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THE HARDY AND HEISENBERG INEQUALITIES IN MORREY SPACES

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Abstract

We use the Morrey norm estimate for the imaginary power of the Laplacian to prove an interpolation inequality for the fractional power of the Laplacian on Morrey spaces. We then prove a Hardy-type inequality and use it together with the interpolation inequality to obtain a Heisenberg-type inequality in Morrey spaces.

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1. Introduction

Inspired by the work of Ciatti *et al.* [1], we are interested in obtaining an estimate for the Morrey norm of the fractional power of the Laplacian, in order to prove Heisenberg's uncertainty inequality in Morrey spaces. Let $(-\Delta)^{z/2}$ be the complex power of the Laplacian, given by

$$[(-\Delta)^{z/2} f]\widehat{\ }(\xi) := |\xi|^z \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \tag{1.1}$$

for suitable functions f on $\mathbb{R}^n$, where the Fourier transform is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n.$$

Our first aim is to show the following Morrey norm estimate for the imaginary power of the Laplacian:

$$\|(-\Delta)^{iu/2} f\|_{\mathcal{M}_q^p} \lesssim (1 + |u|)^{n/2} \|f\|_{\mathcal{M}_q^p}, \quad f \in \mathcal{M}_q^p(\mathbb{R}^n),$$

for every $u \in \mathbb{R}$, provided that $1 < p \le q < \infty$.

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For $1 \le p \le q < \infty$, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{1/q-1/p}\Big(\int_{B(a,r)} |f(y)|^p\,dy\Big)^{1/p}$$

is finite. We refer the reader to [15] for various function spaces built on Morrey spaces.

Based on [9], let us explain why $(-\Delta)^{iu/2}$ should be bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$, for $1 < p \le q < \infty$, with bound $C(u) \lesssim (1+|u|)^{n/2}$. We define $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ to be the closure of $C^\infty_{\rm c}(\mathbb{R}^n)$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ or, equivalently, $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is the closure of $L^q(\mathbb{R}^n)$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ (see [16, page 1846]). We know that $(-\Delta)^{iu/2}$ maps $L^q(\mathbb{R}^n)$ boundedly into $L^q(\mathbb{R}^n)$ [2]. We also establish in Lemma 2.1 that $\|(-\Delta)^{iu/2} f\|_{\mathcal{M}^p_q} \lesssim C(u)\|f\|_{\mathcal{M}^p_q}$ for $f \in C^\infty_{\rm c}(\mathbb{R}^n)$, keeping in mind that $C^\infty_{\rm c}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \subset \mathcal{M}^p_q(\mathbb{R}^n)$ and that $(-\Delta)^{iu/2} f$ makes sense for $f \in C^\infty_{\rm c}(\mathbb{R}^n)$ by (1.1). This means that $(-\Delta)^{iu/2} : \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \to \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is bounded (see Definition 2.2 and Lemma 2.3). From [11, Theorem 4.3], the space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is the dual of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ if $1/p + 1/p' = 1/q + 1/q' = 1$. Here, $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is the set of all functions $f \in L^{q'}(\mathbb{R}^n)$ for which

$$f = \sum_{j=1}^\infty \lambda_j A_j, \tag{1.2}$$

where $\{\lambda_j\}_{j=1}^\infty \in \ell^1$ and $\{A_j\}_{j=1}^\infty$ is a sequence of functions supported on balls with $\|A_j\|_{L^{q'}} \le 1$ for every $j \in \mathbb{N}$. The norm of $f \in \mathcal{H}^{p'}_{q'}$ is defined by

$$\|f\|_{\mathcal{H}^{p'}_{q'}} := \inf\Big\{\sum_{j=1}^\infty |\lambda_j| : \{\lambda_j\}_{j=1}^\infty \text{ and } \{A_j\}_{j=1}^\infty \text{ satisfying the condition for (1.2)}\Big\}.$$

The dual of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is $\mathcal{M}^p_q(\mathbb{R}^n)$ [17]. In general, the dual mapping of a bounded linear mapping T from a Banach space X to Y is bounded from Y^* to X^*. Since $(-\Delta)^{iu/2}$ is formally self-adjoint, we see that the boundedness of $(-\Delta)^{iu/2} : \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \to \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ established above entails that of $(-\Delta)^{iu/2} : \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \to \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ (see Definition 2.4 and Lemma 2.5), which in turn entails the boundedness of $(-\Delta)^{iu/2} : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$ (see Definition 2.6 and Proposition 2.7).

