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THE FAILURE OF APPROXIMATE INNER CONJUGACY FOR STANDARD DIAGONALS IN REGULAR LIMIT ALGEBRAS

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ABSTRACT. AF C*-algebras contain natural AF masas which, here, we call *standard diagonals*. Standard diagonals are unique, in the sense that two standard diagonals in an AF C*-algebra are conjugate by an approximately inner automorphism. We show that this uniqueness fails for non-selfadjoint AF operator algebras. Precisely, we construct two standard diagonals in a particular non-selfadjoint AF operator algebra which are not conjugate by an approximately inner automorphism of the non-selfadjoint algebra.

1. **Introduction.** Approximately finite-dimensional (AF) C*-algebras contain natural AF maximal abelian self-adjoint subalgebras (masas) which, here, we call *standard diagonals*. A standard diagonal arises from a nested matrix unit system for a dense finite-dimensional subalgebra chain and it serves a useful role for coordinatisation, classification and representation theory; see [6] for example. Standard diagonals are regular in the sense, borrowed from von Neumann algebra theory, that the set of partial isometries which normalise the diagonal generates the containing algebra. Indeed, all matrix units normalise the diagonal. In the non-selfadjoint setting it is the so-called *regular* limit algebras that contain (regular) standard diagonals. On the other hand there are limit algebras that are not regular [3].

Just as in the (trivial) finite-dimensional case, any two standard diagonals in an AF C*-algebra are conjugate by an approximately inner automorphism. In other words, there is an automorphism α mapping one standard diagonal onto the other and a sequence of unitaries $\{u_k\}_{k\in\mathbb{N}}$ in the AF C*-algebra so that α has the representation

$$\alpha(a) = \lim_{k\to\infty} u_k a u_k^*.$$

This uniqueness is implicit in the discussions of the uniqueness of AF groupoids given by Renault [5] and Kreiger [2], and the fact that automorphisms of AF C*-algebras that fix K_0 are necessarily approximately inner. For a direct proof, see Theorem 5.7 of [4].

We show that, unexpectedly, this natural uniqueness can fail for non-selfadjoint approximately finite operator algebras and even in the purely algebraic setting, for regular algebraic direct limits of digraph algebras. We do this by constructing two standard diagonals in a particular non-selfadjoint AF operator algebra for which this uniqueness fails. However, there is an automorphism (*not* approximately inner) that maps one standard diagonal to the other. Hence our example leaves the following question open: for

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two standard diagonals in a non-selfadjoint AF operator algebra, is there always a star-extendible automorphism that maps one to the other?

We now give a precise formulation of the main result.

Let $\{B_k, \beta_k\}_{k=1}^{\infty}$ be a direct system of finite-dimensional C*-algebras B_k and C*-algebra homomorphisms $\beta_k: B_k \longrightarrow B_{k+1}$. There are many ways to choose matrix unit systems for $B_1, B_2, \ldots$ so that each β_k maps matrix units to sums of matrix units. Each such choice gives rise to an abelian algebra D_0 in the algebraic direct limit $B_0 = \operatorname{alg} \varinjlim B_k$, namely $D_0 = \operatorname{alg} \varinjlim D_k$, where D_k is the subalgebra spanned by the diagonal matrix units. Since matrix units are mapped to sums of matrix units by each β_k, it follows that these matrix units normalise D_0 and so D_0 is regular in B_0. We refer to D_0 as a *standard diagonal* of B_0. Standard diagonals, even when arising from different direct systems for B_0, are conjugate by an approximately inner star automorphism of B_0. The same is also true for standard diagonals $D = \varinjlim D_k$ in the AF C*-algebra $B = \varinjlim B_k$.

