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PARTIAL CHARACTERS WITH RESPECT TO A NORMAL SUBGROUP

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Abstract

Suppose that G is a π-separable group. Let N be a normal π'-subgroup of G and let H be a Hall π-subgroup of G. In this paper, we prove that there is a canonical basis of the complex space of the class functions of G which vanish off G-conjugates of HN. This implies the existence of a canonical basis of the space of class functions of G defined on G-conjugates of HN. When $N = 1$ and π is the complement of a prime p, these bases are the projective indecomposable characters and set of irreducible Brauer characters of G.

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1. Introduction

Let G be a finite group, let π be a set of prime numbers, and let $N \lhd G$ be a π'-subgroup of G. We consider the set $G^0 = \{x \in G | x_{\pi'} \in N\}$ and the space of complex class functions $\mathrm{cf}(G^0)$ of G defined on G^0. Also, if $\chi \in \mathrm{cf}(G)$ is a class function of G, then we denote by χ^0 the restriction of χ to G^0.

THEOREM 1.1. *Suppose that G is π-separable. Then there exists a canonical basis $\mathrm{I}_\pi(G|N)$ of $\mathrm{cf}(G^0)$ such that if $\chi \in \mathrm{Irr}(G)$, then*

$$\chi^0 = \sum_{\phi \in \mathrm{I}_\pi(G|N)} d_{\chi\phi}\phi$$

for uniquely determined nonnegative integers $d_{\chi\phi}$.

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Let us write $\mathrm{vcf}_\pi(G|N) = \{\tau \in \mathrm{cf}(G) | \tau(x) = 0 \text{ whenever } x \in G - G^0\}$. For $\phi \in \mathrm{I}_\pi(G|N)$, let

$$\Phi_\phi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\phi}\chi.$$

Also, if $\theta, \phi \in \mathrm{cf}(G) \cup \mathrm{cf}(G^0)$, write

$$[\theta, \phi]^0 = \frac{1}{|G|} \sum_{x \in G^0} \theta(x)\overline{\phi(x)}.$$

THEOREM 1.2. *Suppose that G is π-separable. Then the set $\{\Phi_\phi | \phi \in \mathrm{I}_\pi(G|N)\}$ is a basis of* $\mathrm{vcf}_\pi(G|N)$. *In fact, if H is any Hall π-subgroup of G, this is the unique basis $\mathscr{B}$ of* $\mathrm{vcf}_\pi(G|N)$ *satisfying the following two conditions.*

(I) *If $\eta \in \mathscr{B}$, then there exists $\alpha \in \mathrm{Irr}(NH)$ such that*

$$\alpha^G = \eta.$$

(D) *If $\gamma \in \mathrm{Irr}(NH)$, then*

$$\gamma^G = \sum_{\eta \in \mathscr{B}} a_\eta \eta$$

for uniquely determined nonnegative integers a_η.

Furthermore,

$$[\Phi_\phi, \theta]^0 = \delta_{\phi,\theta}$$

for $\phi, \theta \in \mathrm{I}_\pi(G|N)$.

When $N = 1$, Theorems 1.1 and 1.2 are well-known consequences of Isaacs π-theory, and $\mathrm{I}_\pi(G|N) = \mathrm{I}_\pi(G)$ is the set of irreducible Isaacs π-partial characters of the group G. Of course, when $N = 1$ and $\pi = p'$, then $\mathrm{I}_\pi(G|N) = \mathrm{IBr}(G)$ is the set of irreducible p-Brauer characters of G. In the other extreme case, when $G^0 = G$ (that is, when N is a normal π-complement of G), then $\mathrm{I}_\pi(G|N) = \mathrm{Irr}(G)$. If G is a π'-group, then N is any normal subgroup of G and in this case $\mathrm{I}_\pi(G|N)$ is the set of sums of the orbits of the action of G on $\mathrm{Irr}(N)$.

The set $\mathrm{I}_\pi(G|N)$ of 'relative π-partial characters with respect to N,' is described in Section 6 below.

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2. Good bases

We do Sections 2 and 3 of this paper in a general setting for further use.

If G is a finite group, we denote by $\mathrm{cf}(G)$ the space of complex class functions defined on G. Let H be a subgroup of G and write $G^0 = \bigcup_{g \in G} H^g$.

If $X \subseteq \mathrm{cf}(H)$ is any subset, we write X^G to denote $\{\xi^G | \xi \in X\}$. Note that $X^G \subseteq \mathrm{cf}(G)$ and that if X is a subspace of $\mathrm{cf}(H)$, then X^G is a subspace of $\mathrm{cf}(G)$. In particular, $\mathrm{cf}(H)^G = \{\delta^G | \delta \in \mathrm{cf}(H)\} \subseteq \mathrm{cf}(G)$. Also, we write

$$\mathrm{vcf}(G|H) = \{\alpha \in \mathrm{cf}(G) | \alpha(x) = 0 \text{ for } x \in G - G^0\}.$$

If $G^0 = \bigcup_{K \in \mathscr{K}} K$, where $\mathscr{K}$ is the set of conjugacy classes K of G such that $K \cap H \neq \emptyset$, notice that

$$\dim(\mathrm{vcf}(G|H)) = |\mathscr{K}|.$$

LEMMA 2.1. *If H is a subgroup of G, then* $\mathrm{cf}(H)^G = \mathrm{vcf}(G|H)$.

PROOF. It is clear by the induction formula that $\mathrm{cf}(H)^G \subseteq \mathrm{vcf}(G|H)$. Now, let $\mathscr{K}$ be the set of conjugacy classes K of G such that $K \cap H \neq \emptyset$. Hence, $G^0 = \bigcup_{K \in \mathscr{K}} K$. If χ_K is the characteristic function of $K \in \mathscr{K}$, it is easy to check that $\{\chi_K\}_{K \in \mathscr{K}}$ is a basis of $\mathrm{vcf}(G|H)$. Now, let $K \in \mathscr{K}$ and let C be a conjugacy class of H contained in $K \cap H$. If χ_C is the characteristic function (in H) of C, then $(\chi_C)^G$ is a nonzero multiple of χ_K, and the proof of the lemma follows. □

Now, let N be a normal subgroup of G contained in H. If $\theta \in \mathrm{Irr}(N)$, then we write $\mathrm{Irr}(G|\theta)$ for the set of irreducible constituents of θ^G. Also, $\mathrm{cf}(G|\theta)$ is the $\mathbb{C}$-span of the set $\mathrm{Irr}(G|\theta)$. Now, let Θ be a complete set of representatives of the orbits of the action of G on $\mathrm{Irr}(N)$. It is clear, then, that

$$\mathrm{cf}(G) = \bigoplus_{\theta \in \Theta} \mathrm{cf}(G|\theta)$$

because

$$\mathrm{Irr}(G) = \bigcup_{\theta \in \Theta} \mathrm{Irr}(G|\theta)$$

is a disjoint union (by Clifford's theorem).