We note that $|\cdot|^{iu}\widehat{f}$ does not make sense for some $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. As indicated above, the operator $(-\Delta)^{iu/2}$ which is initially defined on $C^\infty_{\rm c}(\mathbb{R}^n)$ is then defined on $\mathcal{M}^p_q(\mathbb{R}^n)$ by the duality relation

$$\langle (-\Delta)^{iu/2} f, g\rangle = \langle f, (-\Delta)^{-iu/2} g\rangle, \quad g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n),$$

because the dual of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is $\mathcal{M}^p_q(\mathbb{R}^n)$ (see [17, Proposition 5] and Definition 2.4). We claim that this definition of $(-\Delta)^{iu/2} f$ coincides with the one given by the Fourier transform, whenever the Fourier transform of f makes sense. Indeed, we show that

$$\overline{\psi(\xi)}\mathcal{F}[(-\Delta)^{iu/2} f](\xi) = \overline{\psi(\xi)}|\xi|^{iu}\mathcal{F} f(\xi)$$

for every $\psi \in C_c^\infty(\mathbb{R}^n)$ and $0 \notin \operatorname{supp}\psi$, where $\mathcal{F}$ denotes the Fourier transform. Observe that if $g \in \mathcal{H}_{q'}^{p'}(\mathbb{R}^n)$, then $\mathcal{F}^{-1}[\psi\mathcal{F}g] \in \mathcal{H}_{q'}^{p'}(\mathbb{R}^n)$. In fact,

$$\mathcal{F}^{-1}[\psi\mathcal{F}g](x) = (2\pi)^n\mathcal{F}^{-1}\psi * g(x) = (2\pi)^n \int_{\mathbb{R}^n} \mathcal{F}^{-1}\psi(y)g(x-y)\,dy.$$

As a result,

$$\begin{aligned}\|\mathcal{F}^{-1}[\psi\mathcal{F}g]\|_{\mathcal{H}_{q'}^{p'}} &\le (2\pi)^n \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\psi(y)|\,\|g(\cdot - y)\|_{\mathcal{H}_{q'}^{p'}}\,dy \\ &\le (2\pi)^n \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\psi(y)|\,\|g\|_{\mathcal{H}_{q'}^{p'}}\,dy = C\|g\|_{\mathcal{H}_{q'}^{p'}} < \infty\end{aligned}$$

and $\mathcal{F}^{-1}[\psi\mathcal{F}g] \in \mathcal{H}_{q'}^{p'}(\mathbb{R}^n)$. It follows that

$$\langle(-\Delta)^{iu/2}f, \mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle = \langle f, (-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle$$

or, equivalently,

$$\langle\mathcal{F}^{-1}[\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f]], g\rangle = \langle f, (-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle.$$

Since $g \in L^{q'}(\mathbb{R}^n)$,

$$(-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g] = \mathcal{F}^{-1}[|\cdot|^{-iu}\psi\mathcal{F}g].$$

Consequently,

$$\langle f, (-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle = \langle f, \mathcal{F}^{-1}[|\cdot|^{-iu}\psi\mathcal{F}g]\rangle = \langle\mathcal{F}^{-1}[\overline{\psi}|\cdot|^{iu}\mathcal{F}f], g\rangle$$

and therefore

$$\langle\mathcal{F}^{-1}[\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f]], g\rangle = \langle\mathcal{F}^{-1}[\overline{\psi}|\cdot|^{iu}\mathcal{F}f], g\rangle.$$

Since g is arbitrary, $\mathcal{F}^{-1}[\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f]] = \mathcal{F}^{-1}[\overline{\psi}|\cdot|^{iu}\mathcal{F}f]$, so that we obtain $\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f] = \overline{\psi}|\cdot|^{iu}\mathcal{F}f$, as claimed.

In the following sections, we prove the Morrey norm estimate for the imaginary power of the Laplacian and its consequence for the fractional power of the Laplacian. We also prove a Hardy-type inequality and use it together with the estimate for the fractional power of the Laplacian to obtain Heisenberg's uncertainty inequality in Morrey spaces.

2. Morrey norm estimates for the fractional power of the Laplacian

For each $u \in \mathbb{R} \setminus \{0\}$ and on $L^p(\mathbb{R}^n)$ for $1 \le p \le 2$, the operator $(-\Delta)^{iu/2}$ (defined by (1.1)) admits an integral kernel K_u given by

$$K_u(x) := \frac{\pi^{-n/2}\Gamma(\frac{n+iu}{2})}{2^{-iu}\Gamma(\frac{-iu}{2})}|x|^{-n-iu} = C(u)|x|^{-n-iu}, \quad x \in \mathbb{R}^n$$

(see [14, page 51]). Here, $\widehat{K_u}(\xi) = |\xi|^{iu}$ in the distribution sense. A close inspection of the above constant shows that

$$|C(u)| \lesssim (1+|u|)^{n/2}, \quad u \in \mathbb{R}.$$

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As shown in [2, 13],

$$\|(-\Delta)^{iu/2} f\|_{L^p} \lesssim (1+|u|)^{|n/p-n/2|}\|f\|_{L^p} \lesssim (1+|u|)^{n/2}\|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n)$$

for every $u \in \mathbb{R}$, provided that $1 < p \le 2$. By duality, the same inequality also holds for $2 < p < \infty$.

Based on the discussion in Section 1, we shall now prove that the inequality above also holds in Morrey spaces (see [9] for similar results). We need several lemmas and definitions.