Consider the intermediate algebra A_0 with $D_0 \subseteq A_0 \subseteq B_0$ given by $A_0 = \operatorname{alg} \varinjlim A_k$ where the A_k are algebras, which are not necessarily self-adjoint, such that, for all k,

$$D_k \subseteq A_k \subseteq B_k \text{ and } \beta_k(A_k) \subseteq A_{k+1}.$$

The finite-dimensional algebras A_k are known variously as poset algebras, incidence algebras, CSL algebras and digraph algebras. We adopt the digraph terminology, which is motivated by the observation that if B_k has matrix unit system $\left\{e_{ij}^{(k)} \mid (i,j) \in I_k\right\}$ then the matrix units $e_{ij}^{(k)}$ in A_k are labelled by the edges (j, i) of a reflexive transitive digraph. A subalgebra E_0 of the intermediate algebra A_0 is called a *standard diagonal* if there exists a presentation $A_0 = \operatorname{alg} \varinjlim A_k$, as above, so that $E_0 = D_0$. The following nonconjugacy result contrasts with both the self-adjoint context and the finite-dimensional case.

THEOREM 1. *There exists a regular limit algebra $A_0 = \operatorname{alg} \varinjlim A_k$ which contains standard diagonals C_0 and C_0' which are not conjugate by an approximately inner automorphism of A_0. Moreover the norm closures of C_0 and C_0' in the operator algebra $A = \varinjlim A_k$ are standard diagonals which are not conjugate by an approximately inner automorphism of A.*

The key to the proof is the construction, in finite dimensions, of homomorphisms of digraph algebras, $\alpha: A_1 \longrightarrow A_2$ and $\beta: A_2 \longrightarrow A_3$, so that neither α nor β map matrix units to sums of matrix units (for any choices of matrix units), yet the composition $\beta \circ \alpha$ is nondegenerate and does map matrix units to sums of matrix units.

Relevant theory for non-selfadjoint limit algebras is given in [4]. Nevertheless, for completeness we give the needed terminology and basic facts in the next section.

2. **Terminology and Preliminaries.** The limit algebras A_0, A may be defined, ab initio, without reference to the containing C*-algebras B_0, B, by means of *regular* homomorphisms, the most natural morphisms for digraph algebras.

DEFINITION 2. Let $A_1 \subseteq M_{n_1}$ and $A_2 \subseteq M_{n_2}$ be digraph algebras. An algebra homomorphism $\varphi: A_1 \longrightarrow A_2$ is *star-extendible* if there exists a C*-algebra homomorphism $\tilde{\phi}: C^*(A_1) \longrightarrow C^*(A_2)$ for which $\varphi = \tilde{\phi}_{|A_1}$.

Given a star-extendible homomorphism $\varphi: A_1 \longrightarrow A_2$ it may not be possible to choose matrix units for A_1 (and M_{n_1}) and for A_2 (and M_{n_2}) so that φ maps matrix units to sums of matrix units. The homomorphisms for which this is so are the star-extendible *regular* homomorphisms.

DEFINITION 3. A star-extendible digraph algebra homomorphism $\varphi: A_1 \longrightarrow A_2$ is of *multiplicity one* if $\mathrm{rank}\big(\varphi(a)\big) \leq \mathrm{rank}(a)$ for all a in A_1, and is *regular* if it is a direct sum of multiplicity one star-extendible homomorphisms.

(It is convenient to allow the zero map to have multiplicity one.) The rank condition need only be specified for rank one projections in A_1. In general, regularity for a digraph algebra homomorphism is defined in terms of normalisers. However, for star-extendible maps, this is equivalent to the definition above; see Exercise 4.1 of [4].

The restriction morphisms $\beta_k: A_k \longrightarrow A_{k+1}$ given in the introduction are easily seen to be regular. But conversely, if $\{A_k, \varphi_k\}$ is a direct system of digraph algebras with regular star-extendible homomorphisms $\varphi_k: A_k \longrightarrow A_{k+1}$, then it is possible to successively choose matrix unit systems $\left\{e^{(k)}_{i,j} : (i,j) \in I_k\right\}$ for each $B_k = C^*(A_k)$ so that each A_k is spanned by $\left\{e^{(k)}_{i,j} : e^{(k)}_{i,j} \in A_k\right\}$ and each φ_k maps matrix units to sums of matrix units.

DEFINITION 4. Let C be a self-adjoint subalgebra of a C*-subalgebra A. Then the (partial isometry) *normaliser* of C in A is the set $N_C(A)$ of partial isometries v in A for which vcv^* and v^*cv belong to C for all c in C.

