Canad. Math. Bull.Vol. **34** (1), 1991 pp. 128–135

K-UNIFORM ROTUNDITY OF SEQUENCE ORLICZ SPACES

WANG TINGFU AND CHEN SHUTAO

ABSTRACT. This paper presents a criterion of KUR for sequence Orlicź spaces with Luxemburg's norm. The result also indicates that for any integer $k \geq 1$, there exists a $k+1$-uniformly rotund Banach space not being k-uniformly rotund.

Let $M: (-\infty, +\infty) \longrightarrow [0, +\infty)$ be convex, even, continuous and $M(u) = 0 \Leftrightarrow u = 0$. For a given sequence $x = (x_n)$, denote $\rho(x) = \sum_n M(x_n)$ and $\ell_M = \{x = (x_n) : \exists \lambda > 0, \rho(\lambda x) < \infty\}$, $\|x\| = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}$ for $x \in \ell_M$, then $(\ell_M, \| \quad \|)$ is a Banach space.

We say $M \in \delta_2$, if $\overline{\lim}_{u \to 0} M(2u)/M(u) < \infty$,

LEMMA 1 [2,3]. *If $M \in \delta_2$, then*

(a) $\|x^n\| \longrightarrow 0 \Leftrightarrow \rho(x^n) \longrightarrow 0$,

(b) $\|x^n\| \longrightarrow 1 \Leftrightarrow \rho(x^n) \longrightarrow 1$ *and*

(c) for any $1 > 0, \epsilon > 0$, *there exists* $\delta > 0$ *such that* $\rho(x) \leq 1$, $\rho(y) \leq \delta$ *imply*

$$|\rho(x+y) - \rho(x)| < \epsilon$$

$M(u)$ is said to be *uniformly convex* on some interval $[a, b]$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that u, v in $[a, b]$, $|u - v| \geq \epsilon \max\{u, v\}$ imply

$$M\left[\frac{1}{2}(u+v)\right] \leq \frac{1}{2}(1-\delta)[M(u) + M(v)]$$

LEMMA 2. *The following are equivalent*

(I) $M(u)$ *is uniformly convex on* $[0, a]$,

(II) for any $\beta \in [0, a)$, $b \geq a$ *and* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\max\{u, v\} \leq b$, $0 \leq \min\{u, v\} \leq \beta$ *and* $|u - v| \geq \epsilon \max\{u, v\}$ *imply*

$$M\left[\frac{1}{2}(u+v)\right] \leq \frac{1}{2}(1-\delta)[M(u) + M(v)]$$

(III) for any integer $m \geq 2$, $\beta \in [0, a)$, $b \geq a$ *and* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\max_{1 \leq j \leq m} u_j \leq b$, $0 \leq \min_{1 \leq j \leq m} u_j \leq \beta$ *and* $\max_{1 \leq i,j \leq m} |u_i - u_j| \geq \epsilon \max_{1 \leq j \leq m} u_j$ *imply*

$$M(\sum_j u_j/m) \leq (1-\delta) \sum_j M(u_j)/m$$

Received by the editors March 2, 1988, revised December 20, 1988.
AMS subject classification: Primary: 46B30, Secondary: 46B20.

PROOF. (I)$\Rightarrow$(II). If (2) is not true, then there exist $\beta < a$, $b \geq a$, $\epsilon > 0$ and u_n, $v_n \leq b$, $u_n - v_n \geq \epsilon u_n > 0$ such that

(1) $$M\left[\frac{1}{2}(u_n + v_n)\right] > \frac{1}{2}(1 - \frac{1}{n})[M(u_n) + M(v_n)] \quad (n = 1, 2, \ldots).$$

Without loss of generality, we may assume $u_n \longrightarrow u'$ and $v_n \longrightarrow v'$; then by (I), $u' \geq a > \beta \geq v'$ and from (1) it is easy to see $M[\frac{1}{2}(u' + v')] = \frac{1}{2}[M(u') + M(v')]$ which shows that $M(u)$ is linear on $[v', u']$ contradicting (I).

(II)$\Rightarrow$(III). For given $\beta < a$, $b \geq a$ and $\epsilon > 0$, select $\delta > 0$ satisfying (*II*). For any integer $m \geq 2$ and real numbers $\{u_j\}$ satisfying the conditions in (III), we may assume $u_1 = \max_{1 \leq j \leq m} u_j$, $u_2 = \min_{1 \leq j \leq m} u_j$.

