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ON THE BRÜCK CONJECTURE

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Abstract

The Brück conjecture states that if a nonconstant entire function f with hyper-order $\sigma_2(f) \in [0, +\infty)\setminus\mathbb{N}$ shares one finite value a (counting multiplicities) with its derivative f', then $f' - a = c(f - a)$, where c is a nonzero constant. The conjecture has been established for entire functions with order $\sigma(f) < +\infty$ and hyper-order $\sigma_2(f) < \frac{1}{2}$. The purpose of this paper is to prove the Brück conjecture for the case $\sigma_2(f) = \frac{1}{2}$ by studying the infinite hyper-order solutions of the linear differential equations $f^{(k)} + A(z)f = Q(z)$. The shared value a is extended to be a 'small' function with respect to the entire function f.

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1. Introduction and main results

In this paper, a meromorphic function is analytic at all points in the complex plane except possibly at a set of poles. We say that two nonconstant meromorphic functions f and g share a meromorphic function h provided that $f(z) - h(z) = 0$ if and only if $g(z) - h(z) = 0$. The functions f and g share h CM if $f - h$ and $g - h$ have the same zeros with the same multiplicities. In 1926, Nevanlinna [20] established the Second Main Theorem concerning the counting function $N(r, f)$, proximity function $m(r, f)$ and characteristic function $T(r, f)$ of a meromorphic function f, and proved the five-value theorem which states that two nonconstant meromorphic functions having the same inverse images (ignoring multiplicities) for five distinct values in the complex plane are identically equal. In 1977, Rubel and Yang proved the following result.

Theorem 1.1 [21]. *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f(z) \equiv f'(z)$: that is, $f(z) = ce^z$, where c is a nonzero constant.*

Example 1.2 [2]. It is easy to check that the entire function

$$f(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)\,dt$$

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satisfies the equation

$$\frac{f'-1}{f-1}=e^z.$$

This means that f and f' share 1 CM. However, $f \not\equiv f'$. Thus the number of shared values in Theorem 1.1 cannot be reduced to one.

This naturally leads to the following question.

Question 1.3 [2, 23]. What can be said when a nonconstant entire function f shares one finite value CM with f'?

The order and hyper-order of an entire function f are defined respectively by

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log^+ T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log^+\log^+ M(r,f)}{\log r},$$

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log^+\log^+ T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log^+\log^+\log^+ M(r,f)}{\log r},$$

where $\log^+ x$ means $\max\{\log x, 0\}$ and $M(r,f)$ denotes the maximum modulus of f on the circle $|z|=r$ centred at the origin. Another entire function h is said to be 'small' with respect to f if $T(r,h)=o(T(r,f))$ as $r\to+\infty$ (thus, $|h(z)|=o(|f(z)|)$ as $|z|=r\to+\infty$ by the definition of the characteristic functions of the entire functions of f and h). For example, polynomials are 'small' with respect to any transcendental entire function.

Note that the function f in Example 1.2 satisfies $\sigma_2(f)=1$. Similarly, one can construct entire functions f satisfying the equations

$$\frac{f'-1}{f-1}=e^{z^n}\quad\text{where } \sigma_2(f)=n\in\mathbb{N}$$

$$\frac{f'-1}{f-1}=e^{e^z}\quad\text{where } \sigma_2(f)=+\infty$$

(see [2]). In 1996, Brück [2] proposed the following conjecture.

Conjecture 1.4 [2]. Let f be a nonconstant entire function such that its hyper-order is finite but not a positive integer. If f and f' share one finite value a CM, then $f'-a=c(f-a)$, where c is a nonzero constant.

The conjecture for the case $a=0$ was affirmed by Brück [2]. In this case, we have $f=c_1e^{cz}$, where c_1 and c are two nonzero constants. In 1998, Gundersen and Yang [11] affirmed the conjecture for the case where f is of finite order. Chen and Shon [8] affirmed it when f is of hyper-order strictly less than $\frac{1}{2}$.