We denote by

$$\mathrm{vcf}(G|H, \theta) = \mathrm{vcf}(G|H) \cap \mathrm{cf}(G|\theta).$$

LEMMA 2.2. *Let $N \lhd G$ and let $N \subseteq H \subseteq G$. Let Θ be a complete set of representatives of the action of G on* $\mathrm{Irr}(N)$. *Then*

$$\mathrm{vcf}(G|H) = \bigoplus_{\theta\in\Theta} \mathrm{vcf}(G|H,\theta).$$

PROOF. It is clear that the sum on the right is direct and contained in $\mathrm{vcf}(G|H)$. Since $\mathrm{cf}(H)^G = \mathrm{vcf}(G|H)$ by Lemma 2.1, it suffices to prove that if $\alpha \in \mathrm{Irr}(H)$, then $\alpha^G \in \sum_{\theta\in\Theta} \mathrm{vcf}(G|H,\theta)$. Now, let $\mu \in \mathrm{Irr}(N)$ be an irreducible constituent of α_N. Hence $\mu^g = \theta$ for some $\theta \in \Theta$ and $g \in G$. Now, if, as usual, α^g denotes the character of H^g satisfying $\alpha^g(h^g) = \alpha(h)$ for $h \in H$, then $\alpha^G = (\alpha^g)^G \in \mathrm{vcf}(G|H) \cap \mathrm{cf}(G|\theta)$, and the proof of the lemma follows. □

DEFINITION 2.3. Suppose that $N \lhd G$ is contained in $H \subseteq G$. Let $\theta \in \mathrm{Irr}(N)$ and let $T = I_G(\theta)$ be the inertia group of θ in G. We say that θ is *H-good* (with respect to G), if for every $g \in G$, we have that $H^g \cap T$ is contained in some T-conjugate of $H \cap T$. In other words, θ is H-good if $G^0 \cap T = T^0$ where $T^0 = \bigcup_{t\in T}(H \cap T)^t$.

LEMMA 2.4. *Suppose that N is a normal subgroup of G contained in $H \subseteq G$, let $\theta \in \mathrm{Irr}(N)$ be H-good and let $T = I_G(\theta)$. Then induction defines an isomorphism*

$$\mathrm{vcf}(T|T\cap H,\theta) \to \mathrm{vcf}(G|H,\theta).$$

PROOF. By the Clifford correspondence, we know that induction defines a bijection $\mathrm{cf}(T|\theta) \to \mathrm{cf}(G|\theta)$. So it suffices to show that if $\psi \in \mathrm{cf}(T|\theta)$, then $\psi \in \mathrm{vcf}(T|T \cap H)$ if and only if $\psi^G \in \mathrm{vcf}(G|H)$.

If we assume that $\psi \in \mathrm{vcf}(T|T\cap H)$, then, by the induction formula, it is clear that $\psi^G \in \mathrm{vcf}(G|H)$.

Now, let $\psi \in \mathrm{cf}(T|\theta)$ and assume that $\psi^G \in \mathrm{vcf}(G|H)$. We claim that $(\psi^G)_T \in \mathrm{vcf}(T|T\cap H)$. Let $t \in T - T^0$. Hence, $t \in T - G^0$ and therefore, $\psi^G(t) = 0$. Thus $(\psi^G)_T \in \mathrm{vcf}(T|T\cap H)$, as claimed. Now, let Λ be a complete set of representatives of the orbits of the action of T on $\mathrm{Irr}(N)$. (Of course, $\theta \in \Lambda$ because θ is T-invariant.) Hence, by Lemma 2.2, we have that

$$\mathrm{vcf}(T|T\cap H) = \bigoplus_{\lambda\in\Lambda} \mathrm{vcf}(T|T\cap H,\lambda).$$

Write $\psi = \sum_{\tau\in\mathrm{Irr}(T|\theta)}[\psi,\tau]\tau$. If $\tau \in \mathrm{Irr}(T|\theta)$, then, by the Clifford correspondence, we know that

$$(\tau^G)_T = \tau + \Xi_\tau,$$

where Ξ_τ is a character of T none of whose irreducible constituents lie over θ. Thus

$$(\psi^G)_T = \sum_{\tau \in \mathrm{Irr}(T|\theta)} [\psi, \tau](\tau^G)_T = \sum_{\tau \in \mathrm{Irr}(T|\theta)} [\psi, \tau](\tau + \Xi_\tau) = \psi + \Xi,$$

where

$$\Xi = \sum_{\tau \in \mathrm{Irr}(T|\theta)} [\psi, \tau]\Xi_\tau$$

does not have any irreducible constituent lying over θ. In other words,

$$\Xi \in \sum_{\lambda \in \Lambda - \{\theta\}} \mathrm{cf}(T|\lambda).$$

Since

$$\mathrm{cf}(T) = \bigoplus_{\lambda \in \Lambda} \mathrm{cf}(T|\lambda),$$

by Lemma 2.2 we conclude that necessarily $\psi \in \mathrm{vcf}(T|T \cap H, \theta)$, as desired. □

In [6], we defined what it means for a basis of $\mathrm{vcf}(G|H)$ to be 'good'.

DEFINITION 2.5. A basis $\mathscr{B}$ of $\mathrm{vcf}(G|H)$ is *good* if it satisfies the following two conditions.

(I) If $\eta \in \mathscr{B}$, then there exists $\alpha \in \mathrm{Irr}(H)$ such that $\alpha^G = \eta$.
(D) If $\gamma \in \mathrm{Irr}(H)$, then $\gamma^G = \sum_{\eta \in \mathscr{B}} a_\eta \eta$ for uniquely determined nonnegative integers a_η.

It is easy to show that good bases are necessarily unique.

THEOREM 2.6. *If $\mathscr{B}$ and $\mathscr{C}$ are good bases of* $\mathrm{vcf}(G|H)$, *then $\mathscr{B} = \mathscr{C}$.*

PROOF. See [6, Theorem 2.2]. □

We will denote by $P(G|H)$ the unique good basis (if it exists) of $\mathrm{vcf}(G|H)$.

Here, we are interested in good bases 'over' an irreducible character of a suitable normal subgroup.

DEFINITION 2.7. Let $N \lhd G$, let $\theta \in \mathrm{Irr}(N)$ and let $N \subseteq H \subseteq G$. A basis $\mathscr{B}$ of $\mathrm{vcf}(G|H, \theta)$ is *good* if it satisfies the following conditions.

(I) If $\eta \in \mathscr{B}$, then there exists $\alpha \in \mathrm{Irr}(H|\theta)$ such that $\alpha^G = \eta$.
(D) If $\gamma \in \mathrm{Irr}(H|\theta)$, then $\gamma^G = \sum_{\eta \in \mathscr{B}} a_\eta \eta$ for uniquely determined integers a_η.

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The same elementary argument as in the proof of Theorem 2.6 shows that good bases 'over' irreducible characters are necessarily unique. We will denote by $P(G|H, \theta)$ the unique good basis (if it exists) of $\mathrm{vcf}(G|H, \theta)$.

We may form a good basis for $\mathrm{vcf}(G|H)$ from good bases over normal irreducible constituents. To prove this result, we need a key property of H-good characters.

LEMMA 2.8. *Suppose that N is a normal subgroup of G contained in H and let $\theta \in \mathrm{Irr}(N)$ be H-good. If $\beta \in \mathrm{Irr}(H|\theta^g)$ for some $g \in G$, then $\beta^G = \gamma^G$ for some character $\gamma \in \mathrm{cf}(H|\theta)$.*