Lemma 2.1. *Let $u \in \mathbb{R}$ and $1 < p \le q < \infty$. Then, for every $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\|(-\Delta)^{iu/2} f\|_{\widetilde{\mathcal{M}}^p_q} \lesssim (1+|u|)^{n/2}\|f\|_{\widetilde{\mathcal{M}}^p_q}.$$

Proof. To prove the inequality, it is sufficient to establish that

$$|B(a,r)|^{1/q-1/p}\Big(\int_{B(a,r)} |(-\Delta)^{iu/2} f(x)|^p\,dx\Big)^{1/p} \lesssim (1+|u|)^{n/2}\|f\|_{\mathcal{M}^p_q}$$

for all fixed balls $B = B(a,r)$. To do so, we adopt the technique used in [6]. For a fixed ball $B = B(a,r)$, we decompose $f := f_1 + f_2$, where $f_1 := f\chi_{B(a,2r)}$ and $f_2 := f - f_1$. By the boundedness of $(-\Delta)^{iu/2}$ on $L^p(\mathbb{R}^n)$,

$$\begin{aligned}
&|B(a,r)|^{1/q-1/p}\Big(\int_{B(a,r)} |(-\Delta)^{iu/2} f_1(x)|^p\,dx\Big)^{1/p}\\
&\quad\le |B(a,r)|^{1/q-1/p}\Big(\int_{\mathbb{R}^n} |(-\Delta)^{iu/2} f_1(x)|^p\,dx\Big)^{1/p}\\
&\quad\lesssim (1+|u|)^{n/2}|B(a,r)|^{1/q-1/p}\Big(\int_{\mathbb{R}^n} |f_1(x)|^p\,dx\Big)^{1/p}\\
&\quad\sim (1+|u|)^{n/2}|B(a,2r)|^{1/q-1/p}\Big(\int_{B(a,2r)} |f(x)|^p\,dx\Big)^{1/p}\\
&\quad\lesssim (1+|u|)^{n/2}\|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

For each $x \in B$,

$$\begin{aligned}
|(-\Delta)^{iu/2} f_2(x)| &\le |C(u)| \int_{\mathbb{R}^n\setminus B(x,r)} \frac{|f(y)|}{|x-y|^n}\,dy \le |C(u)| \sum_{k=0}^\infty \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(y)|}{|x-y|^n}\,dy\\
&\lesssim |C(u)| \sum_{k=0}^\infty \frac{1}{(2^k r)^n} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} |f(y)|\,dy\\
&\lesssim |C(u)| \sum_{k=0}^\infty \Big(\frac{1}{(2^k r)^n} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} |f(y)|^p\,dy\Big)^{1/p}\\
&\lesssim |C(u)|\,\|f\|_{\mathcal{M}^p_q} \sum_{k=0}^\infty (2^k r)^{-n/q} \lesssim r^{-n/q}|C(u)|\,\|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

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Thus,

$$\begin{aligned}
&|B(a,r)|^{1/q-1/p}\left(\int_{B(a,r)}|(-\Delta)^{iu/2}f_2(x)|^p\,dx\right)^{1/p}\\
&\quad\lesssim |B(a,r)|^{1/q-1/p}\left(\int_{B(a,r)}(r^{-n/q}|C(u)|\,\|f\|_{\mathcal{M}^p_q})^p\,dy\right)^{1/p}\\
&\quad= |B(a,r)|^{1/q}r^{-n/q}|C(u)|\,\|f\|_{\mathcal{M}^p_q}\\
&\quad\sim |C(u)|\,\|f\|_{\mathcal{M}^p_q}\lesssim (1+|u|)^{n/2}\|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Combining the two estimates, we obtain the desired inequality. □

Using Lemma 2.1 and density, we give the following natural definition.

Definition 2.2. Given $f\in\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, we define

$$(-\Delta)^{iu/2}f:=\lim_{j\to\infty}(-\Delta)^{iu/2}f_j,$$

where $f_j\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $f_j\to f$ in the $\mathcal{M}^p_q$-norm.

The next lemma is a direct consequence of Lemma 2.1 and Definition 2.2.

Lemma 2.3. *Let $u\in\mathbb{R}$ and $1<p\le q<\infty$. Then, for every $f\in\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$,*

$$\|(-\Delta)^{iu/2}f\|_{\widetilde{\mathcal{M}}^p_q}\lesssim(1+|u|)^{n/2}\|f\|_{\widetilde{\mathcal{M}}^p_q}.$$

Definition 2.4. For every $g\in\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, we define

$$\langle(-\Delta)^{iu/2}g,h\rangle=\langle g,(-\Delta)^{-iu/2}h\rangle\quad\text{for every } h\in\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n).$$