These conclusions on the Brück conjecture have been extended in two directions. One replaces the shared value by a nonconstant function: Li [17, Corollary 1.4] proved that the result of Gundersen and Yang [11] is true for a shared polynomial; Chang and Zhu [5] considered the case where the order of a shared function is strictly less than the order of f; and Wang [22] showed that the conclusion of Gundersen and Yang [11]

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is true for a shared function that is 'small' with respect to f. In all these papers, the order of f is finite.

The other direction is to consider the case of arbitrary kth derivatives $f^{(k)}$ instead of f'. Thus, Yang [23] and Chen and Shon [7] respectively extended the results of [11] and [8] to kth derivatives $f^{(k)}$. In [9], Chen and Zhang considered the case where f with hyper-order $<\frac{1}{2}$ shares fixed points with $f^{(k)}$. Li and Gao [18, Theorem 1.2] and Cao [3, Theorem 5.1] studied the case of a polynomial shared by f and $f^{(k)}$.

For meromorphic functions of finite order, the Brück conjecture fails in general. For example [11], the meromorphic function $f(z) = (2e^z + z + 1)/(e^z + 1)$ shares the value 1 CM with f', while $(f' - 1)/(f - 1)$ is not a constant.

The main purpose of this paper is to confirm the Brück conjecture for the case when the hyper-order of f is equal to $\frac{1}{2}$. Furthermore, the shared value is extended to entire functions that are 'small' with respect to f. We obtain the following result on the Brück conjecture, which improves and generalises all the results mentioned above.

Theorem 1.5. *Let f be a nonconstant entire function with hyper-order $\le\frac{1}{2}$, and let a_1 and a_2 be entire functions that are 'small' with respect to f. If $f - a_1$ and $f^{(k)} - a_2$ share the same zeros with the same multiplicities, then $f^{(k)} - a_2(z) = c(f - a_1(z))$, where c is a nonzero constant.*

Set $f(z) = e^{2z} - (z - 1)e^z$ and $a(z) = e^{2z} - ze^z$. Then $T(r, a) = T(r, f) = O(r)$, while $(f' - a(z))/(f - a(z)) = e^z$ is not a constant (see [22]). This example shows that it is necessary for a_1, a_2 to be 'small' functions with respect to f in Theorem 1.5.

The following corollary follows immediately from Theorem 1.5 for the special case when a_1 and a_2 are the same constant.

Corollary 1.6. *Let f be a nonconstant entire function with hyper-order $\le\frac{1}{2}$. If f shares one finite value a CM with its kth derivative, then $f^{(k)} - a = c(f - a)$, where c is a nonzero constant.*

The Brück conjecture remains open when the hyper-order of f is in $(\frac{1}{2}, +\infty)\backslash\mathbb{N}$.

To handle the case of hyper-order $\sigma_2(f) = \frac{1}{2}$ in Theorem 1.5, we first study, in Section 2, the infinite hyper-order solutions of the linear differential equations $f^{(k)} + A(z)f = Q(z)$, where the hyper-order of A is less than or equal to $\frac{1}{2}$. Theorem 1.5 is proved in Section 3.

2. Results on the differential equations $f^{(k)} + A(z)f = Q(z)$

In 1982, Bank and Laine [1] proved that any nonzero solution of the differential equation $f'' + A(z)f = 0$ with a polynomial coefficient A is an entire function with order $\sigma(f) = \frac{1}{2}(\deg(A) + 2)$, where $\deg(A)$ denotes the degree of A. If A is a transcendental entire function, then all solutions f of $f'' + A(z)f = 0$ satisfy $\sigma(f) = +\infty$ by the lemma of the logarithmic derivative. For the nonhomogeneous linear differential equation

$$f^{(k)} + A(z)f = Q(z) \quad (k \in \mathbb{N}), \tag{2.1}$$

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where $k \geq 2$, $Q(\not\equiv 0)$ is an entire function of finite order and A is a transcendental entire function, Chen and Gao [6] showed that every solution f is an entire function of infinite order, with at most one possible exception.