PROOF. We have that there is a G-conjugate K of H with a character $\eta \in \mathrm{Irr}(K|\theta)$ such that $\eta^G = \beta^G$. Now, $\eta = \psi^K$ for some $\psi \in \mathrm{Irr}(T \cap K|\theta)$ by the Clifford correspondence. Since θ is H-good (with respect to G), it follows that $T \cap K$ is contained in some T-conjugate of $H \cap T$. So there is a $t \in T$ such that $U = (T \cap K)^t \subseteq T \cap H$. Now, $\psi^t \in \mathrm{Irr}(U|\theta)$, and therefore $\gamma = (\psi^t)^H$ is a character of H such that all irreducible constituents lie over θ. Since

$$\gamma^G = (\psi^t)^G = \psi^G = \eta^G = \beta^G,$$

the proof of the lemma is complete. □

LEMMA 2.9. *Suppose that $N \lhd G$ and let $N \subseteq H \subseteq G$. Let Θ be a complete set of representatives of the action of G on $\mathrm{Irr}(N)$ and assume that each $\theta \in \Theta$ is H-good (with respect to G). For each $\theta \in \Theta$, suppose that $P(G|H, \theta)$ is a good basis of $\mathrm{vcf}(G|H, \theta)$. Then*

$$\bigcup_{\theta \in \Theta} P(G|H, \theta) = P(G|H).$$

PROOF. By elementary linear algebra and Lemma 2.2, we have that $\bigcup_{\theta \in \Theta} P(G|H, \theta)$ is a basis of $\mathrm{vcf}(G|H)$. To complete the proof of this lemma, we have to prove that given $\gamma \in \mathrm{Irr}(H)$, then

$$\gamma^G = \sum_{\theta \in \Theta} \sum_{\eta \in P(G|H,\theta)} a_{\theta\eta} \eta$$

for some nonnegative integers $a_{\theta\eta}$. Now, there exists $g \in G$ and $\theta \in \Theta$ such that γ lies over θ^g. By Lemma 2.8, there exists a character β of H all of whose irreducible constituents lie in $\mathrm{Irr}(H|\theta)$ and such that $\gamma^G = \beta^G$. Since

$$\beta^G = \sum_{\eta \in P(G|H,\theta)} a_{\theta\eta} \eta$$

for some nonnegative integers $a_{\theta\eta}$, the proof of the lemma is complete. □

There is a 'Clifford correspondence' for good bases over normal irreducible constituents which easily follows from Lemma 2.4.

LEMMA 2.10. *Suppose that* $N \lhd G$ *is contained in* $H \subseteq G$. *Let* $\theta \in \mathrm{Irr}(N)$ *be* H*-good and let* $T = I_G(\theta)$. *If* $P(T|T \cap H, \theta)$ *is a good basis of* $\mathrm{vcf}(T|T \cap H, \theta)$, *then* $P(T|T \cap H, \theta)^G$ *is a good basis of* $\mathrm{vcf}(G|H, \theta)$.

PROOF. It is clear by the definition of good bases, the Clifford correspondence and Lemma 2.4. □

LEMMA 2.11. *Suppose that* N *is a normal subgroup of* G *contained in* $H \subseteq G$ *and let* $\theta \in \mathrm{Irr}(N)$ *be* H*-good. Then*

$$\mathrm{cf}(H|\theta)^G = \mathrm{vcf}(G|H, \theta).$$

PROOF. If $\alpha \in \mathrm{Irr}(H|\theta)$, then it is clear that $\alpha^G \in \mathrm{vcf}(G|H, \theta)$. Hence,

$$\mathrm{cf}(H|\theta)^G \subseteq \mathrm{vcf}(G|H, \theta)$$

and we now prove the reverse inclusion. Let $\phi \in \mathrm{vcf}(G|H, \theta)$ and write $\phi = \eta^G$ for some $\eta \in \mathrm{cf}(H)$. Decompose $\eta = \eta_1 + \eta_2$ where η_1 is a linear combination of irreducible characters of H lying over G-conjugates of θ and no irreducible constituent of η_2 lies over a G-conjugate of θ. Then

$$(\eta_2)^G = \phi - (\eta_1)^G \in \mathrm{cf}(G|\theta).$$

This easily implies that $(\eta_2)^G = 0$ and $\phi = (\eta_1)^G$.

To complete the proof of the lemma, it suffices to apply Lemma 2.8. □

3. Partial characters

Our next objective is to associate to the basis $P(G|H)$ of $\mathrm{vcf}(G|H)$, a natural basis $I(G|H)$ of $\mathrm{cf}(G^0)$, where $\mathrm{cf}(G^0)$ is the set of complex class functions of G defined on G^0.

Note that if $G^0 = \bigcup_{K \in \mathscr{K}} K$, where $\mathscr{K}$ is the set of conjugacy classes K of G such that $K \cap H \neq \emptyset$, then

$$\dim(\mathrm{cf}(G^0)) = |\mathscr{K}| = \dim(\mathrm{vcf}(G|H)).$$

If $\phi, \theta \in \mathrm{cf}(G^0) \cup \mathrm{cf}(G)$, we write

$$[\phi, \theta]^0 = \frac{1}{|G|} \sum_{x \in G^0} \phi(x)\overline{\theta(x)}.$$

We may view $[\cdot,\cdot]^0$ as a bilinear pairing

$$[\cdot,\cdot]^0 : \mathrm{cf}(G^0) \times \mathrm{vcf}(G|H) \to \mathbb{C}.$$

We claim that this pairing is non-degenerate. By elementary linear algebra, it suffices to prove that any $\eta \in \mathrm{vcf}(G|H)$ is zero if $[\phi, \eta]^0 = 0$ for every $\phi \in \mathrm{cf}(G^0)$. Now, if $\chi_K \in \mathrm{cf}(G^0)$ is the characteristic function of $K \in \mathscr{K}$, where $\mathscr{K}$ has the same significance as before, and $x_K \in K$, then we have that

$$0 = [\chi_K, \eta]^0 = \frac{|K|}{|G|}\overline{\eta(x_K)}.$$

This proves the claim.

Given a basis $\mathscr{B} = \{\eta_1, \dots, \eta_k\}$ of $\mathrm{vcf}(G|H)$, then it follows that there exists a unique basis $\mathscr{I} = \{\phi_1, \dots, \phi_k\}$ of $\mathrm{cf}(G^0)$ satisfying

$$[\phi_i, \eta_j]^0 = \delta_{i,j}.$$

If $\chi \in \mathrm{cf}(G)$, then $\chi^0 \in \mathrm{cf}(G^0)$ denotes the restriction of χ to G^0.

THEOREM 3.1. *Let $P(G|H) = \{\eta_1, \dots, \eta_k\}$ be the good basis of* $\mathrm{vcf}(G|H)$ *and let $I(G|H) = \{\phi_1, \dots, \phi_k\}$ be the unique basis of* $\mathrm{cf}(G^0)$ *satisfying*

$$[\eta_i, \phi_j]^0 = \delta_{i,j}.$$

If χ is a character of G, then

$$\chi^0 = \sum_{\phi \in I(G|H)} d_{\chi\phi}\phi$$

for uniquely determined nonnegative integers $d_{\chi\phi}$.

PROOF. This is [6, Theorem 2.4]. □

We may view the basis $I(G|H)$ as the set of 'irreducible Brauer characters' of G with respect to H. We view the integers $d_{\chi\phi}$ as the 'decomposition numbers' and the elements in a good basis $P(G|H)$ as the 'projective indecomposable characters.' If $\phi \in I(G|H)$, then we denote by Φ_ϕ the unique element in $P(G|H)$ such that

$$[\Phi_\phi, \mu]^0 = \delta_{\phi,\mu}$$

for $\mu \in I(G|H)$.

LEMMA 3.2. *If $\phi \in I(G|H)$, then*

$$\Phi_\phi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\phi}\chi.$$

PROOF. For each $\mu \in I(G|H)$, let $\gamma_\mu \in \mathrm{Irr}(H)$ be such that $(\gamma_\mu)^G = \Phi_\mu$. Now, if $\chi \in \mathrm{Irr}(G)$, then

$$\chi^0 = \sum_{\mu \in I(G|H)} d_{\chi\mu}\mu.$$

Since $H \subseteq G^0$, we have

$$\chi_H = \sum_{\mu \in I(G|H)} d_{\chi\mu}\mu_H.$$

Let $\tilde{\mu} \in \mathrm{cf}(G)$ be any extension of $\mu \in \mathrm{cf}(G^0)$. Then

$$[\mu_H, \gamma_\phi] = [\tilde{\mu}_H, \gamma_\phi] = [\tilde{\mu}, (\gamma_\phi)^G] = [\tilde{\mu}, \Phi_\phi] = [\mu, \Phi_\phi]^0 = \delta_{\mu,\phi}.$$