Lemma 2.5. *Let $u\in\mathbb{R}$ and $1<p\le q<\infty$. Then, for every $g\in\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$,*

$$\|(-\Delta)^{iu/2}g\|_{\mathcal{H}^{p'}_{q'}}\lesssim(1+|u|)^{n/2}\|g\|_{\mathcal{H}^{p'}_{q'}}.$$

Proof. For every $h\in\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$,

$$|\langle(-\Delta)^{iu/2}g,h\rangle|=|\langle g,(-\Delta)^{-iu/2}h\rangle|\le\|g\|_{\mathcal{H}^{p'}_{q'}}\|(-\Delta)^{-iu/2}h\|_{\widetilde{\mathcal{M}}^p_q}\lesssim(1+|u|)^{n/2}\|g\|_{\mathcal{H}^{p'}_{q'}}\|h\|_{\widetilde{\mathcal{M}}^p_q}.$$

Since $(\widetilde{\mathcal{M}}^p_q)^*(\mathbb{R}^n)\simeq\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ [17], we get the desired result. □

We use Lemma 2.5 to give the following definition.

Definition 2.6. For every $f\in\mathcal{M}^p_q(\mathbb{R}^n)$, we define

$$\langle(-\Delta)^{iu/2}f,g\rangle=\langle f,(-\Delta)^{-iu/2}g\rangle\quad\text{for every } g\in\mathcal{H}^{p'}_{q'}(\mathbb{R}^n).$$

Proposition 2.7. *Let $u\in\mathbb{R}$ and $1<p\le q<\infty$. Then, for every $f\in\mathcal{M}^p_q(\mathbb{R}^n)$,*

$$\|(-\Delta)^{iu/2}f\|_{\mathcal{M}^p_q}\lesssim(1+|u|)^{n/2}\|f\|_{\mathcal{M}^p_q}.$$

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PROOF. For every $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$,

$$|\langle(-\Delta)^{iu/2} f, g\rangle| = |\langle f, (-\Delta)^{-iu/2} g\rangle| \le \|f\|_{\mathcal{M}^p_q} \|(-\Delta)^{-iu/2} g\|_{\mathcal{H}^{p'}_{q'}} \lesssim (1 + |u|)^{n/2} \|f\|_{\mathcal{M}^p_q} \|g\|_{\mathcal{H}^{p'}_{q'}}.$$

Since $(\mathcal{H}^{p'}_{q'})^*(\mathbb{R}^n) \simeq \mathcal{M}^p_q(\mathbb{R}^n)$, we get the desired result. □

As a corollary of Proposition 2.7, we obtain the following result for the fractional power of the Laplacian, which is analogous to the interpolation inequality in [1]. See also [4] for further results on interpolation of Morrey spaces.

THEOREM 2.8. *Let* $\alpha \ge 0$. *Then, for* $0 \le \theta \le 1$,

$$\|(-\Delta)^{\alpha\theta/2} f\|_{\mathcal{M}^p_q} \lesssim \|f\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}} \|(-\Delta)^{\alpha/2} f\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}}, \quad f \in C_c^\infty(\mathbb{R}^n), \tag{2.1}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

with $1 < p_0 \le q_0 < \infty$ *and* $1 < p_1 \le q_1 < \infty$.

We remark that [12, Theorem 1.1] is a special case of Theorem 2.8. To prove Theorem 2.8, we use the following observation, which is based on [5].

LEMMA 2.9. *Let* $1 \le w \le \infty$, $0 \le v \le 1$, $\alpha \ge 0$ *and let* B *be any ball in* $\mathbb{R}^n$. *Then, for every* $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|(-\Delta)^{\alpha v/2} f\|_{L^w(B)} \le C,$$

where the constant $C = C(n, \alpha, B, f)$ *is independent of* w *and* v.

PROOF. Let $N := \lfloor n + \alpha \rfloor + 1$. Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} |(-\Delta)^{\alpha v/2} f(x)| &\le \int_{\{|\xi|<1\}} |\xi|^{\alpha v} |\hat{f}(\xi)|\, d\xi + \int_{\{|\xi|\ge 1\}} |\xi|^{\alpha v} |\hat{f}(\xi)|\, d\xi \\ &\le \|\hat{f}\|_{L^\infty} |B(0,1)| + \|\mathcal{F}[(-\Delta)^N f]\|_{L^\infty} \int_{\{|\xi|\ge 1\}} |\xi|^{\alpha - 2N}\, d\xi. \end{aligned} \tag{2.2}$$

Let $E := \operatorname{supp}(f)$. Observe that

$$\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1} \le \|f\|_{L^\infty} |E| \tag{2.3}$$

and

$$\|\mathcal{F}[(-\Delta)^N f]\|_{L^\infty} \le \|(-\Delta)^N f\|_{L^1} \le \|(-\Delta)^N f\|_{L^\infty} |E|. \tag{2.4}$$