Many authors concentrated on the special case when A has no zeros and every solution of (2.1) is of infinite order. For example, in [11, 23] it was proved that every solution of the differential equation

$$f^{(k)} - e^{P(z)} f = 1 \quad (k \in \mathbb{N}) \tag{2.2}$$

is an entire function of infinite order when P is a nonconstant polynomial. In [24, 25], Yang asked whether the hyper-order of every solution f of equation (2.2) is a positive integer or infinite when P is a nonconstant entire function. It was shown later [3] that, when Q is a nonzero polynomial and P is a nonconstant polynomial, every solution of

$$f^{(k)} - e^{P(z)} f = Q(z) \quad (k \in \mathbb{N}) \tag{2.3}$$

has infinite order and its hyper-order is a positive integer less than or equal to the degree of P. It follows from [18, Theorem 1.1] that the hyper-order of f is equal to the degree of P. When P is a transcendental entire function with order $<\frac{1}{2}$ and Q is a nonzero polynomial, all solutions f of (2.3) have infinite hyper-order (see [3]).

For $r \in [0, +\infty)$, define $\exp_1 r = e^r$ and $\exp_{n+1} r = \exp(\exp_n r)$ for $n \in \mathbb{N}$. For all r sufficiently large, define $\log_1 r = \log r$ and $\log_{n+1} r = \log(\log_n r)$ for $n \in \mathbb{N}$. We also write $\exp_0 r = r = \log_0 r$, $\log_{-1} r = \exp_1 r$ and $\exp_{-1} r = \log_1 r$. As in [15, 16], the p-iterated order $\sigma_p(f)$ and p-iterated convergent exponent $\lambda_p(f)$ of an entire function f are respectively defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r},$$
$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N(r,1/f)}{\log r}.$$

The iterated order for an entire function f can also be defined by its central index (see [4, Lemma 6]) as

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p \nu(r,f)}{\log r}.$$

Also, as in [15, 16], the growth index of the iterated order of a meromorphic function f is defined by $i(f) = 0$ if f is rational and, for a transcendental function f,

$$i(f) = \begin{cases} \min\{p \in \mathbb{N} : \sigma_p(f) < +\infty\} & \text{if } \sigma_p(f) < +\infty \text{ for some } p \in \mathbb{N}, \\ +\infty & \text{if } \sigma_p(f) = +\infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

In this section, we continue to consider (2.3) in cases where every solution has infinite order. The first result is concerned with the case when Q is an entire function that is 'small' with respect to the solutions and P is a nonconstant polynomial.

THEOREM 2.1. *Let P be a nonconstant polynomial and let f be a nonzero entire solution of the differential equation (2.3), where Q is an entire function that is 'small' with respect to f. Then the hyper-order of f is equal to the degree of P.*

PROOF. Since P is a nonconstant polynomial, any solution $f(\not\equiv 0)$ of (2.3) is transcendental. By the Wiman–Valiron theory (see, for example, [13, 16]), there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure (that is $\int_E t^{-1}\,dt < +\infty$), such that for some point $z_r = re^{i\theta}$ ($\theta \in [0, 2\pi)$) satisfying $|z_r| = r \notin E$ and $M(r, f) = |f(z_r)|$,

$$\frac{f^{(k)}(z_r)}{f(z_r)} = \Big(\frac{\nu(r, f)}{z_r}\Big)^k (1 + o(1)) \tag{2.4}$$

as $r \to +\infty$, where $\nu(r, f)$ denotes the central index of the entire function f.

Since Q is an entire function that is 'small' with respect to f,

$$\frac{|Q(z)|}{|f(z)|} = o(1) \tag{2.5}$$

as $r \to +\infty$, for sufficiently large $|z| = r \notin E$. (We remark that if Q is identically equal to zero, the proof will still work.)