Therefore

$$[\chi, \Phi_\phi] = [\chi_H, \gamma_\phi] = \sum_{\mu \in I(G|H)} d_{\chi\mu}[\mu_H, \gamma_\phi] = d_{\chi\phi},$$

as required. □

LEMMA 3.3. *Suppose that $\phi \in I(G|H)$. Then ϕ_H is an ordinary character of H.*

PROOF. Since $\phi \in \mathrm{cf}(G^0)$, we have that $\phi_H \in \mathrm{cf}(H)$. Therefore, we may write

$$\phi_H = \sum_{\gamma \in \mathrm{Irr}(H)} [\phi_H, \gamma]\gamma.$$

Let $\gamma \in \mathrm{Irr}(H)$. We prove that $[\phi_H, \gamma]$ is a nonnegative integer. By property (D) of the good bases, we have that

$$\gamma^G = \sum_{\mu \in I(G|H)} a_\mu \Phi_\mu$$

for nonnegative integers a_μ. Let $\tilde{\phi} \in \mathrm{cf}(G)$ be any extension of ϕ to G. Since γ^G vanishes off G^0, we have that

$$\begin{aligned}[\phi_H, \gamma] &= [\tilde{\phi}_H, \gamma] = [\tilde{\phi}, \gamma^G] = [\tilde{\phi}, \gamma^G]^0 = [\phi, \gamma^G]^0 \\ &= \left[\phi, \sum_{\mu \in I(G|H)} a_\mu \Phi_\mu\right]^0 = \sum_{\mu \in I(G|H)} a_\mu [\phi, \Phi_\mu]^0 = a_\phi.\end{aligned}$$

This proves the lemma. □

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Our next objective is to prove a Clifford type theorem for partial characters which we will need later on.

Suppose that $N \lhd G$ is contained in H and let $\theta \in \mathrm{Irr}(N)$. Note that the map ${}^0 : \mathrm{cf}(G) \to \mathrm{cf}(G^0)$ given by $\chi \mapsto \chi^0$ is $\mathbb{C}$-linear and surjective. We denote by

$$\mathrm{cf}(G^0|\theta) = \mathrm{cf}(G|\theta)^0.$$

LEMMA 3.4. *Suppose that N is a normal subgroup of G contained in H and let Θ be a complete set of representatives of the orbits of the action of G on* $\mathrm{Irr}(N)$. *Then*

$$\mathrm{cf}(G^0) = \bigoplus_{\theta\in\Theta} \mathrm{cf}(G^0|\theta).$$

PROOF. Since

$$\mathrm{cf}(G) = \bigoplus_{\theta\in\Theta} \mathrm{cf}(G|\theta),$$

it follows that

$$\mathrm{cf}(G^0) = \sum_{\theta\in\Theta} \mathrm{cf}(G^0|\theta).$$

Suppose that

$$\sum_{\theta\in\Theta} (\mu_\theta)^0 = 0$$

where $\mu_\theta \in \mathrm{cf}(G|\theta)$ for $\theta \in \Theta$. We prove that $(\mu_\theta)^0 = 0$. Since $H \subseteq G^0$, we have that

$$\sum_{\theta\in\Theta} (\mu_\theta)_H = 0.$$

We claim that $(\mu_{\theta_1})_H$ and $(\mu_{\theta_2})_H$ do not have any 'irreducible constituent' in common whenever $\theta_1 \neq \theta_2$. This is because the character $(\mu_\theta)_H$ consists of a linear combination of characters of the form χ_H for $\chi \in \mathrm{Irr}(G|\theta)$. Hence, if θ_1 and θ_2 are not G-conjugate, it follows that $[\chi_H, \eta_H] = 0$ for $\chi \in \mathrm{Irr}(G|\theta_1)$ and $\eta \in \mathrm{Irr}(G|\theta_2)$. We conclude that $(\mu_\theta)_H = 0$ for $\theta \in \Theta$. However, since μ_θ is a class function of G, we see that $(\mu_\theta)_H = 0$ if and only if $(\mu_\theta)^0 = 0$, and the proof of the lemma is complete. □

In several parts of this paper, we use the fact that $[\gamma, \eta]^0 = 0$ for $\gamma \in \mathrm{cf}(G^0|\theta)$ and $\eta \in \mathrm{vcf}(G|H, \mu)$ whenever θ and μ are not G-conjugate. This easily follows from the following argument. If $\chi \in \mathrm{cf}(G|\theta)$ is such that $\chi^0 = \gamma$, then

$$[\gamma, \eta]^0 = [\chi, \eta] = 0$$

because χ and η do not have any common 'irreducible constituent' by Clifford's theorem.

We already know that $\dim(\mathrm{cf}(G^0)) = \dim(\mathrm{vcf}(G|H))$. In fact, there is a natural isomorphism between both complex spaces.

LEMMA 3.5. *The map $\phi \mapsto (\phi_H)^G$ is a natural linear isomorphism $\mathrm{cf}(G^0) \to \mathrm{vcf}(G|H)$. In fact, if N is a normal subgroup of G contained in H and $\theta \in \mathrm{Irr}(N)$, then the map $\phi \mapsto (\phi_H)^G$ maps $\mathrm{cf}(G^0|\theta)$ isomorphically onto $\mathrm{vcf}(G|H,\theta)$. Therefore, if $\{\eta_1, \dots, \eta_k\}$ is any basis of $\mathrm{vcf}(G|H,\theta)$, then there exists a unique basis $\{\gamma_1, \dots, \gamma_k\}$ of $\mathrm{cf}(G^0|\theta)$ satisfying*

$$[\gamma_i, \eta_j]^0 = \delta_{i,j}.$$

PROOF. It is clear that the map $\phi \mapsto (\phi_H)^G$ is a linear map $\mathrm{cf}(G^0) \to \mathrm{vcf}(G|H)$. Since $\dim(\mathrm{cf}(G^0)) = \dim(\mathrm{vcf}(G|H))$, it is enough to show that it is injective to complete the proof of the first part of the lemma.

Suppose that $(\alpha_H)^G = 0$ for some $\alpha \in \mathrm{cf}(G^0)$. Let $\tilde{\alpha} \in \mathrm{cf}(G)$ be an extension of α to G. Then

$$(\alpha_H)^G = (\tilde{\alpha}_H)^G = (\tilde{\alpha}_H 1_H)^G = \tilde{\alpha}(1_H)^G.$$

We have that

$$0 = ((\alpha_H)^G)_H = \tilde{\alpha}_H((1_H)^G)_H = \alpha_H((1_H)^G)_H.$$

Since the character $((1_H)^G)_H$ is never zero, we deduce that $\alpha_H = 0$. Since $\alpha \in \mathrm{cf}(G^0)$, we have that $\alpha = 0$. This proves that the map $\phi \mapsto (\phi_H)^G$ is an isomorphism.

If $\theta \in \mathrm{Irr}(N)$, then we want to show that the map $\phi \mapsto (\phi_H)^G$ carries $\mathrm{cf}(G^0|\theta)$ isomorphically onto $\mathrm{vcf}(G|H,\theta)$. Let Θ be a complete set of representatives of the action of G on $\mathrm{Irr}(N)$ with $\theta \in \Theta$. By Lemma 2.2 and Lemma 3.4, it suffices to show that if $\phi \in \mathrm{cf}(G^0|\theta)$, then $(\phi_H)^G \in \mathrm{vcf}(G|H,\theta)$. Since we already know that $(\phi_H)^G \in \mathrm{vcf}(G|H)$, we have to show that $(\phi_H)^G \in \mathrm{cf}(G|\theta)$. Let $\mu \in \mathrm{cf}(G|\theta)$ be such that $\phi = \mu^0$. Then $\phi_H = \mu_H$ and we prove that $(\mu_H)^G \in \mathrm{cf}(G|\theta)$. However, this reduces to proving that whenever $\chi \in \mathrm{Irr}(G|\theta)$, then $(\chi_H)^G \in \mathrm{cf}(G|\theta)$. Let $\tau \in \mathrm{Irr}(G)$ be an irreducible constituent of $(\chi_H)^G$. Hence, τ is an irreducible constituent of some ξ^G, where $\xi \in \mathrm{Irr}(H)$ is an irreducible constituent of χ_H. Since χ lies over θ, by Clifford's theorem we have that ξ lies over some G-conjugate of θ. Hence, τ lies over θ and the second part of the lemma is complete.