Combining (2.2)–(2.4) and $\int_{\{|\xi|\ge 1\}} |\xi|^{\alpha-2N}\, d\xi = O(1/(2N - \alpha - n))$ gives

$$\|(-\Delta)^{\alpha v/2} f\|_{L^\infty(B)} \le C_{n,\alpha,f},$$

where

$$C_{n,\alpha,f} := \Big(|B(0,1)| \|f\|_{L^\infty} + \frac{D}{2N - \alpha - n} \|(-\Delta)^N f\|_{L^\infty}\Big) |E|$$

with $D \gg 1$. Consequently, for $1 \le w < \infty$,

$$\|(-\Delta)^{\alpha v/2} f\|_{L^w(B)} \le C_{n,\alpha,f} |B|^{1/w} \le C_{n,\alpha,f} \max(1, |B|),$$

as desired. □

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PROOF OF THEOREM 2.8. Let $f \in C_c^\infty(\mathbb{R}^n)$. We prove (2.1) by showing that

$$\left(\int_B |(-\Delta)^{\alpha\theta/2} f(x)|^p \, dx\right)^{1/p} \lesssim |B|^{1/p-1/q} \|f\|_{\mathcal{M}^{p_0}_{q_0}}^{1-\theta} \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}}^{\theta} \tag{2.5}$$

for every fixed ball $B = B(a, r)$. Let p_0', p_1' and p' be defined by

$$\frac{1}{p_0'} := 1 - \frac{1}{p_0}, \quad \frac{1}{p_1'} := 1 - \frac{1}{p_1}, \quad \frac{1}{p'} := 1 - \frac{1}{p},$$

respectively. Define $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and let $\overline{S}$ be its closure. For every $z \in \overline{S}$ and $x \in \mathbb{R}^n$, we define

$$G(z, x) := \begin{cases} 0, & g(x) = 0, \\ \mathrm{sgn}(g(x))|g(x)|^{p'((1-z)/p_0'+z/p_1')}, & g(x) \neq 0, \end{cases}$$

where g is an arbitrary simple function with $\|g\|_{L^{p'}(B)} = 1$. We shall apply the three lines theorem to the function $F(z)$, defined by

$$F(z) := e^{z^2} \int_B (-\Delta)^{\alpha z/2} f(x) G(z, x) \, dx.$$

Note that F is continuous on $\overline{S}$ and holomorphic in S. Let $z = v + iu$, where $v \in [0, 1]$ and $u \in \mathbb{R}$. Define w by $1/w := 1 - (1 - v)/p_0' - v/p_1'$. Then

$$|F(v + iu)| \lesssim e^{-u^2}(1 + \alpha|u|)^{n/2} \|(-\Delta)^{\alpha v/2} f\|_{L^w(B)} \|G(v + iu, \cdot)\|_{L^{w'}(B)}. \tag{2.6}$$

Here, we have used the boundedness of $(-\Delta)^{i\alpha u/2}$ on $L^w(B)$ and the fact that

$$(-\Delta)^{\alpha z/2} = (-\Delta)^{i\alpha u/2}(-\Delta)^{\alpha v/2}.$$

Combining (2.6), Lemma 2.9 and

$$\|G(v + iu, \cdot)\|_{L^{w'}(B)} = \||g|^{p'((1-v)/p_0'+v/p_1')}\|_{L^{w'}(B)} = \|g\|_{L^{p'}(B)}^{p'/w'} = 1$$

yields $\sup_{z\in\overline{S}} |F(z)| < \infty$, that is, F is bounded on $\overline{S}$. Next, we observe that

$$\begin{aligned} |F(iu)| &\lesssim e^{-u^2} \|(-\Delta)^{i\alpha u/2} f\|_{\mathcal{M}^{p_0}_{q_0}} |B|^{1/p_0-1/q_0} \|G(iu, \cdot)\|_{L^{p_0'}(B)} \\ &\lesssim e^{-u^2}(1 + \alpha|u|)^{n/2} \|f\|_{\mathcal{M}^{p_0}_{q_0}} |B|^{1/p_0-1/q_0} \| |g|^{p'/p_0'} \|_{L^{p_0'}(B)} \\ &\lesssim \|f\|_{\mathcal{M}^{p_0}_{q_0}} |B|^{1/p_0-1/q_0} \end{aligned}$$

and similarly

$$|F(1 + iu)| \lesssim \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}} |B|^{1/p_1-1/q_1}.$$

It thus follows from the three lines theorem that

$$\begin{aligned} |F(\theta)| \leq \sup_{u\in\mathbb{R}} |F(\theta + iu)| &\leq \Big(\sup_{u\in\mathbb{R}} |F(iu)|\Big)^{1-\theta} \cdot \Big(\sup_{u\in\mathbb{R}} |F(1 + iu)|\Big)^{\theta} \\ &\lesssim \|f\|_{\mathcal{M}^{p_0}_{q_0}}^{1-\theta} \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}}^{\theta} |B|^{1/p-1/q} \end{aligned}$$

for $0 \leq \theta \leq 1$. Accordingly,

$$\left|\int_B (-\Delta)^{\alpha\theta/2} f(x) g(x) \, dx\right| = e^{-\theta^2} |F(\theta)| \lesssim \|f\|_{\mathcal{M}^{p_0}_{q_0}}^{1-\theta} \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}}^{\theta} |B|^{1/p-1/q}.$$