The normaliser of a standard diagonal in a digraph algebra coincides with the set of partial isometries v which are orthogonal sums of unimodular multiples of matrix units. As mentioned above, a star-extendible homomorphism $\varphi: A_1 \longrightarrow A_2$ is regular if and only if there exist diagonals (masas) $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ such that $\varphi\big(N_{C_1}(A_1)\big) \subseteq N_{C_2}(A_2)$. Furthermore for each diagonal C_1 chosen arbitrarily in A_1 there exists a diagonal C_2 in A_2 for which the inclusion holds.

Repeating this observation, it follows that if $\{A_k, \alpha_k\}$ is a regular direct system of digraph algebras, and if $C_k \subseteq A_k$ are chosen masas (or are given *a priori* as in the introduction) with the normalising property $\alpha_k\big(N_{C_k}(A_k)\big) \subseteq N_{C_{k+1}}(A_{k+1})$ for all k, then the abelian algebra $C_0 = \mathrm{alg}\varinjlim C_k$, which is a masa in A_0, has nontrivial normaliser $N_{C_0}(A_0)$ in $A_0 = \mathrm{alg}\varinjlim A_k$, and in fact A_0 is generated as an algebra by $N_{C_0}(A_0)$. Similar assertions hold in the case of Banach algebra limits. To summarise, standard diagonals are regular in the sense mentioned in the introduction.

Let D_4 be the 4-cycle digraph and $A(D_4) \subseteq M_4(\mathbb{C})$ be the digraph algebra spanned by the standard matrix units $e_{i,j}$ where (j, i) is an edge of D_4, *i.e.*,

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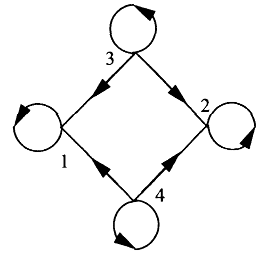
D_4 is 

$$A(D_4) = \begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

The limit algebras of Theorem 1 are based on the building blocks $A(D_4) \otimes M_n(\mathbb{C})$.

There are four symmetries of D_4: the identity, the reflection that exchanges 1 with 2, the rotation that exchanges 1 with 2 and 3 with 4, and the reflection that exchanges 3 with 4. They induce four automorphisms of $A(D_4)$, which we label, respectively, θ_1, θ_2, θ_3 and θ_4. As in [4] we say that a homomorphism $\alpha: A(D_4) \otimes M_n \longrightarrow A(D_4) \otimes M_m$ is *rigid* if it is a direct sum of multiplicity one homomorphisms, each of which is inner conjugate to an embedding γ where $\gamma(a \otimes b) = \theta(a) \otimes (b \oplus 1_{m-n})$ and θ is one of the θ_i. Thus a rigid embedding is a regular embedding without degenerate summands.

Let us say that an ordered set $\{v_1, v_2, v_3, v_4\}$ of rank one partial isometries in $A(D_4) \otimes M_n$ is an *essential 4-cycle* if $v_4^* v_3 v_2^* v_1 = v_1^* v_1$ and each v_i belongs to one of the spaces $e_{13} \otimes M_n, e_{14} \otimes M_n, e_{24} \otimes M_n, e_{23} \otimes M_n$ with no two belonging to the same such space. Then rigid embeddings are characterised by the fact that they map an essential 4-cycle to a 4-cycle of partial isometries which splits into a direct sum of essential 4-cycles.

The significance of rigid embeddings is that they generate a wide class of non-self-adjoint limit algebras (see [4] and [1]) and they are determined by their induced map on $K_0 \oplus H_1$. Here $H_1\big(A(D_4) \otimes M_n\big)$ can be taken to be the integral simplicial homology of the simplical complex of the undirected graph $\bar{D}_4 \times K_n$, where K_n is the complete n-vertex graph and $\bar{D}_4$ is the undirected 4-cycle graph. Thus, $H_1\big(A(D_4) \otimes M_n\big) = \mathbb{Z}$ and, with consistent identifications of $H_1\big(A(D_4) \otimes M_n\big)$ and $H_1\big(A(D_4) \otimes M_m\big)$, $H_1\gamma = [\pm 1]$. Precisely, $H_1\gamma$ is the 1×1 matrix $[+1]$ if θ is the identity θ_1 or the rotation θ_3 and is $[-1]$ if θ is one of the two reflections.