Finally, suppose that $\{\eta_1, \dots, \eta_k\}$ is any basis of $\mathrm{vcf}(G|H,\theta)$. We wish to find a basis $\{\gamma_1, \dots, \gamma_k\}$ of $\mathrm{cf}(G^0|\theta)$ satisfying

$$[\gamma_i, \eta_j]^0 = \delta_{ij}.$$

By elementary linear algebra, it suffices to show that the bilinear pairing

$$[\cdot,\cdot]^0 : \mathrm{cf}(G^0|\theta) \times \mathrm{vcf}(G|H,\theta) \to \mathbb{C}$$

is nondegenerate. If Θ has the same significance as before, this easily follows from Lemma 2.2, Lemma 3.4, the fact that the 'whole' pairing

$$[\cdot,\cdot]^0 : \mathrm{cf}(G^0) \times \mathrm{vcf}(G|H) \to \mathbb{C}$$

is nondegenerate, and the fact that $[\gamma,\eta]^0 = 0$ for $\gamma \in \mathrm{cf}(G^0|\theta_1)$, $\eta \in \mathrm{vcf}(G|H,\theta_2)$ and distinct $\theta_1, \theta_2 \in \Theta$. (See the remark preceding the statement of this lemma.) □

Suppose that $N \triangleleft G$ and $N \subseteq H \subseteq G$. Assume that the good basis $P(G|H,\theta)$ of $\mathrm{vcf}(G|H,\theta)$ exists. Then we denote by $I(G|H,\theta)$ the unique basis of $\mathrm{cf}(G^0|\theta)$ uniquely determined by $P(G|H,\theta)$ by Lemma 3.5.

Next we prove the analogue of Lemma 2.9 for partial characters.

LEMMA 3.6. *Suppose that $N \triangleleft G$ and let $N \subseteq H \subseteq G$. Let Θ be a complete set of representatives of the action of G on* Irr(N) *and assume that each $\theta \in \Theta$ is H-good (with respect to G). For each $\theta \in \Theta$, suppose that $P(G|H,\theta)$ is a good basis of* $\mathrm{vcf}(G|H,\theta)$. *Then*

$$\bigcup_{\theta\in\Theta} I(G|H,\theta) = I(G|H).$$

PROOF. By Lemma 2.9, we have that

$$\bigcup_{\theta\in\Theta} P(G|H,\theta) = P(G|H).$$

Clearly, it is enough to show is that $[\gamma,\eta]^0 = 0$ for $\gamma \in \mathrm{cf}(G^0|\theta)$ and $\eta \in \mathrm{vcf}(G|H,\mu)$ whenever θ and μ are not G-conjugate. We already remarked on this fact before the statement of Lemma 3.5. □

Next, we define induction of partial characters. Suppose that J is a subgroup of G such that $J^0 = G^0 \cap J$, where $J^0 = \bigcup_{x\in J}(J\cap H)^x$. Suppose that $\eta \in \mathrm{cf}(J^0)$. Then we define $\eta^G \in \mathrm{cf}(G^0)$ in the following way. If $x \in G^0$, we set

$$\eta^G(x) = \frac{1}{|J|} \sum_{\substack{g\in G \\ gxg^{-1}\in J}} \eta(gxg^{-1}).$$

It is straightforward to check that this is a well defined class function on G^0. Furthermore, if $\mu \in \mathrm{cf}(J)$ is such that $\mu^0 = \eta$, then $(\mu^G)^0 = \eta^G$.

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LEMMA 3.7. *Suppose that* $N \lhd G$, *where* $N \subseteq H \subseteq G$. *Assume that* $\theta \in \mathrm{Irr}(N)$ *is* H*-good and suppose that* $P(T|T \cap H, \theta)$ *is a good basis of* $\mathrm{vcf}(T|T \cap H, \theta)$, *where* $T = I_G(\theta)$. *Then the map* $\gamma \mapsto \gamma^G$ *is a bijection* $I(T|T \cap H, \theta) \to I(G|H, \theta)$.

PROOF. By Lemma 2.10, we know that $P(T|T \cap H, \theta)^G$ is a good basis of $\mathrm{vcf}(G|H, \theta)$. Hence, it suffices to show that

$$[\gamma^G, (\Phi_\tau)^G]^0 = \delta_{\gamma,\tau}$$

for $\tau, \gamma \in I(T|T \cap H, \theta)$.

If $\tau \in I(T|T \cap H, \theta)$, then we claim that

$$((\Phi_\tau)^G)_T = \Phi_\tau + \Delta,$$

where Δ is a character of T such that none of its irreducible constituents lies over θ. By definition of a good basis of T over θ, it is clear that we may write $\Phi_\tau = \sum_{\psi \in \mathrm{Irr}(T|\theta)} [\Phi_\tau, \psi]\psi$. By the Clifford correspondence, we know that

$$(\psi^G)_T = \psi + \Delta_\psi,$$

where Δ_ψ is a character of T none of whose irreducible constituents lie over θ. Thus

$$((\Phi_\tau)^G)_T = \sum_{\psi \in \mathrm{Irr}(T|\theta)} [\Phi_\tau, \psi](\psi^G)_T = \sum_{\psi \in \mathrm{Irr}(T|\theta)} [\Phi_\tau, \psi](\psi + \Delta_\psi) = \Phi_\tau + \Delta,$$

where

$$\Delta = \sum_{\psi \in \mathrm{Irr}(T|\theta)} [\Phi_\tau, \psi]\Delta_\psi$$

does not have any irreducible constituent lying over θ, as claimed.

Suppose that $\gamma \in I(T|T \cap H, \theta)$ and let $\tilde{\gamma} \in \mathrm{cf}(T|\theta)$ be such that $\tilde{\gamma}^0 = \gamma$. Then $\tilde{\gamma}^G \in \mathrm{cf}(G)$ is such that $(\tilde{\gamma}^G)^0 = \gamma^G$. Now

$$\begin{aligned}[\gamma^G, (\Phi_\tau)^G]^0 &= [\tilde{\gamma}^G, (\Phi_\tau)^G] = [\tilde{\gamma}, ((\Phi_\tau)^G)_T] = [\tilde{\gamma}, \Phi_\tau + \Delta] = \\ &= [\tilde{\gamma}, \Phi_\tau] + [\tilde{\gamma}, \Delta] = [\tilde{\gamma}, \Phi_\tau] = [\gamma, \Phi_\tau]^0 = \delta_{\gamma,\tau},\end{aligned}$$

as desired. □

4. Reviewing π-theory

Suppose that G is a π-separable group, and denote by G^π the set of π-elements of G, so that if H is a Hall π-subgroup of G, then $G^\pi = \bigcup_{g \in G} H^g$. Also, if $\chi \in \mathrm{cf}(G)$, then $\chi^\pi \in \mathrm{cf}(G^\pi)$ denotes the restriction of χ to G^π.

Isaacs proved in [1] the existence of a unique basis $\mathrm{I}_\pi(G)$ of $\mathrm{cf}(G^\pi)$ satisfying the following two properties.

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(D) If $\chi \in \mathrm{Irr}(G)$, then

$$\chi^\pi = \sum_{\phi \in \mathrm{I}_\pi(G)} d_{\chi\phi}\phi$$

for uniquely determined nonnegative integers $d_{\chi\phi}$.