Since g is any simple function with $L^{p'}(B)$-norm 1, we conclude that (2.5) holds. □

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3. A Hardy-type inequality and a Heisenberg-type inequality

We shall now prove a Hardy-type inequality and Heisenberg's uncertainty inequality in Morrey spaces. According to [10],

$$\|W \cdot (-\Delta)^{-\alpha/2} f\|_{\mathcal{M}^p_q} \lesssim \|W\|_{\mathcal{M}^u_v} \|f\|_{\mathcal{M}^p_q} \quad \text{for } f \in \mathcal{M}^p_q(\mathbb{R}^n), \tag{3.1}$$

where $0 < \alpha < n$, $1 < p \le q < n/\alpha$, $u = np/\alpha q$ and $v = n/\alpha$. This inequality goes back to Olsen [8], so we call it Olsen's inequality. Note that the inequality follows from Hölder's inequality and the boundedness of the fractional integral operator $I_\alpha := (-\Delta)^{-\alpha/2}$ from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ for $0 < \alpha < n$, $1 < p \le q < n/\alpha$, $1/s = 1/p - \alpha q/np$ and $s/t = p/q$ (see also [3]). Note that through its Fourier transform, one may recognise $(-\Delta)^{-\alpha/2}$ as the convolution operator whose kernel is a multiple of $|\cdot|^{\alpha-n}$, which is initially defined on $C^\infty_c(\mathbb{R}^n)$ (see [14]).

The next proposition is a consequence of the inequality (3.1).

Proposition 3.1. *Let $1 < p \le q < \infty$ and $0 < \alpha < n/q$. Then, for every $f \in \mathcal{M}^p_q(\mathbb{R}^n)$,*

$$\||\cdot|^{-\alpha} g\|_{\mathcal{M}^p_q} \lesssim \|(-\Delta)^{\alpha/2} g\|_{\mathcal{M}^p_q}. \tag{3.2}$$

Remark 3.2. The inequality (3.2) may be viewed as a Hardy-type inequality in Morrey spaces.

To prove the proposition, we need some lemmas.

Lemma 3.3. *Let $0 < \alpha < n$. If $g \in C^\infty_c(\mathbb{R}^n)$, then*

$$|(-\Delta)^{\alpha/2} g(x)| \lesssim \min(1, |x|^{-\alpha-n}).$$

In particular, $f = (-\Delta)^{\alpha/2} g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. We have already seen that $|(-\Delta)^{\alpha/2} g(x)| \lesssim 1$ in the proof of Lemma 2.9. Now let $\psi \in C^\infty_c(\mathbb{R}^n)$ be such that $\chi_{B(1)} \le \psi \le \chi_{B(2)}$, where $B(r)$ denotes the ball of radius r centred at the origin. Define $\varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)$. We decompose

$$(-\Delta)^{\alpha/2} g(x) = \mathcal{F}^{-1}[|\cdot|^\alpha (1-\psi)\mathcal{F} g](x) + \sum_{j=-\infty}^{0} \mathcal{F}^{-1}[|\cdot|^\alpha \varphi_j \mathcal{F} g](x).$$

Since $h = \mathcal{F}^{-1}[|\cdot|^\alpha (1-\psi)\mathcal{F} g]$ belongs to $\mathcal{S}(\mathbb{R}^n)$, we only need to handle the second term. Using a crude estimate, $\mathcal{F} g \in L^\infty(\mathbb{R}^n)$ and so

$$|\mathcal{F}^{-1}[|\cdot|^\alpha \varphi_j \mathcal{F} g](x)| \lesssim 2^{j\alpha} \||2^{-j} \cdot|^\alpha \varphi_j \mathcal{F} g\|_{L^1} \sim 2^{j(\alpha+n)}.$$

Let $N \in \mathbb{N}$ be sufficiently large. Then, as before,

$$\begin{aligned} |x|^{2N} |\mathcal{F}^{-1}[|\cdot|^\alpha \varphi_j \mathcal{F} g](x)| &= |\mathcal{F}^{-1}[\Delta^N [|\cdot|^\alpha \varphi_j \mathcal{F} g]](x)| \\ &\lesssim \sum_{\beta \in (\mathbb{N}\cup\{0\})^n,\, |\beta| = 2N} \|\partial^\beta [|\cdot|^\alpha \varphi_j \mathcal{F} g]\|_{L^1}. \end{aligned}$$

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Here and below let β be such that $|\beta| = 2N$. Then

$$|\partial^\beta[|\xi|^\alpha \varphi_j(\xi)\mathcal{F}g(\xi)]| \lesssim \sum_{\beta_1+\beta_2+\beta_3=\beta} |\partial^{\beta_1}[|\xi|^\alpha]|\,|\partial^{\beta_2}\varphi_j(\xi)|\,|\partial^{\beta_3}\mathcal{F}g(\xi)|.$$