3. **Irregular morphisms and nonconjugate diagonals.** Let $A_k = A(D_4) \otimes M_{2^{k-1}}$, for $k = 1, 2, \ldots$, and view A_k as the subalgebra of $M_4 \otimes M_{2^{k-1}}$ consisting of block matrices

$$a = \begin{bmatrix} A & & X & Y \\ & B & W & Z \\ & & C & \\ & & & D \end{bmatrix}$$

where the unspecified entries are zero. The cyclical labeling X, Y, Z, W is useful, as we shall see, for keeping track of the homology of images of an essential 4-cycle of matrix

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units in A_k. Define $\alpha_k \colon A_k \longrightarrow A_{k+1}$ by setting

$$\alpha_k(a) = \begin{bmatrix} A & 0 & & & X' & Y' & X' & Y' \\ 0 & B & & & W' & -Z' & -W' & Z' \\ & & A & 0 & X' & -Y' & -X' & Y' \\ & & 0 & B & W' & Z' & W' & Z' \\ & & & & C & 0 & & \\ & & & & 0 & D & & \\ & & & & & & C & 0 \\ & & & & & & 0 & D \end{bmatrix}$$

where $X' = \frac{1}{\sqrt{2}}X$, $Y' = \frac{1}{\sqrt{2}}Y$, $Z' = \frac{1}{\sqrt{2}}Z$, $W' = \frac{1}{\sqrt{2}}W$.

LEMMA 5. *For each k, α_k is a star-extendible algebra injection of multiplicity 2 which is irregular. However, the compositions $\alpha_{k+1} \circ \alpha_k$ are rigid and $H_1(\alpha_{k+1} \circ \alpha_k) = 0$.*

A composition of irregular embeddings may be regular for rather trivial or degenerate reasons. For instance, the composition may map into the self-adjoint subalgebra of the final algebra. The importance of the α_k is that their compositions are rigid and so do not have this kind of degeneracy.

PROOF. It is easy to check that each α_k is a star-extendible algebra injection of multiplicity 2. To see that α_1 is irregular, note that $\alpha_1(e_{1,3})$ is not inner conjugate to the sum of two rank one partial isometries in A_2 whose initial and final projections belong to A_2, and yet this is a necessary condition for regularity. Similar remarks apply to α_k for $k \geq 1$.

It remains only to show that $\alpha_{k+1} \circ \alpha_k$ is rigid. To see this observe first that $\alpha_{k+1} \circ \alpha_k(a)$ is the block matrix

$$\begin{bmatrix} A & 0 & 0 & 0 & & & & & x & y & x & y & x & y & x & y \\ 0 & B & 0 & 0 & & & & & w & -z & -w & z & w & -z & -w & z \\ 0 & 0 & A & 0 & & & & & x & -y & x & -y & -x & y & -x & y \\ 0 & 0 & 0 & B & & & & & w & z & -w & -z & -w & -z & w & z \\ & & & & A & 0 & 0 & 0 & x & y & -x & -y & -x & -y & x & y \\ & & & & 0 & B & 0 & 0 & w & -z & w & -z & -w & z & -w & z \\ & & & & 0 & 0 & A & 0 & x & -y & -x & y & x & -y & -x & y \\ & & & & 0 & 0 & 0 & B & w & z & w & z & w & z & w & z \\ & & & & & & & & C & 0 & 0 & 0 & & & & \\ & & & & & & & & 0 & D & 0 & 0 & & & & \\ & & & & & & & & 0 & 0 & C & 0 & & & & \\ & & & & & & & & 0 & 0 & 0 & D & & & & \\ & & & & & & & & & & & & C & 0 & 0 & 0 \\ & & & & & & & & & & & & 0 & D & 0 & 0 \\ & & & & & & & & & & & & 0 & 0 & C & 0 \\ & & & & & & & & & & & & 0 & 0 & 0 & D \end{bmatrix}$$

where $x = \frac{1}{2}X$, $y = \frac{1}{2}Y$, $z = \frac{1}{2}Z$ and $w = \frac{1}{2}W$. In particular, the image of each of the matrix units $e_{13}, e_{14}, e_{24}, e_{23}$ has the block form

$$\begin{bmatrix} 0 & & V_1 & V_2 \\ & 0 & V_4 & V_3 \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

where *each V_i is a partial isometry*. It is this fortuitous property which is crucial for the construction.