If m is even, then

$$\begin{aligned} M\left[(u_1 + \cdots + u_m)/m\right] &= M\{\left[\frac{1}{2}(u_1 + u_2) + \cdots + \frac{1}{2}(u_{m-1} + u_m)\right]/(m/2)\} \\ &\leq (2/m)\left\{M\left[\frac{1}{2}(u_1 + u_2)\right] + \cdots + M\left[\frac{1}{2}(u_{m-1} + u_m)\right]\right\} \\ &\leq (2/m)\{\frac{1}{2}(1 - \delta)[M(u_1) + M(u_2)] + \frac{1}{2}[M(u_3) + M(u_4)] \\ &\qquad + \cdots + \frac{1}{2}[M(u_{m-1}) + M(u_m)]\} \\ &= \sum_j M(u_j)/m - \delta[M(u_1) + M(u_2)]/m \\ &\leq \sum_j M(u_j)/m - \delta \sum_j M(u_j)/m^2 \\ &= (1 - \delta/m)\sum_j M(u_j)/m \end{aligned}$$

If m is odd, observe $u_1 > \sum_j u_j/m > u_2$, by the result we just obtained, we have

$$\begin{aligned} &M[(u_1 + \cdots + u_m)/m] \\ &\quad = M\left\{[(u_1 + \cdots + u_m + (u_1 + \cdots + u_m)/m]/(m+1)\right\} \\ &\quad \leq [1 - \delta/(m+1)]\{\sum_j M(u_j) + M[(u_1 + \cdots + u_m)/m]\}/(m+1) \\ &\quad \leq [1 - \delta/(m+1)]\sum_j M(u_j)/m \end{aligned}$$

(III)$\Rightarrow$(I). In fact, (I) is the special case of (III) when $m = 2$, $\beta = (1 - \epsilon)a$ and $b = a$.

Now, we turn to the k-uniform rotundity. A Banach space X is *k-uniformly rotund*, provided that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x^1, \ldots, x^{k+1} \in S(X)$, the sphere of X, $\Delta(x^1, \ldots, x^{k+1}) \geq \epsilon$ implies $\|x^1 + \cdots + x^{k+1}\| \leq (1 - \delta)(k+1)$, where

$$\Delta(x^1, \ldots, x^{k+1}) = \sup_{f_j \in S(X^*)} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x^1) & f_1(x^2) & \cdots & f_1(x^{k+1}) \\ \vdots & \vdots & \vdots & \vdots \\ f_k(x^1) & f_k(x^2) & \cdots & f_k(x^{k+1}) \end{vmatrix}$$

(see [1]).

https://doi.org/10.4153/CMB-1991-021-6 Published online by Cambridge University Press

THEOREM 1. *M is k-uniformly rotund ($k \geq 1$) if and only if $(a) M \in \delta_2$ and (b) $M(u)$ is uniformly convex on* $[0, M^{-1}(1/(k+1))]$.

PROOF. Necessity: (a) is trivial since k-uniform rotundity implies reflexivity. If (b) does not hold, then there exists $\epsilon > 0, u_n, v_n \in [0, M^{-1}(1/(k+1))]$, $u_n - v_n \geq \epsilon u_n$ satisfying

$$M\left[\frac{1}{2}(u_n + v_n)\right] > \frac{1}{2}(1 - \frac{1}{n})[M(u_n) + M(v_n)] \quad (n = 1, 2, \ldots).$$

Denote $w_n = [(k-1)u_n + (k+1)v_n]/2k$, then $M(v_n) < M(w_n) < M(u_n) \leq 1/(k+1)$, therefore, $M(u_n) + kM(w_n) < 1$. Choose an integer m_n such that $\frac{1}{2} < m_n[M(u_n) + kM(w_n)] \leq 1$ and $t_n \geq 0$ such that $m_n[M(u_n) + kM(w_n)] + M(t_n) = 1$, define

$$x^{(n,1)} = (\overbrace{w_n, \ldots, w_n}^{km_n}, \overbrace{u_n, \ldots, u_n}^{m_n}, t_n, 0, 0, \ldots)$$

$$x^{(n,2)} = (\overbrace{u_n, \ldots, u_n}^{m_n}, \overbrace{w_n, \ldots, w_n}^{km_n}, t_n, 0, 0, \ldots)$$