We may assume that $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with degree $\deg(P) := n$ and $a_n \neq 0$. Then, for $|z| = r$,

$$|a_n| r^n (1 - o(1)) \leq |P(z)| \leq |a_n| r^n (1 + o(1)). \tag{2.6}$$

On the one hand, it follows from (2.3) that

$$\left|\frac{f^{(k)}}{f}\right| \leq \frac{|Q(z)|}{|f|} + |e^{P(z)}|. \tag{2.7}$$

Substituting (2.4)–(2.6) into (2.7) gives

$$\begin{aligned} k \log \nu(r, f) &\leq \log\Big(\frac{|Q(z_r)|}{f(z_r)} + e^{|P(z_r)|}\Big) + k \log r + o(1) \\ &\leq |P(z_r)| + k \log r + O(1) \\ &\leq |a_n| r^n (1 + o(1)) + k \log r + O(1), \end{aligned}$$

and thus

$$\log \log \nu(r, f) \leq n \log r + \log \log r + O(1)$$

for sufficiently large $r = |z_r| \notin E$. Since $M(r, f) = |f(z_r)|$, we have $\sigma_2(f) \leq n = \deg(P)$.

On the other hand, rewrite (2.3) as

$$e^{P(z)} = \frac{f^{(k)}}{f} - \frac{Q(z)}{f}. \tag{2.8}$$

Taking the principal branch of the logarithm, (2.8) becomes

$$P(z) = \log\Big(\frac{f^{(k)}}{f} - \frac{Q(z)}{f}\Big) = \log\left|\frac{f^{(k)}}{f} - \frac{Q(z)}{f}\right| + i \arg\Big(\frac{f^{(k)}}{f} - \frac{Q(z)}{f}\Big). \tag{2.9}$$

Substituting (2.4)–(2.6) into (2.9), we obtain

$$\begin{aligned}|a_n|r^n(1-o(1)) &\le |P(z_r)|\\ &\le \log\left|\frac{f^{(k)}}{f}(z_r)\right| + \log\left(\left|\frac{Q(z_r)}{f(z_r)}\right| + e\right) + O(1)\\ &\le k\log\frac{\nu(r,f)}{r} + O(1)\end{aligned}$$

and thus

$$n\log r \le \log\log \nu(r,f) - \log\log r + O(1)$$

for sufficiently large $r = |z_r| \notin E$. Since $M(r,f) = |f(z_r)|$, we have $\deg(P) = n \le \sigma_2(f)$.

Therefore, $\sigma_2(f) = \deg(P)$. □

Next, we adapt the method of Rossi [19] to consider the case when the p-iterated order of all solutions of (2.1) is infinite and A is a transcendental entire function with $i(A) = p$ and $Q(\not\equiv 0)$ is a 'small' function with respect to solutions f.

THEOREM 2.2. *Let A be a transcendental entire function with $i(A) = p$ $(0 < p < +\infty)$, and let f be an entire solution of the differential equation (2.1), where Q is a nonzero entire function that is 'small' with respect to f. Then either $\sigma_p(f) = +\infty$ or*

$$\frac{1}{\sigma_p(A)} + \frac{1}{\sigma_p(f)} \le 2.$$

In particular, if $\sigma_p(A) \le \frac{1}{2}$, then $\sigma_p(f) = +\infty$.

For the proof of Theorem 2.2, we introduce three lemmas as follows.