(FS) If $\phi \in \mathrm{I}_\pi(G)$, then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^\pi = \phi$.

Of course, in the 'classical case' where $\pi = p'$, then $\mathrm{I}_\pi(G) = \mathrm{IBr}(G)$ by the Fong-Swan theorem.

As the reader may easily check, any basis of $\mathrm{cf}(G^\pi)$ satisfying (D) and (FS) is necessarily equal to $\mathrm{I}_\pi(G) = \{\chi^\pi | \chi \in \mathrm{Irr}(G),\, \chi^\pi$ is not of the form $\chi^\pi = \alpha^\pi + \beta^\pi$ for characters α and β of $G\}$. Isaacs calls $\mathrm{I}_\pi(G)$ the set of *irreducible π-partial characters* of G, while the restrictions χ^π of the characters χ of G are simply called the *π-partial characters* of G.

There are two known proofs of the theorem above. The original proof in [1] constructed a canonical subset $\mathrm{B}_\pi\, G \subseteq \mathrm{Irr}(G)$ such that the map $\chi \mapsto \chi^\pi$ turned out to be a bijection $\mathrm{B}_\pi\, G \to \mathrm{I}_\pi(G)$. Another easier proof (which, however, does not allow development of Clifford theory for I_π-characters, among other things) was given later in [3].

An important role in π-theory is played by the so called Fong characters. If $\phi \in \mathrm{I}_\pi(G)$, then an irreducible constituent $\alpha \in \mathrm{Irr}(H)$ of ϕ_H is a *Fong character* of H associated with ϕ, if $\alpha(1) = \phi(1)_\pi$. Fong characters always exist and if $\alpha_\phi \in \mathrm{Irr}(H)$ is a Fong character associated with $\phi \in \mathrm{I}_\pi(G)$, then

$$[\phi_H, \alpha_\mu] = \delta_{\phi,\mu}$$

for $\phi, \mu \in \mathrm{I}_\pi(G)$ (see [2, Section 2]). If

$$\Phi_\phi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\phi}\chi,$$

then it easily follows that any Fong character α associated to ϕ satisfies $\alpha^G = \Phi_\phi$.

In the next result, we use the notation of Sections 2 and 3.

LEMMA 4.1. *If H is a Hall π-subgroup of a π-separable group G, then the set $\{\Phi_\phi | \phi \in \mathrm{I}_\pi(G)\}$ is the good basis $P(G|H)$ of* $\mathrm{vcf}(G|H)$. *Furthermore $I(G|H) = \mathrm{I}_\pi(G)$.*

PROOF. The first part is [6, Theorem 3.5]. For each $\phi \in \mathrm{I}_\pi(G)$, let $\alpha_\phi \in \mathrm{Irr}(H)$ be such that $(\alpha_\phi)^G = \Phi_\phi$ and let $\chi_\phi \in \mathrm{Irr}(G)$ be such that $(\chi_\phi)^0 = \phi$. If $\theta, \phi \in \mathrm{I}_\pi(G)$, then

$$[\theta, \Phi_\phi]^0 = [(\chi_\theta)^0, \Phi_\phi]^0 = [\chi_\theta, \Phi_\phi] = [(\chi_\theta)_H, \alpha_\phi] = [\theta_H, \alpha_\phi] = \delta_{\theta,\phi},$$

as desired. □

5. Main results

We say that $\chi \in \mathrm{Irr}(G)$ is a *π-character* if $\chi(1)$ and $o(\chi)$ (the order of the determinantal character $\det\chi$ in the group of linear characters of G) are π-numbers. The key result on π-characters is due to Gallagher ([4, Corollary 8.16]). It asserts that if $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ is a G-invariant π-character and $|G : N|$ is a π'-number, then θ has a unique extension χ to G such that χ is a π-character.

If U is a subgroup of G and $\alpha \in \mathrm{cf}(U)$, then we say that α is *G-stable* if $\alpha(x) = \alpha(y)$ whenever $x, y \in U$ are G-conjugate.

The proof of our main results heavily depends on the following lemma.

LEMMA 5.1. *Let $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be a G-invariant π-character. Suppose that $N \subseteq U \subseteq G$ is such that $|U : N|$ is a π'-number. If $\alpha \in \mathrm{Irr}(U)$ is the unique π-character of U extending θ, then α is G-stable.*

PROOF. Let $x, y \in U$ and suppose that $x = y^g$ for some $g \in G$. We wish to prove that $\alpha(x) = \alpha(y)$. Let $K = N\langle x\rangle \subseteq U$ and let $J = N\langle y\rangle \subseteq U$. Note that $J^g = K$. Also, note that α_K is the unique π-character of K extending α and that α_J is the unique π-character of J extending α. Write $\beta = \alpha_J$ and consider the character β^g of $J^g = K$ defined by $\beta^g(j^g) = \beta(j)$ for $j \in J$. Since θ is G-invariant, observe that

$$\beta^g(n) = \beta(n^{g^{-1}}) = \theta(n^{g^{-1}}) = \theta(n).$$

Hence, β^g is a character of K extending θ. Also, $o(\beta^g) = o(\beta)$ is a π-number. By the uniqueness of the π-character extension, we deduce that

$$\beta^g = \alpha_K.$$

Now,

$$\alpha(y) = \alpha_J(y) = \beta(y) = \beta^g(y^g) = \alpha_K(y^g) = \alpha_K(x) = \alpha(x),$$

as desired. □

LEMMA 5.2. *Let N be a normal π-subgroup of a π-separable group G and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Let H be a Hall π-complement of G and let $\hat{\theta} \in \mathrm{Irr}(NH)$ be the π-character of NH extending θ. Then the character $\hat{\theta}_H$ is never zero.*

PROOF. Let $h \in H$ and write $K = N\langle h\rangle$. Since $(\hat{\theta})_K$ is the unique π-character of K extending θ, we may assume that $K = G$. If θ^* is the Glauberman correspondent of θ with respect to $\langle h\rangle$, it follows by [4, Theorem 13.6] that there is $\epsilon = \pm 1$ such that

$$\hat{\theta}(h) = \epsilon\theta^*(1) \neq 0,$$

as desired. □

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If H is a π-subgroup of G and N is a normal π'-subgroup of G, note that we may view the characters of H as characters of HN with N in contained in their kernel. In fact, given α a character of H there is a unique character $\hat{\alpha}$ of NH with $N \subseteq \ker\hat{\alpha}$ such that $\hat{\alpha}_H = \alpha$. We will use this notation in several parts below. Furthermore, if as usual we identify the characters of $\bar{G} = G/N$ with the characters of G with N contained in their kernel (suppose that $\chi \mapsto \bar{\chi}$ is the natural bijection $\{\chi \in \mathrm{Irr}(G) | N \subseteq \ker\chi\} \to \mathrm{Irr}(\bar{G})$), we have that

$$\hat{\alpha}^G = \sum_{\bar{\chi} \in \mathrm{Irr}(\bar{G})} [\bar{\hat{\alpha}}^{\bar{G}}, \bar{\chi}]\chi.$$

THEOREM 5.3. *Suppose that G is a π-separable group. Let H be a Hall π-subgroup of G and for each $\phi \in \mathrm{I}_\pi(G)$, let $\alpha_\phi \in \mathrm{Irr}(H)$ be a Fong character associated with ϕ. Let N be a normal π'-subgroup of G and suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant. If $\hat{\theta} \in \mathrm{Irr}(NH)$ is the π-character of HN extending θ, then*

$$P(G|HN, \theta) = \{(\hat{\theta}\widehat{\alpha_\phi})^G | \phi \in \mathrm{I}_\pi(G)\}.$$

PROOF. By Lemma 5.1, we know that $\hat{\theta}$ is G-stable. Hence, we may find $\tilde{\theta} \in \mathrm{cf}(G)$ extending $\hat{\theta}$.