$$x^{(n,3)} = (\overbrace{w_n, \ldots, w_n,}^{m_n}, \overbrace{u_n, \ldots, u_n}^{m_n}, \overbrace{w_n, \ldots, w_n}^{(k-1)m_n}, t_n, 0, 0, \ldots)$$

$$\vdots$$

$$x^{(n,k)} = (\overbrace{w_n, \ldots, w_n}^{(k-2)m_n}, \overbrace{u_n, \ldots, u_n}^{m_n}, \overbrace{w_n, \ldots, w_n}^{2m_n}, t_n, 0, 0, \ldots)$$

$$x^{(n,k+1)} = (\overbrace{w_n, \ldots, w_n}^{(k-1)m_n}, \overbrace{u_n, \ldots, u_n}^{m_n}, \overbrace{w_n, \ldots, w_n}^{2m_n}, t_n, 0, 0, \ldots) \quad (n = 1, 2, \cdots),$$

then for each $n \geq 1$ and $j \leq k+1$, $\rho(x^{(n,j)}) = m_n M(u_n) + km_n M(w_n) + M(t_n) = 1$; therefore, $\|x^{(n,j)}\| = 1$. Furthermore

$$\begin{aligned}
\rho\left(\sum_j x^{n,j}/(k+1)\right) &= (k+1)m_n M\left[(u_n + kw_n)/(k+1)\right] + M(t_n) \\
&= (k+1)m_n M\left[\frac{1}{2}(u_n + v_n)\right] + M(t_n) \\
&> (k+1)m_n(1 - \frac{1}{n})\frac{[M(u_n) + M(v_n)]}{2} + M(t_n) \\
&= (1 - \frac{1}{n})\left\{m_n M(u_n) + km_n \frac{[(k-1)M(u_n) + (k+1)M(v_n)]}{2k} + M(t_n)\right\} \\
&\geq \left(1 - \frac{1}{n}\right)\{m_n M(u_n) + km_n M(w_n) + M(t_n)\} \\
&= (1 - \frac{1}{n}) \longrightarrow 1
\end{aligned}$$

it follows $\|\sum_j x^{(n,j)}\| \longrightarrow k+1$, $(n \longrightarrow \infty)$.

Set $c_n = [m_n M^{-1}(\frac{1}{mn})]^{-1}$ and

$$f^{(n,j)} = (\overbrace{0,\ldots,0}^{(j-1)m_n},\overbrace{c_n,\ldots,c_n}^{m_n}0,0,\ldots) \quad (n = 1,2,\ldots;j = 1,2,\ldots,k)$$

then $f^{(n,j)} \in \ell_M^*$ and $\|f^{(n,j)}\| = 1 \quad (n = 1,2,\ldots;j = 1,2,\ldots,k)$, (see [4,5]) thus

$$\begin{aligned}
\Delta(x^{(n,1)},\ldots,x^{(n,k+1)}) &\geq \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f^{(n,1)}(x^{(n,1)}) & f^{(n,1)}(x^{(n,2)}) & \cdots & f^{(n,1)}(x^{(n,k+1)}) \\ \vdots & \vdots & \vdots & \vdots \\ f^{(n,k)}(x^{(n,1)}) & f^{(n,k)}(x^{(n,2)}) & \cdots & f^{(n,k)}(x^{(n,k+1)}) \end{vmatrix} \\
&= \begin{vmatrix} c_n m_n(u_n - w_n) & 0 & 0 & \cdots & 0 \\ 0 & c_n m_n(u_n - w_n) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_n m_n(u_n - w_n) \end{vmatrix} \\
&= [c_n m_n(u_n - w_n)]^k \\
&= [c_n m_n(k+1)(u_n - v_n)/2k]^k \\
&\geq [(k+1)\epsilon c_n m_n u_n/2k]^k
\end{aligned}$$

By (1), $m_n(k+1)M(u_n) > \frac{1}{2}$, therefore

$$\begin{aligned}
c_n m_n u_n &= u_n/M^{-1}(1/m_n) \\
&> u_n/M^{-1}[2(k+1)M(u_n)] \\
&> \frac{1}{2(k+1)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta(x^{(n,1)},\ldots,x^{(n,k+1)}) &\geq [(k+1)\epsilon/4k(k+1)]^k \\
&= (\epsilon/4k)^k
\end{aligned}$$

a contradiction.