If $u \in A_{k+2}$ is the block diagonal unitary whose four diagonal entries are each

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

then $u\big(\alpha_{k+1} \circ \alpha_k(a)\big)u^*$ has the form

$$\left[\begin{array}{cccccccccccccccc}
A & 0 & 0 & 0 & & & & & X & 0 & 0 & 0 & 0 & Y & 0 & 0 \\
0 & B & 0 & 0 & & & & & 0 & 0 & -W & 0 & 0 & 0 & 0 & Z \\
0 & 0 & A & 0 & & & & & 0 & -Y & 0 & 0 & -X & 0 & 0 & 0 \\
0 & 0 & 0 & B & & & & & 0 & 0 & 0 & -Z & 0 & 0 & W & 0 \\
 & & & & A & 0 & 0 & 0 & 0 & 0 & -X & 0 & 0 & 0 & 0 & Y \\
 & & & & 0 & B & 0 & 0 & W & 0 & 0 & 0 & 0 & Z & 0 & 0 \\
 & & & & 0 & 0 & A & 0 & 0 & 0 & 0 & Y & 0 & 0 & -X & 0 \\
 & & & & 0 & 0 & 0 & B & 0 & Z & 0 & 0 & W & 0 & 0 & 0 \\
 & & & & & & & & C & 0 & 0 & 0 & & & & \\
 & & & & & & & & 0 & D & 0 & 0 & & & & \\
 & & & & & & & & 0 & 0 & C & 0 & & & & \\
 & & & & & & & & 0 & 0 & 0 & D & & & & \\
 & & & & & & & & & & & & C & 0 & 0 & 0 \\
 & & & & & & & & & & & & 0 & D & 0 & 0 \\
 & & & & & & & & & & & & 0 & 0 & C & 0 \\
 & & & & & & & & & & & & 0 & 0 & 0 & D
\end{array}\right]$$

The minus signs are easily removed, by a further diagonal conjugation, and it follows that $\alpha_{k+1} \circ \alpha_k$ is a regular embedding. Furthermore $\alpha_{k+1} \circ \alpha_k$ is rigid, since the image of an essential 4-cycle can be seen to be the direct sum of 4 essential 4-cycles, and inspecting the orientation of these cycles shows that $H_1(\alpha_{k+1} \circ \alpha_k) = 0$. ■

Consider now the two regular direct systems

$$\mathcal{S}_{\text{even}} = \{A_{2k}, \alpha_{2k+1} \circ \alpha_{2k}\}, \quad \mathcal{S}_{\text{odd}} = \{A_{2k-1}, \alpha_{2k} \circ \alpha_{2k-1}\}.$$

Note that they determine the same limit algebra, call it $\mathcal{A}$, and yet the identity isomorphism $\mathcal{A} \longrightarrow \mathcal{A}$ is not induced by a regular isomorphism between $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$. By a *regular* isomorphism, we mean one for which there is a commmuting diagram between $\mathcal{S}_{\text{even}}$ and $\mathcal{S}_{\text{odd}}$ in which all the interlacing maps are regular. However in this case the connecting maps of such a commuting diagram must be compositions of an odd number of consecutive α_k and these will be irregular. The next theorem shows that we can construct the required counterexamples in $\mathcal{A}$, proving Theorem 1.

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THEOREM 6. *The limit algebra* $\mathcal{A} = \varinjlim\big(A(D_4)\otimes M_{2^{k-1}}, \alpha_k\big)$ *contains two standard diagonals which are not conjugate by an approximately inner automorphism of* $\mathcal{A}$.