Let $\gamma \in \mathrm{Irr}(HN|\theta)$. By Gallagher [4, Corollary 6.17], we have that $\gamma = \hat{\theta}\hat{\mu}$ for some $\mu \in \mathrm{Irr}(H)$.

Notice that for $\phi \in \mathrm{I}_\pi(G)$, we have that $\widehat{\alpha_\phi}$ is a Fong character of G/N when considered as a character of HN/N. By Lemma 4.1 (applied in G/N) and the comments preceding the statement of this theorem, it is easy to check that

$$\hat{\mu}^G = \sum_{\phi \in \mathrm{I}_\pi(G)} a_\phi (\widehat{\alpha_\phi})^G$$

for some nonnegative integers a_ϕ. Then

$$\begin{aligned}\gamma^G &= (\hat{\theta}\hat{\mu})^G = (\tilde{\theta}_{NH}\hat{\mu}^G) = \tilde{\theta}\hat{\mu}^G \\ &= \tilde{\theta} \sum_{\phi \in \mathrm{I}_\pi(G)} a_\phi (\widehat{\alpha_\phi})^G = \sum_{\phi \in \mathrm{I}_\pi(G)} a_\phi (\hat{\theta}\widehat{\alpha_\phi})^G.\end{aligned}$$

By Lemma 2.11, we have that the set $\{(\hat{\theta}\widehat{\alpha_\phi})^G | \phi \in \mathrm{I}_\pi(G)\}$ spans $\mathrm{vcf}(G|HN, \theta)$.

To complete the proof of the theorem, it remains to show that the set $\{(\hat{\theta}\widehat{\alpha_\phi})^G | \phi \in \mathrm{I}_\pi(G)\}$ is linearly independent. If

$$0 = \sum_{\phi \in \mathrm{I}_\pi(G)} b_\phi (\hat{\theta}\widehat{\alpha_\phi})^G$$

for some complex numbers b_ϕ, then we have that

$$0 = \tilde{\theta} \sum_{\phi \in \mathrm{I}_\pi(G)} b_\phi (\widehat{\alpha_\phi})^G.$$

Therefore

$$0 = \hat{\theta}_H \sum_{\phi \in \mathrm{I}_\pi(G)} b_\phi ((\widehat{\alpha_\phi})^G)_H.$$

By Lemma 5.2, we conclude that

$$\sum_{\phi \in \mathrm{I}_\pi(G)} b_\phi ((\widehat{\alpha_\phi})^G)_H = 0.$$

In fact, using that the characters $(\widehat{\alpha_\phi})^G$ have N contained in their kernel, and that they are induced from characters of HN, this easily implies that

$$\sum_{\phi \in \mathrm{I}_\pi(G)} b_\phi (\widehat{\alpha_\phi})^G = 0.$$

By Lemma 4.1 (applied in G/N), we conclude that $b_\phi = 0$ for all ϕ, proving the theorem. □

Now we are ready to prove Theorems 1.1 and 1.2. First, we unify the notation we used in the introduction and in the previous sections.

Suppose that G is a π-separable group and let N be a normal π'-subgroup of G. Let H be a Hall π-subgroup of G and notice that

$$G^0 = \{x \in G | x_{\pi'} \in N\} = \bigcup_{g \in G} (HN)^g.$$

Also,

$$\mathrm{vcf}_\pi(G|N) = \{\tau \in \mathrm{cf}(G) | \tau(x) = 0 \text{ for } x \in G - G^0\} = \mathrm{vcf}(G|HN).$$

PROOF OF THEOREMS 1.1 AND 1.2. Let H be a Hall π-subgroup of G. Firstly, we prove that there exists a good basis for $\mathrm{vcf}(G|HN)$.

Given $\theta \in \mathrm{Irr}(N)$, we claim that there exists $x \in G$ such that θ^x is HN-good. Recall that $\eta \in \mathrm{Irr}(N)$ is HN-good if for every $g \in G$, we have that $H^g N \cap T$ is contained in some T-conjugate of $HN \cap T$, where $T = I_G(\eta)$. Let P be a Hall π-subgroup of $I_G(\theta)$. Hence $P \subseteq H^{x^{-1}}$ for some $x \in G$. Thus P^x is a Hall π-subgroup of $T = I_G(\theta^x)$ contained in H. Therefore, $H \cap T = P^x$ is a Hall π-subgroup of T and thus $(H \cap T)N/N$ is a Hall π-subgroup of T/N. Write $\eta = \theta^x$. We prove that η is HN-good. Let $g \in G$. Then $(H^g N \cap T)/N = (T \cap H^g)N/N$ is a π-subgroup of

T/N. Hence, there exists $t \in T$ such that $(H^g N \cap T)/N \subseteq (H \cap T)^t N/N$. Therefore, $H^g N \cap T \subseteq (H \cap T)^t N = (HN \cap T)^t$. This proves the claim.

By the claim, we may find a complete set Θ of representatives of the orbits of the action of G on $\mathrm{Irr}(N)$ such that each $\theta \in \Theta$ is HN-good.

By Lemma 2.9, it suffices to show that there exists a good basis $P(G|HN, \theta)$ of $\mathrm{vcf}(G|HN, \theta)$ for every $\theta \in \Theta$. (We write $\mathrm{I}_\pi(G|N, \theta)$ for the unique basis of $\mathrm{cf}(G^0|\theta)$ uniquely determined by $P(G|HN, \theta)$ by applying Lemma 3.5.)

We fix $\theta \in \Theta$ and, by Lemma 2.10, note that we may assume that θ is G-invariant. In this case, there exists a good basis of $\mathrm{vcf}(G|HN, \theta)$ by Theorem 5.3. This proves that there exists a good basis $P(G|HN)$ of $\mathrm{vcf}(G|HN)$ by Lemma 2.9.

Let $\mathrm{I}_\pi(G|N) = I(G|HN)$ be the unique basis of $\mathrm{cf}(G^0)$ determined by $P(G|HN)$ by using Theorem 3.1. (Note also that $\mathrm{I}_\pi(G|N) = \bigcup_{\theta \in \Theta} \mathrm{I}_\pi(G|N, \theta)$ by Lemma 3.6.) Also, again by Theorem 3.1, we know that whenever $\chi \in \mathrm{Irr}(G)$, then

$$\chi^0 = \sum_{\phi \in \mathrm{I}_\pi(G|N)} d_{\chi\phi}\phi$$

for uniquely determined integers $d_{\chi\phi}$. Furthermore, if

$$\Phi_\phi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\phi}\chi,$$

then, by Lemma 3.2,

$$P(G|HN) = \{\Phi_\phi | \phi \in \mathrm{I}_\pi(G|N)\}.$$

This completes the proof of Theorems 1.1 and 1.2. □

6. The set $\mathrm{I}_\pi(G|N)$

In this final section, we give a complete description of the set $\mathrm{I}_\pi(G|N)$.

Suppose that N is a normal π'-subgroup of a π-separable group G. As before, we let $G^0 = \{x \in G | x_{\pi'} \in N\}$ and, as in Section 4, write G^π for the set of π-elements of G. If $\theta \in \mathrm{Irr}(N)$ is G-invariant and $\phi \in \mathrm{cf}(G^\pi)$, we define a class function $\theta * \phi \in \mathrm{cf}(G^0)$ as follows. Let H be any Hall π-subgroup of G and let $\hat{\theta} \in \mathrm{Irr}(HN)$ be the unique extension of θ to HN which is a π'-character. By Lemma 5.1, we know that $\hat{\theta}$ is G-stable. If $x \in G^0$, then x is G-conjugate to some hn for some $h \in H$ and $n \in N$. By elementary group theory, note that h is then determined up to G-conjugacy. We define

$$(\theta * \phi)(x) = \hat{\theta}(hn)\phi(h).$$

Observe that this is a well defined function by Lemma 5.1. Also, note that $\theta * \phi \in \mathrm{cf}(G^0)$ does not depend on H.