LEMMA 2.3 [10]. *Let f be a transcendental meromorphic function. Let $\alpha > 1$ be a constant and k, j integers satisfying $k > j \ge 0$.*

(i) *There exist a set $E_1 \subset (1,+\infty)$ of finite logarithmic measure and a constant $K > 0$ such that, for all z satisfying $|z| = r \notin E_1 \cup [0,1]$,*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le K\left[\frac{T(\alpha r,f)}{r}(\log r)^{\alpha}\log T(\alpha r,f)\right]^{k-j}. \tag{2.10}$$

(ii) *There exists a set $E_2 \subset [0,2\pi)$ of zero linear measure such that, if $\theta \in [0,2\pi)\backslash E_2$, then there is a constant $R(= R(\theta) > 0)$ such that (2.10) holds for all z satisfying $\arg z = \theta$ and $|z| \ge R$.*

Let D be a region in $\mathbb{C}$. For $r \in \mathbb{R}^+$, set $\theta_D^*(r) = \theta^*(r) = +\infty$, if the entire circle $|z| = r$ lies in D. Otherwise, let $\theta_D^*(r) = \theta^*(r)$ be the measure of the set of $\theta \in [0,2\pi)$ such that $re^{i\theta} \in D$.

LEMMA 2.4 [19]. *Let u be a subharmonic function in $\mathbb{C}$ and let D be an open component of* $\{z : u(z) > 0\}$. *Set* $\rho(u) := \limsup_{r\to+\infty} \log M(r,u)/\log , r$, *where* $M(r,u)$ *is the maximum modulus of the function u on a circle of radius r. Then*

$$\rho(u) \geq \limsup_{R\to+\infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{t\theta_D^*(t)}. \tag{2.11}$$

Furthermore, given $\varepsilon > 0$, define $F = \{r : \theta_D^* \leq \varepsilon\pi\}$. Then

$$\limsup_{R\to+\infty} \frac{1}{\log R} \int_{F\cap[1,R]} \frac{dt}{t} \leq \varepsilon\rho(u). \tag{2.12}$$

LEMMA 2.5 [19]. *Let $l_1(t), l_2(t) > 0$ $(t \geq t_0)$ be two measurable functions on $(0, +\infty)$ with $l_1(t) + l_2(t) \leq (2 + \varepsilon)\pi$, where $\varepsilon > 0$. If $G \subseteq (0, +\infty)$ is any measurable set and*

$$\pi \int_G \frac{dt}{tl_1(t)} \leq \alpha \int_G \frac{dt}{t}, \quad \alpha \geq \frac{1}{2},$$

then

$$\pi \int_G \frac{dt}{tl_2(t)} \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1} \int_G \frac{dt}{t}.$$

PROOF OF THEOREM 2.2. Since Q is a nonzero entire function that is 'small' with respect to f and A is transcendental, (2.1) implies that f is transcendental. From

$$A(z) = \frac{f^{(k)}}{f} - \frac{Q(z)}{f}$$

and the basic Nevanlinna theory (see, for example, [12]),

$$\begin{aligned} T(r, A) = m(r, A) &\leq m(r, 1/f) + m(r, Q) + m(r, f^{(k)}/f) + O(1) \\ &= T(r, 1/f) - N(r, 1/f) + T(r, Q) + O(\log(rT(r, f))) \\ &= T(r, f) - N(r, 1/f) + T(r, Q) + O(\log(rT(r, f))) \end{aligned}$$

for all sufficiently large r possibly outside a set F with finite linear measure. Since Q is a 'small' function with respect to f,

$$(1 + o(1))T(r, f) \geq T(r, A) + N(r, 1/f), \quad r \notin F.$$

This implies that $i(f) \geq i(A) = p$ and $\sigma_p(f) \geq \sigma_p(A)$.

Now we may assume that $\sigma_p(f) < +\infty$. By Lemma 2.3(ii) and the definition of the iterated order, there exists a constant $C = C(\varepsilon)$ such that

$$\left|\frac{f^{(k)}}{f}(re^{i\theta})\right| \leq O\left(\frac{T(\alpha r, f)}{r} (\log r)^\alpha \log T(\alpha r, f)\right)^k \leq \exp_{p-1}(r^C) \tag{2.13}$$

for all $r > r_0 = R(\theta)$ and $\theta \notin J(r)$, where $J(r)$ is a set with zero linear measure. Note that $m(J(r)) < \varepsilon\pi$ for any given $\varepsilon > 0$, which may be arbitrarily small.