Sufficiency: let $x^{(n,1)},\ldots,x^{(n,k+1)} \epsilon S(\ell_M)$ and $\rho\left[\frac{\sum_j x^{(n,j)}}{(k+1)}\right] \longrightarrow 1$ as $n \longrightarrow \infty$. We need to prove that for any $\tau > 0$, we have $\Delta_n \equiv \Delta(x^{(n,1)},\ldots,x^{(n,k+1)}) \leq O(\tau)$ for all n large enough.

By Lemma 1, we can select $\epsilon > 0$ satisfying $(k+1)\epsilon < 1$ and

(2) $$\rho(x) \leq 1, \rho(y) \leq (k+2)\epsilon \Rightarrow |\rho(x+y) - \rho(x)| < \tau,$$

(3) $$\rho(z) \leq (k+1)\epsilon \Rightarrow \|z\| < \tau.$$

Choose $\beta \in (0, M^{-1}[\frac{1}{(k+1)}])$ such that

(4) $$kM(\beta) + M[(1+\epsilon)\beta] > 1$$

https://doi.org/10.4153/CMB-1991-021-6 Published online by Cambridge University Press

and denote

$$I_n = \{\, i : \max_{j,1}\{\, |x_i^{(n,j)} - x_i^{(n,1)}| \,\} < \epsilon \max_j |x_i^{(n,j)}| \,\},$$

$$J_n = \{\, i : \max_{j,1}\{\, |x_i^{(n,j)} - x_i^{(n,1)}| \,\} \geq \epsilon \max_j |x_i^{(n,j)}| \,\}; \quad \text{or } (x_i^{(n,j)})_j$$

$$\text{have different signs or they have same sign but } \min_j |x_i^{(n,j)}| \leq \beta \,\}$$

$$K_n = \{\, i : \max_{j,1}\{\, |x_i^{(n,j)} - x_i^{(n,1)}| \,\} \geq \epsilon \max_j |x_i^{(n,j)}| \,\};\ (x_i^{(n,j)})_j$$

$$\text{have same sign and } \max_j |x_i^{(n,j)}| > \beta \,\} \quad (n = 1, 2, \ldots).$$

Since $M \in \delta_2$, for $f_n \in S(\ell_M^*)$, there exists $(y_i^{(n)})_i \in \ell_N$ such that $f_n(x) = \sum_i x_i y_i^{(n)}$, where $N(v) = \sup_u\{\, uv - M(u) \,\}$ (see[4,5]). Therefore

$$\Delta_n = \sup_{\substack{\|y^j\|_N \leq 1; \\ (j=1,\ldots,k)}} \begin{vmatrix} \sum_i y_i^{(1)}(x_i^{(n,2)} - x_i^{(n,1)}) & \cdots & \sum_i y_i^{(1)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \\ \vdots & \vdots & \vdots \\ \sum_i y_i^{(k)}(x_i^{(n,2)} - x_i^{(n,1)}) & \cdots & \sum_i y_i^{(k)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \end{vmatrix}$$

(5)

$$= \sup \begin{vmatrix} \sum_{I_n+J_n+H_n} y_i^{(1)}(x_i^{(n,2)} - x_i^{(n,1)}) & \cdots & \sum_{I_n+J_n+H_n} y_i^{(1)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \\ \vdots & \vdots & \vdots \\ \sum_{I_n+J_n+H_n} y_i^{(k)}(x_i^{(n,2)} - x_i^{(n,1)}) & \cdots & \sum_{I_n+J_n+H_n} y_i^{(k)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \end{vmatrix}$$

Denote $e^i = (0, \ldots, 0, \overbrace{1}^{i\text{th}}, 0, \ldots)$. By the choice of I_n and $(K+1)\epsilon < 1$, we have

(6)

$$\begin{aligned} \rho\left(\sum_{I_n}(x_i^{(n,j)} - x_i^{(n,1)})e^i\right) &= \sum_{I_n} M(x_i^{(n,j)} - x_i^{(n,1)}) \\ &\leq \sum_{I_n} M(\epsilon(|x_i^{(n,1)}| + \cdots + |x_i^{(n,k+1)}|)) \\ &< (k+1)\epsilon \sum_{I_n} M\left(\frac{\sum_j |x_i^{(n,j)}|}{(k+1)}\right) \\ &\leq \epsilon \sum_j \sum_i M(x_i^{(n,j)}) \\ &= (k+1)\epsilon \quad (j = 1, \ldots, k+1) \end{aligned}$$