If $\theta \in \mathrm{Irr}(N)$, recall that $\mathrm{cf}(G^0|\theta) = \{\chi^0 | \chi \in \mathrm{cf}(G|\theta)\}$, where χ^0 denotes the restriction of χ to G^0.

LEMMA 6.1. *Suppose that N is a normal π'-subgroup of a π-separable group G. Let $\theta \in \mathrm{Irr}(N)$ be G-invariant and suppose that $\phi \in \mathrm{cf}(G^\pi)$. Then $\theta * \phi \in \mathrm{cf}(G^0|\theta)$.*

PROOF. We know that $\theta * \phi \in \mathrm{cf}(G^0)$. Let Λ be a complete set of representatives of the orbits of the action of G on $\mathrm{Irr}(N)$. By Lemma 3.4, we have that

$$\mathrm{cf}(G^0) = \mathrm{cf}(G^0|\theta) \oplus \left(\sum_{\lambda \in \Lambda - \{\theta\}} \mathrm{cf}(G^0|\lambda) \right).$$

Hence, we may write $\theta * \phi = \chi^0 + \psi^0$, where $\chi \in \mathrm{cf}(G|\theta)$ and $\psi \in \sum_{\lambda \in \Lambda - \{\theta\}} \mathrm{cf}(G|\lambda)$. Then

$$(\theta * \phi)_{NH} = \chi_{NH} + \psi_{NH}.$$

Now, $(\theta * \phi)_{NH} = \hat{\theta}\widehat{\phi_H}$ is a character of NH all of whose irreducible constituents lie over θ. (Recall that ϕ_H is a character of H and that $\widehat{\phi_H}$ is the unique extension of ϕ_H to NH containing N in its kernel.) Also, χ_{NH} only involves irreducible characters of NH lying over θ. On the other hand, no 'irreducible constituent' of ψ_{NH} lies over θ. Therefore, by the linear independence of $\mathrm{Irr}(NH)$, we conclude that $\psi_{NH} = 0$. Hence, $\psi^0 = 0$ and the proof of the lemma is complete. □

Recall that we are writing $\mathrm{I}_\pi(G)$ for the set of the Isaacs irreducible π-partial characters of G.

THEOREM 6.2. *Suppose that G is a π-separable group. Let N be a normal π'-subgroup of G and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Then*

$$\{\theta * \phi | \phi \in \mathrm{I}_\pi(G)\} = \mathrm{I}_\pi(G|N, \theta).$$

Therefore, if $\chi \in \mathrm{Irr}(G|\theta)$, then

$$\chi^0 = \sum_{\phi \in \mathrm{I}_\pi(G)} a_{\chi\phi}(\theta * \phi)$$

for some nonnegative integers $a_{\chi\phi}$.

PROOF. Let H be a Hall π-subgroup of G. By Theorem 5.3 (and Lemma 3.5), it is enough to show that

$$[\theta * \phi, (\hat{\theta}\widehat{\alpha_\mu})^G]^0 = \delta_{\phi,\mu}$$

for $\phi, \mu \in \mathrm{I}_\pi(G)$. Let $\chi \in \mathrm{B}_\pi\, G$ with $\chi^\pi = \phi$ and let $\widehat{\theta * \phi}$ be any class function of G extending the class function $\theta * \phi$. Now

$$\begin{aligned}[\theta * \phi, (\hat{\theta}\widehat{\alpha_\mu})^G]^0 &= [\widehat{\theta * \phi}, (\hat{\theta}\widehat{\alpha_\mu})^G]^0 = [\widehat{\theta * \phi}, (\hat{\theta}\widehat{\alpha_\mu})^G] = [(\widehat{\theta * \phi})_{NH}, \hat{\theta}\widehat{\alpha_\mu}] \\ &= [(\theta * \phi)_{NH}, \hat{\theta}\widehat{\alpha_\mu}] = [\hat{\theta}\chi_{NH}, \hat{\theta}\widehat{\alpha_\mu}].\end{aligned}$$

Now, N is in the kernel of χ [1, Corollary 5.3] and therefore, all irreducible constituents of χ_{NH} have N in its kernel. By Gallagher [4, Corollary 6.17], (and using that $N \subseteq \ker \chi$), we have that

$$[\hat{\theta}\chi_{NH}, \hat{\theta}\widehat{\alpha_\mu}] = [\chi_{NH}, \widehat{\alpha_\mu}] = [\chi_H, \alpha_\mu].$$

Now,

$$[\chi_H, \alpha_\mu] = [\phi_H, \alpha_\mu] = \delta_{\phi,\mu}$$

by [2, Theorem 2.2]. □

Since it is always possible to choose a complete set Θ of representatives of the action of G on $\mathrm{Irr}(N)$ such that each member $\theta \in \Theta$ is HN-good (as we already did in the proof of Theorems 1.1 and 1.2), it follows that Theorem 6.2, together with Lemma 3.6 and Lemma 3.7 completely describes the set $\mathrm{I}_\pi(G|N)$.

There is a relationship between the sets $\mathrm{I}_\pi(G)$ and $\mathrm{I}_\pi(G|N)$. The general fact is the next result. If $\chi \in \mathrm{cf}(G)$, we are denoting by χ^π the restriction of χ to G^π, the set of π-elements of G.

THEOREM 6.3. *Suppose that $\chi \in \mathrm{Irr}(G)$ is such that $\chi^\pi \in \mathrm{I}_\pi(G)$. Then $\chi^0 \in \mathrm{I}_\pi(G|N)$ for every normal π'-subgroup N of G.*

PROOF. By Theorem 1.1, write

$$\chi^0 = \sum_{\phi \in \mathrm{I}_\pi(G|N)} d_{\chi\phi}\phi,$$

where the $d_{\chi\phi}$ are nonnegative integers. Let H be a Hall π-subgroup of G. Then

$$\chi_H = \sum_{\phi \in \mathrm{I}_\pi(G|N)} d_{\chi\phi}\phi_H$$

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and note that ϕ_H is a G-invariant character of H because ϕ_{HN} is a character of HN by Lemma 3.3. Hence, by [5, Theorem B], there exists a character ψ_ϕ of G such that $(\psi_\phi)_H = \phi_H$. Now,

$$\chi_H = \sum_{\phi \in \mathrm{I}_\pi(G|N)} d_{\chi\phi}(\psi_\phi)_H$$

and we deduce that

$$\chi^\pi = \sum_{\phi \in \mathrm{I}_\pi(G|N)} d_{\chi\phi}(\psi_\phi)^\pi .$$

Since $\chi^\pi \in \mathrm{I}_\pi(G)$, this implies that there is a unique $\phi \in \mathrm{I}_\pi(G|N)$ such that $d_{\chi\phi} = 1$, while $d_{\chi\mu} = 0$ for $\phi \neq \mu \in \mathrm{I}_\pi(G|N)$. This proves the theorem. □

We say that $\chi, \psi \in \mathrm{Irr}(G)$ are *linked* if there exists $\phi \in \mathrm{I}_\pi(G|N)$ such that $d_{\chi\phi} \neq 0 \neq d_{\psi\phi}$. Of course, the connected components in $\mathrm{Irr}(G)$ of the graph defined by linking partitions $\mathrm{Irr}(G)$ into 'blocks' associated with the normal π'-subgroup N. These blocks, the associated Cartan matrices and some other relevant features are studied in [7].

A natural question when dealing with canonical bases of certain normal subsets of a finite group, is whether or not there is a 'Fong-Swan theorem' for them; that is, if the elements of the canonical basis can be extended to characters of the group. Contrary to the case where $N = 1$, this is not true here.

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