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Let $N \in \mathbb{N}$ such that $N > C = C(\varepsilon)$, and

$$\log_p M(2, A) < N \log 2.$$

Since A is transcendental and $+\infty > i(A) = p > 0$, there exists z_0 ($|z_0| > 2$) such that

$$\log_p |A(z_0)| > N \log |z_0|.$$

Let D_1 be the component of the set $\{z : \log_p |A(z)| - N \log |z| > 0\}$ containing z_0. Observe that D_1 is an open set. So, $\log_p |A(z)| - N \log |z|$ is subharmonic in D_1 and identically zero on ∂D_1. If we define

$$u(z) = \begin{cases} \log_p |A(z)| - N \log |z|, & z \in D, \\ 0, & z \in \mathbb{C} \backslash D, \end{cases}$$

then $u(z)$ is subharmonic in $\mathbb{C}$ with

$$\rho(u) \le \sigma_p(A). \tag{2.14}$$

Set $D_2 := \{z : \log_p |f(z)| - \log_p |Q(z)| > 0\}$ and $D_3 := \{re^{i\theta} : \theta \in J(r)\}$. Observe that if $(D_1 \cap D_2)\backslash D_3$ contains an unbounded sequence $\{r_n e^{i\theta_n}\}$, then

$$\begin{aligned} \exp_{p-1}(r_n^N) &< |A(r_n e^{i\theta_n})| \\ &\le \left| \frac{f^{(k)}}{f}(r_n e^{i\theta_n}) \right| + \frac{|Q(r_n e^{i\theta_n})|}{|f(r_n e^{i\theta_n})|} \\ &\le \left| \frac{f^{(k)}}{f}(r_n e^{i\theta_n}) \right| + o(1) \\ &\le \exp_{p-1}(r_n^C) + o(1) \end{aligned}$$

holds for sufficiently large r_n. But this contradicts $N > C = C(\varepsilon)$. Thus, for arbitrary ε, we may assume that $(D_1 \cap D_2)\backslash D_3$ is bounded. This means that, for $r \ge r_1 \ge r_0$ (where r_0 is defined as above),

$$K_r = \{\theta : re^{i\theta} \in D_1 \cap D_2\} \subseteq J(r).$$

Therefore,

$$m(K_r) \le m(J(r)) < \varepsilon\pi. \tag{2.15}$$

(We remark here that the proof of Theorem 2.2 would now follow easily from (2.11) and Lemma 2.5 if D_1 and D_2 were disjoint. As we shall see, (2.12), (2.13) and (2.15) imply that these sets are 'essentially' disjoint.)

For $j = 1, 2$, define

$$l_j(t) = \begin{cases} 2\pi & \text{if } \theta^*_{D_j}(t) = +\infty, \\ \theta^*_{D_j}(t) & \text{otherwise.} \end{cases}$$

Since $+\infty > i(A) = p > 0$ and Q is 'small' with respect to f, it follows that D_1 and D_2 are unbounded open sets. Thus, $l_1(t), l_2(t) > 0$ for t sufficiently large, and

$$l_1(t) + l_2(t) \le 2\pi + \varepsilon\pi.$$

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Now let

$$\alpha := \limsup_{R\to+\infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{tl_1(t)}. \tag{2.16}$$

Since $l_1(t) \le 2\pi$, we have $\alpha \ge \frac{1}{2}$. If

$$\pi \int_1^R \frac{dt}{tl_1(t)} \le \alpha \log R = \alpha \int_1^R \frac{dt}{t},$$

then by Lemma 2.5,

$$\pi \int_1^R \frac{dt}{tl_2(t)} \ge \frac{\alpha}{(2+\varepsilon)\alpha - 1} \int_1^R \frac{dt}{t} = \frac{\alpha}{(2+\varepsilon)\alpha - 1} \log R,$$

and thus,

$$\limsup_{R\to+\infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{tl_2(t)} \ge \frac{\alpha}{(2+\varepsilon)\alpha - 1}. \tag{2.17}$$