Since $\varphi_j(\xi)$ vanishes outside $\{2^{j-2} \le |\xi| \le 2^{j+2}\}$,

$$\partial^{\beta_1}[|\xi|^\alpha] = \mathrm{O}(|\xi|^{\alpha-|\beta_1|}), \quad \partial^{\beta_2}\varphi_j(\xi) = \mathrm{O}(|\xi|^{-|\beta_2|}), \quad |\partial^{\beta_3}\mathcal{F}g(\xi)| \lesssim 1 \lesssim 2^{-j|\beta_3|}$$

as $\xi \to 0$. Thus,

$$\begin{aligned}|\partial^\beta[|\xi|^\alpha \varphi_j(\xi)\mathcal{F}g(\xi)]| &\lesssim \sum_{\beta_1+\beta_2+\beta_3=\beta} |\xi|^{\alpha-|\beta_1|}|\xi|^{-|\beta_2|}2^{-j|\beta_3|}\chi_{\{2^{j-2}\le|\xi|\le 2^{j+2}\}}(\xi)\\ &\lesssim 2^{j(\alpha-2N)}\chi_{\{|\xi|\le 2^{j+2}\}}(\xi)\end{aligned}$$

and hence

$$\|\partial^\beta[|\cdot|^\alpha \varphi_j \mathcal{F}g]\|_{L^1} = \mathrm{O}(2^{j(\alpha+n-2N)})$$

as $j \to -\infty$. As a result,

$$\begin{aligned}|(-\Delta)^{\alpha/2}g(x)| &\lesssim |x|^{-\alpha-n} + \sum_{j=-\infty}^{0} \min(|x|^{-2N}2^{j(\alpha+n-2N)}, 2^{j(\alpha+n)})\\ &\le |x|^{-\alpha-n} + |x|^{-\alpha-n}\sum_{j=-\infty}^{\infty} \min(|x|^{\alpha+n-2N}2^{j(\alpha+n-2N)}, |x|^{\alpha+n}2^{j(\alpha+n)}).\end{aligned}$$

Now

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} \min(|x|^{\alpha+n-2N}2^{j(\alpha+n-2N)}, |x|^{\alpha+n}2^{j(\alpha+n)})\\ &\quad\le \sum_{j=-\infty,\,2^j|x|\le 1}^{\infty} (2^j|x|)^{\alpha+n} + \sum_{j=-\infty,\,2^j|x|>1}^{\infty} (2^j|x|)^{\alpha+n-N}\\ &\quad\lesssim \sum_{j=-\infty,\,2^j|x|\le 1}^{\infty} \int_{2^j|x|}^{2^{j+1}|x|} t^{\alpha+n-1}\,dt + \sum_{j=-\infty,\,2^j|x|>1}^{\infty} \int_{2^{j-1}|x|}^{2^j|x|} t^{\alpha+n-N-1}\,dt\\ &\quad\le \int_0^2 t^{\alpha+n-1}\,dt + \int_{1/2}^{\infty} t^{\alpha+n-N-1}\,dt \lesssim 1\end{aligned}$$

and we conclude that $|(-\Delta)^{\alpha/2}g(x)| \lesssim |x|^{-\alpha-n}$, as desired. □

Lemma 3.4. *Let $1 \le p \le q < \infty$ and $0 < \alpha < n$. For $g \in C_c^\infty(\mathbb{R}^n)$, define $f := (-\Delta)^{\alpha/2}g$. Then $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ and $(-\Delta)^{-\alpha/2}f = g$ pointwise.*

Proof. We have proved that $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Consequently,

$$\|f\|_{\mathcal{M}_q^p} \le \|f\|_{L^q} \le \|f\|_{L^\infty}^{1-1/q}\|f\|_{L^1}^{1/q} < \infty.$$

(This justifies the right-hand side of (3.2).)

Next, $|\cdot|^\alpha\widehat{g} \in L^1(\mathbb{R}^n)$ and $f = \mathcal{F}^{-1}(|\cdot|^\alpha\widehat{g}) \in L^1(\mathbb{R}^n)$. Hence, $\widehat{f} = |\cdot|^\alpha\widehat{g}$ pointwise and so $|\cdot|^{-\alpha}\widehat{f} = \widehat{g}$ pointwise. Thus, $(-\Delta)^{-\alpha/2}f = g$ pointwise. □

Now we come to the proof of Proposition 3.1.