It follows from (3) that $\left\|\left(\sum_{I_n}\left(x_i^{(n,j)} - x_i^{(n,1)}\right)e^i\right)\right\| < \tau$, therefore

$$\begin{aligned} \left|\sum_{I_n} y_i^{(h)}\left(x_i^{(n,j)} - x_i^{(n,1)}\right)\right| &\leq \|y^{(h)}\|_N \left\|\sum_{I_n}\left(x_i^{(n,j)} - x_i^{(n,1)}\right)e^i\right\| \\ &< \tau \quad (n = 1, 2, \ldots; j = 2, \ldots, k+1; h = 1, \ldots, k). \end{aligned}$$

Next we show

(8) $$\rho\left(\sum_{J_n}\left(x_i^{(n,j)} - x_i^{(n,1)}\right)e^i\right) \to 0 \quad (n \to \infty)$$

https://doi.org/10.4153/CMB-1991-021-6 Published online by Cambridge University Press

By Lemma 2, for $i \in J_n$ and all $\left(x_i^{(n,j)}\right)_j$ have same sign; besides $\min_j |x_i^{(n,j)}| \leq \beta$, then

$$M\left(\sum_j x_i^{(n,j)}/(k+1)\right) \leq (1-\delta)\sum_j M(x_i^{(n,j)})/(k+1) \tag{9}$$

Otherwise $(x_i^{(n,j)})_j$ have different signs since $i \in J_n$. Say $x_i^{(n,1)}, \ldots, x_i^{(n,p)}$ have same sign, $x_i^{(n,p+1)}, \ldots, x_i^{(n,k+1)}$ have same sign and $\sum_{j\leq p}\left|x_i^{(n,j)}\right| \geq \sum_{j>p}\left|x_i^{(n,j)}\right|$. Consider the $K+1$ numbers $x_i^{(n,1)}, \ldots, x_i^{(n,p)}, 0, \ldots, 0$; obviously, they satisfy all the conditions in Lemma 2, for $b = M^{(-1)}(1)$, $a = M^{-1}(\frac{1}{k+1})$ and $m = K+1$. Therefore, there exists $\delta > 0$ such that

$$\begin{aligned} M\left(\sum_{j\leq k+1}\frac{x_i^{(n,j)}}{k+1}\right) &< M\left(\sum_{j\leq p}\frac{x_i^{(n,j)}}{k+1}\right) \\ &\leq (1-\delta)\sum_{j\leq p}\frac{M\left(x_i^{(n,j)}\right)}{k+1} \\ &< (1-\delta)\sum_{j\leq k+1}\frac{M\left(x_i^{(n,j)}\right)}{k+1}. \end{aligned}$$

If (8) is not true, we can find a subsequence of $\{n\}$, again denoted by $\{n\}$, such that $\sum_{J_n} M(x_i^{(n,j)} - x_i^{(n,1)}) \geq \sigma > 0\,(n \geq 1)$. It follows from $M \in \delta_2$ that there exists a positive constant c such that $\sum_{J_n} M\left(\frac{x_i^{(n,j)} - x_i^{(n,1)}}{2}\right) \geq c\sigma \quad (n \geq 1)$. Therefore by (9)

$$\begin{aligned} 0 \leftarrow 1 - \sum_{i\geq 1} M\left(\sum_{j\leq k+1}\frac{x_i^{(n,j)}}{k+1}\right) \\ = \sum_{i\geq 1}\left\{\sum_{j\leq k+1}\left(M\frac{x_i^{(n,j)}}{k+1}\right) - M\left(\sum_{j\leq k+1}\frac{x_i^{(n,j)}}{k+1}\right)\right\} \\ \geq \sum_{J_n}\left\{\sum_{j\leq k+1} M\left(x_i^{(n,j)}\right)/(k+1) - M\left(\sum_{j\leq k+1} x_i^{(n,j)}/(k+1)\right)\right\} \\ > \delta\sum_{J_n}\left\{\sum_{j\leq k+1} M(x_i^{(n,j)})/(k+1)\right\} \\ \geq \delta\sum_{J_n} M\left(\frac{x_i^{(n,j)} - x_i^{(n,1)}}{2}\right)/(k+1) \\ \geq \frac{\delta c\sigma}{k+1}. \end{aligned}$$