For $j = 1, 2$, define $B_j = \{r : \theta^*_{D_j}(r) = +\infty\}$. If $r \in B_1$ and $r \ge r_1$, then $\theta^*_{D_2}(r) \le \varepsilon\pi$ by (2.15) and so $B_1 \subseteq \{r : \theta^*_{D_2}(r) \le \varepsilon\pi\}$. It follows from Lemma 2.4 that

$$\limsup_{R\to+\infty} \frac{1}{\log R} \int_{B_1\cap[1,R]} \frac{dt}{t} \le \varepsilon\rho(\log_p |f| - \log_p |Q|) \le \varepsilon\sigma_p(f). \tag{2.18}$$

Let $\tilde{B}_j := \mathbb{R}^+ \backslash B_j$ for $j = 1, 2$. Then it follows from (2.11), (2.16), (2.18) that

$$\begin{aligned}
\rho(u) &\ge \limsup_{R\to+\infty} \frac{\pi}{\log R} \int_1^R \frac{dt}{t\theta^*_{D_1}(t)} \\
&= \limsup_{R\to+\infty} \frac{\pi}{\log R} \int_{\tilde{B}_1\cap[1,R]} \frac{dt}{t\theta^*_{D_1}(t)} \\
&= \limsup_{R\to+\infty} \frac{1}{\log R} \cdot \left[\pi \int_1^R \frac{dt}{tl_1(t)} - \frac{1}{2} \int_{B_1\cap[1,R]} \frac{dt}{t}\right] \\
&\ge \alpha - \frac{\varepsilon}{2}\sigma_p(f),
\end{aligned}$$

which together with (2.14) shows that

$$\sigma_p(A) \ge \alpha - \frac{\varepsilon}{2}\sigma_p(f). \tag{2.19}$$

By a similar discussion as above for B_2 instead of B_1, if $r \in B_2$ and $r \ge r_1$, then $\theta^*_{D_1}(r) \le \varepsilon\pi$ and $B_2 \subseteq \{r : \theta^*_{D_1}(r) \le \varepsilon\pi\}$. Then we obtain, also from Lemma 2.4, that

$$\limsup_{R\to+\infty} \frac{1}{\log R} \int_{B_2\cap[1,R]} \frac{dt}{t} \le \varepsilon\rho(u) \le \varepsilon\sigma_p(A). \tag{2.20}$$

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It follows from (2.11), (2.17), (2.20) that

$$\begin{aligned}\sigma_p(f) &\geq \rho(\log_p|f| - \log_p|Q|)\\ &\geq \limsup_{R\to+\infty}\frac{\pi}{\log R}\int_1^R \frac{dt}{t\theta^*_{D_2}(t)}\\ &= \limsup_{R\to+\infty}\frac{\pi}{\log R}\int_{\tilde{B}_2\cap[1,R]}\frac{dt}{t\theta^*_{D_2}(t)}\\ &= \limsup_{R\to+\infty}\frac{1}{\log R}\cdot\left[\pi\int_1^R\frac{dt}{tl_2(t)} - \frac{1}{2}\int_{B_2\cap[1,R]}\frac{dt}{t}\right]\\ &\geq \frac{\alpha}{(2+\varepsilon)\alpha - 1} - \frac{\varepsilon}{2}\sigma_p(A).\end{aligned}$$

Substituting (2.19) into the above inequality and eliminating α, gives

$$\sigma_p(f) \geq \frac{\sigma_p(A) + \frac{1}{2}\varepsilon\sigma_p(f)}{(2+\varepsilon)(\sigma_p(A) + \frac{1}{2}\varepsilon\sigma_p(f)) - 1} - \frac{\varepsilon}{2}\sigma_p(A),$$

where $\varepsilon > 0$ may be arbitrarily small. Letting $\varepsilon \to 0$ and rearranging yields

$$\frac{1}{\sigma_p(A)} + \frac{1}{\sigma_p(f)} \leq 2.$$

□

From Theorem 2.2 and [15, Theorem 2.3], we have the following result.

