$. e is continuous, and by an argument similar to that above one obtains a continuous extension $\bar{e}: I^* \to I^*$. Thus

$$i = e(i) = \bar{e}(i) = \bar{e}(i_0s) = (\bar{e}i_0)s = 0.$$

It is clear that $\operatorname{Ker}(R \to S) = T(R)$. Thus we have shown that S_R is an essential extension of $R/T(R)$. In order to show that S_R is a subring of $Q(R)$ (i.e., $Q_{\mathscr{D}}(R)$) it suffices to show that $S(R)/\kappa(R)$ is $\mathscr{D}$-torsion. We know $S_R/\kappa(R) \cong i_0S_R/i_0R$. Suppose

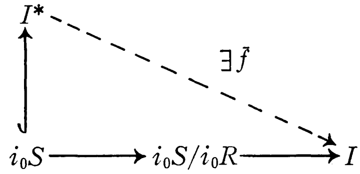
$$i_0S_R/i_0R \xrightarrow{f} I, \qquad f \neq 0.$$

We know there is an

$$I^* \xrightarrow{\bar{f}} I$$

extending the map

$$i_0S \xrightarrow{\pi} i_0S/i_0R \xrightarrow{f} I.$$



We claim $\bar{f}(I^*) \subseteq I^*$ and $\bar{f}$ is continuous. Take $i \in I^*$ and $U \in \mathscr{V}$. Then there is a $V \in \mathscr{V}$, such that $iV \subseteq \Gamma(i_0U) \subseteq Q_{\mathscr{D}}(i_0R)$. Thus $\bar{f}(iV) = 0$, since

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$Q_{\mathscr{D}}(i_0R)/i_0R$ is $\mathscr{D}$-torsion. Since $\bar{f}(i)V = 0$ we certainly have $\bar{f}(i) \in I^*$. Similarly, $\bar{f}(\Gamma(i_0V)) \subseteq \Gamma(\bar{f}(i_0V)) = \Gamma(0)$, so $\bar{f}$ is continuous. Now for any $s \in S, f \cdot \pi(i_0s) = \bar{f}(i_0s) = (\bar{f}i_0)s = 0$. Therefore $f = 0$. Thus we may think of S and Q^* as two subrings of $Q(R)$ containing $R/T(R)$. To show that $Q^* \subseteq S$, it will suffice to show that $g: I^* \to I^*: i \to iq$ is a well defined and continuous E-homomorphism. We know I is a $Q(R)$ module. Let $i \in I^*$ and $q \in Q^*$. Take $U \in \mathscr{U}$ and $V \in \mathscr{V}$ such that $iV \subseteq U$. Take $W \in \mathscr{V}$ such that $qW \subseteq \Gamma(V)$. Then $(iq)W \subseteq i\Gamma(V) \subseteq \Gamma(iV) \subseteq \Gamma(U) = U$. Therefore $iq \in I^*$. To check continuity, $V' \in \mathscr{V}$. Then there is a $W \in \mathscr{V}$ such that $Wq \subseteq \Gamma(V')$.

$$\Gamma(i_0W)q \subseteq \Gamma(i_0Wq) \subseteq \Gamma(i_0\Gamma(V')) \subseteq \Gamma(\Gamma(_{i0}V')) = \Gamma(i_0V').$$

It remains to check that g is an E-endomorphism. Take $e \in E$, $I \in I^*$, and $D \in \mathscr{D}$ such that $q(D) \subseteq R/T(R)$. Then $(e(iq) - (ei)q)d = e(iqd) - (ei)qd = (ei)qd - (ei)qd = 0$, each $d \in D$. Since I^* is $\mathscr{D}$-torsionfree, $e(iq) = (ei)q$.

We now show $S \subseteq Q^*$. Take $V \in \mathscr{V}$. We know $i_0\Gamma(V) \subseteq \Gamma(i_0)\Gamma(V) \subseteq \Gamma(i_0V) \subseteq Q(i_0R) = i_0Q(R)$. Let

$$i_0R \xrightarrow{\mathbf{f}} R: i_0 \to 1.$$

$Q(f)(\Gamma(i_0V)) \subseteq \Gamma(f(i_0V)) = \Gamma(V)$. Thus $\Gamma(i_0V) \subseteq i_0\Gamma(V)$, i.e., $\Gamma(i_0V) = i_0\Gamma(V)$. Since s is a continuous endomorphism, there is a $W \in \mathscr{V}$, such that

$$(i_0W)s \subseteq \Gamma(i_0W)s \subseteq \Gamma(i_0V) = (i_0\Gamma(V)).$$

Thus $Ws \subseteq \Gamma(V)$. On the right side, since $i_0s \in I^*$, there is a $W' \in \mathscr{V}$, such that $(i_0s)W' \subseteq \Gamma(i_0V) = i_0\Gamma(V)$, i.e., $sW' \subseteq \Gamma(V)$. Therefore $s \in Q^*$.

Finally it remains to show that the topology of pointwise convergence on S coincides with the topology of Q^*. If $i \in I^*$, and $V \in \mathscr{V}$, a typical neighborhood of 0 in S is $X = \{s \in S: is \in \Gamma(i_0V)\}$. If $i = i_0$ then this is just $\Gamma(V)$ so the topology of S is finer than that of Q^*. Conversely, we know $i = ei_0$ some $e \in E$. Thus $X = \{s \in S: i_0s \in e^{-1}(\Gamma(i_0V))\}$. Since e is continuous, $\Gamma(i_0V)$ is a neighborhood of 0 in I^*, i.e. there is a $W \in \mathscr{V}$ such that $e^{-1}(\Gamma(i_0V)) \supseteq \Gamma(i_0W) = i_0\Gamma(W)$. Thus $X \supseteq \Gamma(W)$.

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