This contradiction verifies (8). Hence from (3), for all n large enough

$$\left|\sum_{J_n} y_i^{(h)}(x_i^{(n,j)} - x_i^{(n,1)})\right| \leq \|y^{(h)}\|_N \left\|\sum_{J_n}(x_i^{(n,j)} - x_i^{(n,1)})e^i\right\| < \tau \tag{10}$$

From (10), (7) and (5), it is easy to see that

$$\Delta_n = \sup\begin{vmatrix} \sum_{H_n} y_i^{(1)}(x_i^{(n,2)} - x_i^{(n,1)}) & \cdots & \sum_{H_n} y_i^{(1)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \\ \vdots & \vdots & \vdots \\ \sum_{H_n} y_i^{(k)}(x_i^{(n,2)} - x_i^{(n,1)}) & \cdots & \sum_{H_n} y_i^{(k)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \end{vmatrix} + O(\tau) \tag{11}$$

https://doi.org/10.4153/CMB-1991-021-6 Published online by Cambridge University Press

Now, we show that H_n contains no more than K different numbers. In fact, otherwise, we may assume $1, \dots, K+1 \in H_n$, $x_1^{(n,2)} = \min_j |x_1^{(n,j)}|$, $x_1^{(n,1)} = \max_j |x_1^{(n,j)}|$. From the definition of H_n, we have $x_1^{(n,2)} \geq \beta$ and $x_1^{(n,1)} - x_1^{(n,2)} \geq \epsilon x_1^{(n,1)}$, hence $x_1^{(n,1)} \geq (1+\epsilon)\beta$. Combined with (4) we obtain a contradiction

$$1 = \rho(x^{(n,1)}) \geq \sum_{j \leq k+1} M(x_j^{(n,1)}) > M((1+\epsilon)\beta) + kM(\beta) > 1.$$

We may assume, without loss of generality, $H_n = \{1, \dots, k\}$ for all $n \geq 1$. (11) can be written as

$$(12) \qquad \Delta_n = \sup \begin{vmatrix} \sum_{i \leq k} y_i^{(1)}(x_i^{(n,2)} - x_i^{(n,1)}) & \dots & \sum_{i \leq k} y_i^{(1)}(x_i^{(n,k+1)} - x_i^{(n,1)}) \\ \vdots & \vdots & \vdots \\ \sum_{i \leq k} y_i^k(x_i^{(n,2)} - x_i^{(n,1)}) & \dots & \sum_{i \leq k} y_i^k(x_i^{(n,k+1)} - x_i^{(n,1)}) \end{vmatrix} + O(\tau)$$

Select a subsequence of $\{\Delta_n\}$, once more symbolized by $\{\Delta_n\}$, such that corresponding sequence $\{x_i^{(n,j)}\}_n$ converges, say the limit is $a_i^j (i = 1, 2, \dots, k; j = 1, \dots, k+1)$. Then from (12) it is easy to deduce

$$(13) \qquad \Delta_n = \sup \begin{vmatrix} \sum_{i \leq k} y_i^{(1)}(a_i^2 - a_i^1) & \dots & \sum_{i \leq k} y_i^{(1)}(a_i^{k+1} - a_i^1) \\ \vdots & \vdots & \vdots \\ \sum_{i \leq k} y_i^{(k)}(a_i^2 - a_i^1) & \dots & \sum_{i \leq k} y_i^{(k)}(a_i^{k+1} - a_i^1) \end{vmatrix} + O(\tau)$$

for n large enough.

Notice that $\sum_{i \geq 1} \left\{ \sum_{j \leq k+1} M\left(x_i^{(n,j)}\right) / (k+1) - M\left(\sum_{j \leq k+1} x_i^{(n,j)} / (k+1)\right) \right\} \longrightarrow 0$, so $\sum_{j \leq k+1} M\left(x_i^{(n,j)}\right) / (k+1) - M(\sum_{j \leq k+1} x_i^{(n,j)} / (k+1)) \longrightarrow 0$. Let $n \longrightarrow \infty$; we have

$$\sum_{j \leq k+1} M(a_i^j) / (k+1) = M\left(\sum_{j \leq k+1} a_i^j / (k+1) \right) \quad (i = 1, 2, \dots, k)$$

which shows $(a_i^j)_j$ are in the same interval on which $M(u)$ is linear, say

$$(14) \qquad M(u) = P_i u + Q_i; \quad (i \leq k)$$

From (6) and (8), when n is large enough, we have $\sum_{I_n \cup J_n} M(x_i^{(n,j)} - x_i^{(n,1)}) < (k+2)\epsilon$ $(j \leq k+1)$ it follows from (2)

$$\left| \sum_{I_n \cup J_n} M(x_i^{(n,j)}) - \sum_{I_n \cup J_n} M(x_i^{(n,1)}) \right| < \tau \quad (j \leq k+1)$$