PROOF. Recall $A_k = A(D_4)\otimes M_{2^{k-1}}$. Choose masas $C_{2k} \subseteq A_{2k}$ successively so that the normaliser of C_{2k} contains the image under $\alpha_{2k-1}\circ\alpha_{2k-2}$ of the normaliser of C_{2k-2}. Similarly choose $C_1, C_3, \ldots$. Then $C = \varinjlim C_{2k}$ and $C' = \varinjlim C_{2k+1}$ are standard diagonals in $\mathcal{A}$.

For notational convenience we can henceforth regard the given maps α_k as inclusion maps.

Suppose, by way of contradiction, that $\gamma\colon \mathcal{A}\longrightarrow\mathcal{A}$ is an automorphism so that

$$\gamma(C) = C' \text{ and } \gamma(a) = \lim_{k\to\infty} u_k a u_k^*,$$

for some sequence of unitaries $u_k \in \mathcal{A}$. If a unitary in $\mathcal{A}$ is close to an element of the subalgebra $A(D_4)\otimes M_{2^{k-1}}$, then it is close to a unitary in $A(D_4)\otimes M_{2^{k-1}}$. For this reason we may assume that $u_k \in A_k$ for all k.

Since γ is isometric it follows that γ maps the normaliser of C in $\mathcal{A}$ into the normaliser of C' in $\mathcal{A}$ (see Proposition 7.1 of [4]). In view of the structure of normalising partial isometries (*i.e.*, Lemma 5.5 of [4]), there exists m_1 such that

$$\gamma(A_1) \subseteq \tilde{A}_{2m_1}$$

where $\tilde{A}_{2m_1}$ is the algebra generated by A_{2m_1} and C'. Increasing m_1 if necessary, we can express the resulting restriction map $\gamma_r\colon A_1 \longrightarrow \tilde{A}_{2m_1}$, as a composition $(\operatorname{Ad} c_1)\circ\gamma_1$ where $\gamma_1\colon A_1 \longrightarrow A_{2m_1}$ is regular, mapping C_1 into C_{2m_1}, and c_1 is a unitary element of C. (This follows immediately from the elementary C*-algebraic fact that the extension $\tilde{\gamma}_r\colon\ C^*(A_1)\longrightarrow\ C^*(\tilde{A}_{2m_1})$ has such a factored form.) Increasing m_1 again, if necessary, γ_r is close to $(\operatorname{Ad} c_2)\circ\gamma_1$ for some unitary element c_2 of A_{2m_1}. But by hypothesis, we may also assume that γ_r is close to $\operatorname{Ad} u$ with $u = u_{2m_1}$ a unitary in A_{2m_1}. Thus the inclusion map $i\colon A_1 \longrightarrow A_{2m_1}$ is close to the regular map $(\operatorname{Ad} u^*)\circ(\operatorname{Ad} c_2)\circ\gamma_1$. In particular viewing $i(e_{13})$ as a 4×4 block matrix in A_{2m_1} we deduce that the $(1,3)$ block is close to a partial isometry, and in particular has norm close to one.

On the other hand, $i = \beta\circ\alpha_1$, where β, being an even composition of the given maps α_k, is a rigid embedding. Since β is a direct sum of multiplicity one rigid embeddings it becomes clear that the norm of the $(1,3)$ block of $i(e_{13})$ equals the norm of the $(1,3)$ block of $\alpha_1(e_{13})$, *i.e.*, $1/\sqrt{2}$. This contradiction completes the proof. ∎

REMARKS. 1. This theorem has some purely C*-algebraic consequences for extensions of approximately inner automorphisms.

Let $\mathcal{E} = \mathcal{A}\cap\mathcal{A}^*$ and let $\mathcal{F} = C^*(\mathcal{A})$, both AF C*-algebras. Then there is an approximately inner automorphism of $\mathcal{E}$, say ϕ, that carries C to C'. However, if $\{u_k\}$ is any sequence of unitaries in $\mathcal{E}$ satisfying $\phi(x) = \lim_k u_k x u_k^*$ for all $x\in\mathcal{E}$, then this limit representation does not extend to $\mathcal{F}$. That is, there must be some $y\in\mathcal{F}$ so that $\lim_k u_k y u_k^*$ does not exist. Otherwise, we would have an approximately inner automorphism of $\mathcal{F}$

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that restricts to an approximately inner automorphism of $\mathcal{A}$ (as $u_k, u_k^* \in \mathcal{A}$), contradicting Theorem 6.