THEOREM 2.6. *Let P be a transcendental entire function of order $\sigma(P) \leq \frac{1}{2}$ and f a nonzero entire solution of the differential equation (2.3), where Q is an entire function that is 'small' with respect to f. Then $\sigma_2(f) = +\infty$.*

PROOF. *Case 1.* $Q \equiv 0$. Since P is a transcendental entire function, it follows from [15, Theorem 2.3] that every nonzero solution f of (2.3) where $Q \equiv 0$ satisfies $i(f) = 3 = i(e^P) + 1$, and thus $\sigma_2(f) = +\infty$.

Case 2. $Q \not\equiv 0$. Since P is an entire transcendental function with $\sigma(P) \leq \frac{1}{2}$, we have $i(e^P) = 2$ and $\sigma_2(e^P) \leq \frac{1}{2}$. Then it follows from Theorem 2.2 that all solutions f of (2.3) satisfy $\sigma_2(f) = +\infty$. □

3. Proof of Theorem 1.5

Since f is a nonconstant entire function with hyper-order $\sigma_2(f) \leq \frac{1}{2}$, and since a_1, a_2 are 'small' functions with respect to f, we have $\sigma_2((f^{(k)} - a_2(z))/(f - a_1(z))) \leq \frac{1}{2}$. By the assumption that $f^{(k)} - a_2$ and $f - a_1$ share 0 CM, it follows from the essential part of the factorisation theorem for meromorphic functions of finite iterated order [14, Satz 12.4], that

$$\frac{f^{(k)} - a_2(z)}{f - a_1(z)} = e^{P(z)},$$

where P is an entire function with $\sigma(P) = \sigma_2(e^P) \leq \frac{1}{2}$.

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We may assume that P is not a constant. Set $F := f - a_1$, which is not identically equal to zero. Then $f^{(k)} = F^{(k)} + a_1^{(k)}$ and (2.3) becomes

$$F^{(k)} - e^{P(z)}F = Q(z),$$

where $Q(z) := a_2(z) - a_1^{(k)}(z)$ is an entire function that is 'small' with respect to F. Since P is a nonconstant entire function with $\sigma(P) \leq \frac{1}{2}$, it follows from Theorems 2.1 and 2.6 that $\sigma_2(F)$, and thus $\sigma_2(f)$ is equal to a positive integer or infinite. This contradicts the assumption of $\sigma_2(f) \leq \frac{1}{2}$. Therefore, P must be a constant and consequently e^P is a nonzero constant. This completes the proof of Theorem 1.5.

4. Remark

Consider again the equation $f^{(k)} - a = c(f - a)$ for $k \in \mathbb{N}$, where c is a nonzero constant and a is a constant (or even a 'small' function of f). Set $F := f - a$, so that

$$F^{(k)} - cF = a - a^{(k)}.$$

By the Wiman–Valiron theory as in the proof of Theorem 2.1, it is not difficult to see that all solutions of the differential equation $F^{(k)} - cF = Q$, where Q is a 'small' function of F, satisfy $\sigma(F) = k$. Thus, Corollary 1.6 leads to the following result.

THEOREM 4.1. *There is no entire function f with hyper-order $\sigma_2(f) \leq \frac{1}{2}$ which shares a finite value CM with its kth derivative $f^{(k)}$, unless the order of f satisfies $\sigma(f) = k$.*

We do not know whether there exists an entire function f with $\sigma_2(f) \in (\frac{1}{2}, +\infty)\backslash\mathbb{N}$ that shares a finite value CM with its kth derivative $f^{(k)}$ and does not satisfy the linear differential equation $f^{(k)} - a = c(f - a)$, where c is a nonzero constant.

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