Therefore, by $\rho(x^{(n,j)}) = 1 (j \leq k+1)$, we have $\left| \sum_{H_n} M(x_i^{(n,j)}) - \sum_{H_n} M(x_i^{(n,1)}) \right| < \tau$ $(j \leq k+1)$. Let $n \longrightarrow \infty$, we obtain $\left| \sum_{i \leq k} P_i(a_i^j - a_i^1) \right| \leq \tau$. Thus

$$a_i^j - a_1^1 \leq -\frac{1}{P_1} \sum_{i=2}^{k} P_i(a_i^j - a_i^1) + \frac{\tau}{P_1} \quad (j \leq k+1)$$

https://doi.org/10.4153/CMB-1991-021-6 Published online by Cambridge University Press

Notice that $0 < M'(\beta) \le M'(a_i^j) = P_i \le M'(M^{-1}(1))$ $(i = 1,2,\ldots,k)$, it is easy to deduce

$$\Delta_n = \sup \begin{vmatrix} \sum_{i=2}^k (y_i^{(1)} - y_1^{(1)} Pi/P_1)(a_i^2 - a_i^1) & \ldots & \sum_{i=2}^k (y_i^{(1)} - y_1^{(1)} Pi/P_1)(a_i^{k+1} - a_i^1) \\ \vdots & \vdots & \vdots \\ \sum_{i=2}^k (y_i^{(k)} - y_1^{(k)} Pi/P_1)(a_i^2 - a_i^1) & \ldots & \sum_{i=2}^k (y_i^{(k)} - y_1^{(k)} Pi/P_1)(a_i^{k+1} - a_i^1) \end{vmatrix}$$

$$(15) \quad + O(\tau)$$

Finally, we expand the last determinant into $(K-1)^{k-1}$ many determinants of order K; each of them has the form

$$\begin{vmatrix} (y_{s_1}^{(1)} - y_1^{(1)} P_{s_1}/P_1)(a_{s_1}^2 - a_{s_1}^1) & \ldots & (y_{s_k}^{(1)} - y_1^{(1)} P_{s_k}/P_1)(a_{s_k}^{k+1} - a_{s_k}^1) \\ \vdots & \vdots & \vdots \\ (y_{s_1}^{(k)} - y_1^{(k)} P_{s_1}/P_1)(a_{s_1}^2 - a_{s_1}^1) & \ldots & (y_{s_k}^{(k)} - y_1^{(k)} P_{s_k}/P_1)(a_{s_k}^{k+1} - a_{s_k}^1) \end{vmatrix}$$

Since each s_i is between 2 and K, such determinant has at least two proportional columns, therefore it vanishes, i.e., (15) becomes $\Delta_n = O(\tau)$.

For any integer $K \ge 1$, we define $M(u) = u^2$ on $(-M^{-1}(\frac{1}{k+1}), M^{-1}(\frac{1}{k+1}))$ and $M(u) = 2M^{-1}(1/(k+1))|u| - (M^{-1}(1/(k+1)))^2$ otherwise; then it is easily verified by the theorem that ℓ_M, generated by $M(u)$, is kUR but not $(k-1)$ UR.

REFERENCES

1. F. Sullivan, *A generalization of uniformly rotund Banach spaces*, Canad. J. Math. **31**(1979), 628–636.
2. A. Kaminska, *On uniform convexity of Orlicz spaces*, Indag. Math. **A85**(1982), 27–36.
3. W. Congxin, W. Tingfu, C. Shutao, W. Yuwen, *Geometry of Orlicz spaces*. Harbin, 1986.
4. J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I*. Springer-Verlag, 1977.
5. M. A. Krasnoselskii, Ya. B. Rutickii, *Convex functions and orlicz spaces*. Groningen, 1961.

Department of Mathematics
Harbin University of Science and Technology
Harbin, Heilongjiang
China

Harbin Teachers University
Harbin, Heilongjiang
China