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PROOF OF PROPOSITION 3.1. Denote $u = np/\alpha q$ and $v = n/\alpha$. For $1 < p < q < \infty$ and $0 < \alpha < n/q$, it follows that $u < v$. By computing its Morrey norm directly, we see that $W(\cdot) := |\cdot|^{-\alpha} \in \mathcal{M}^u_v(\mathbb{R}^n)$. Hence, for $g \in C^\infty_c(\mathbb{R}^n)$, we take $f := (-\Delta)^{\alpha/2} g$, which is a function in $\mathcal{M}^p_q(\mathbb{R}^n)$ by Lemma 3.4. Moreover, $g = (-\Delta)^{-\alpha/2} f \in \mathcal{M}^s_t(\mathbb{R}^n)$, where $1/s = 1/p - \alpha q/np$ and $s/t = p/q$, so that Olsen's inequality (3.1) gives

$$\| |\cdot|^{-\alpha} g\|_{\mathcal{M}^p_q} \lesssim \|W\|_{\mathcal{M}^u_v} \|(-\Delta)^{\alpha/2} g\|_{\mathcal{M}^p_q}.$$

For $1 \le p = q < n/\alpha$, we use the fact that $f \in L^q(\mathbb{R}^n)$ and $g = (-\Delta)^{-\alpha/2} f \in wL^t(\mathbb{R}^n)$ for $1/t = 1/q - \alpha/n$ with $\|(-\Delta)^{-\alpha/2} f\|_{wL^t} \lesssim \|f\|_{L^q}$ (where $wL^t(\mathbb{R}^n)$ denotes the weak Lebesgue space of exponent t). From [7, Proposition 4.1],

$$\| |\cdot|^{-\alpha} g\|_{wL^q} = \|W(-\Delta)^{-\alpha/2} f\|_{wL^q} \lesssim \|W\|_{wL^v} \|(-\Delta)^{-\alpha/2} f\|_{wL^t} \lesssim \|W\|_{wL^v} \|f\|_{L^q}$$

(where $v = n/\alpha$). This inequality holds for every q with $1 \le q < n/\alpha$. By the Marcinkiewicz interpolation theorem,

$$\| |\cdot|^{-\alpha} g\|_{L^q} \lesssim \|W\|_{wL^v} \|f\|_{L^q} = \|W\|_{wL^v} \|(-\Delta)^{\alpha/2} g\|_{L^q}$$

for $1 < q < n/\alpha$. This completes the proof. □

As a corollary of Proposition 3.1, we obtain the following result (which is analogous to [1, Corollary 5.2]).

THEOREM 3.5. *Suppose that* $1 < p \le q < \infty$, $1 \le p_2 \le q_2 < \infty$, $\beta > 0$ *and* $0 < \gamma < n/q$. *If* $(\beta + \gamma)/p_0 = \beta/p + \gamma/p_2$ *and* $(\beta + \gamma)/q_0 = \beta/q + \gamma/q_2$, *then*

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \| |\cdot|^\beta g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\gamma/2} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^p_q}$$

for every $g \in C^\infty_c(\mathbb{R}^n)$.

PROOF. Write $g(x) = [|x|^\beta g(x)]^{\gamma/(\beta+\gamma)} [|x|^{-\gamma} g(x)]^{\beta/(\beta+\gamma)}$. By Hölder's inequality and Proposition 3.1,

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \le \| |\cdot|^\beta g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \| |\cdot|^{-\gamma} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^p_q} \lesssim \| |\cdot|^\beta g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\gamma/2} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^p_q},$$

as desired. □

Finally, we use our estimate for the fractional power of the Laplacian in Theorem 2.8 to prove the following Heisenberg uncertainty inequality (which is analogous to [1, Theorem 5.4]).

THEOREM 3.6. *Suppose that* $1 < p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$ *and* $\beta, \delta > 0$. *If the conditions* $(\beta + \delta)/p_0 = \beta/p_1 + \delta/p_2$ *and* $(\beta + \delta)/q_0 = \beta/q_1 + \delta/q_2$ *hold, then*

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \| |\cdot|^\beta g\|^{\delta/(\beta+\delta)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\delta/2} g\|^{\beta/(\beta+\delta)}_{\mathcal{M}^{p_1}_{q_1}}$$

for every $g \in C^\infty_c(\mathbb{R}^n)$.

PROOF. The idea of the proof is the same as in [1]. If $\delta < n/q_1$, we do not have to do anything because the inequality is the same as in Theorem 3.5. Otherwise, we set $\gamma = \delta\theta$ and apply the interpolation inequality

$$\|(-\Delta)^{\delta\theta/2} g\|_{\mathcal{M}^p_q} \lesssim \|g\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}} \|(-\Delta)^{\delta/2} g\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}}$$

for $0 < \theta < n/\delta q_1$, so that the inequality in Theorem 3.5 becomes

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \| |\cdot|^\beta g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\delta/2} g\|^{\beta\theta/(\beta+\gamma)}_{\mathcal{M}^{p_1}_{q_1}} \|g\|^{\beta(1-\theta)/(\beta+\gamma)}_{\mathcal{M}^{p_0}_{q_0}}.$$

Rearranging the expression gives the desired inequality. □

REMARK 3.7. Note that the value of δ in the above proposition can be as large as possible. This is the benefit we obtain from the interpolation inequality for the fractional power of the Laplacian.

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