If we do not require that both C and C' are regular in $\mathcal{F}$, then it is possible to construct standard diagonals C, C' in the AF C*-algebra $\mathcal{E}$ so that there is an approximately inner automorphism of $\mathcal{E}$ carrying C to C' but no such automorphism of $\mathcal{F}$. Obviously, the automorphism of $\mathcal{E}$ does not extend to $\mathcal{A}$ in this case. The automorphism of $\mathcal{E}$ in the setting of Theorem 6 fails to extend for different reasons, as C and C' are regular in $\mathcal{F}$ and so there is an approximately inner automorphism of $\mathcal{F}$ that carries C to C'. By the previous paragraph, the implementing unitaries of this automorphism of $\mathcal{F}$ cannot be chosen in $\mathcal{E}$.

2. We give another direct system for $\mathcal{A}$, showing $\mathcal{A}$ can be given by quite conventional regular embeddings. Let $\beta_k: A(D_4) \otimes M_{4^{k-1}} \longrightarrow A(D_4) \otimes M_{4^k}$ be regular embeddings for which

$$\beta_k(a \otimes b) = \theta_1(a) \otimes (b \oplus 0 \oplus 0 \oplus 0) + \theta_2(a) \otimes (0 \oplus b \oplus 0 \oplus 0) \\ + \theta_3(a) \otimes (0 \oplus 0 \oplus b \oplus 0) + \theta_4(a) \otimes (0 \oplus 0 \oplus 0 \oplus b),$$

where $\theta_1, \ldots, \theta_4$ are the permutation automorphisms of $A(D_4)$. The embeddings β_k are in fact rigid embeddings with

$$K_0\beta_k = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \quad H_1\beta_k = [0].$$

Hence $\alpha_{2k+1} \circ \alpha_{2k}$ is inner conjugate to β_k, for all k. With the notation above, $\mathcal{A}$ is isomorphic to $\varinjlim\bigl(A(D_4) \otimes M_{4^{k-1}}, \beta_k\bigr)$.

3. The diagonals C, C' above are automorphically conjugate, by which we mean that there is a star-extendible automorphism of $\mathcal{A}$ which maps C to C'. To see this one can appeal to Theorem 11.21 of [4] or, much more simply, argue as follows.

Recall that θ_1 is the identity automorphism of $A(D_4)$ and θ_2 and θ_4 are the two reflections. Let $\eta_1: A_1 \longrightarrow A_2$ be the standard regular embedding from $A(D_4) \otimes \mathbb{C}$ to $A(D_4) \otimes M_2$ with

$$\eta_1(a \otimes b) = \theta_1(a) \otimes \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} + \theta_2(a) \otimes \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}.$$

Replacing η_1 by an inner conjugate we can arrange $\eta_1(C_1) \subseteq C_2$. Let $\eta_2: A_2 \longrightarrow A_3$ be the standard regular embedding from $A(D_4) \otimes M_2$ to $A(D_4) \otimes M_4$ with

$$\eta_2(a \otimes b) = \theta_1(a) \otimes \begin{bmatrix} b & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} + \theta_4(a) \otimes \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & b \end{bmatrix}.$$

Now $\eta_2 \circ \eta_1$ is rigid, and furthermore has the same $K_0 \oplus H_1$ data as β_1. We may therefore replace η_2 by an inner conjugate embedding so that $\eta_2 \circ \eta_1 = \beta_1$. Continue in this way (with η_3 defined in terms of θ_1 and θ_2, η_4 in terms of θ_1 and θ_4, and so on) to obtain an automorphism $\lim \eta_{2k-1}$ from the pair $(\mathcal{A}, C)$ to the pair $(\mathcal{A}, C')$.

It is an important open problem whether standard diagonals are automorphically conjugate in general. A positive answer would show, for example, that homology groups considered in [4, Chapter 11] and [1] are invariants for the non-selfadjoint AF operator algebra as well as for the pair of algebra and standard diagonal.

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