BULL. AUSTRAL. MATH. SOC. 60A05
VOL. 61 (2000) [263–266]

VERIFYING THE INDEPENDENCE OF PARTITIONS OF A PROBABILITY SPACE

S.B. MULAY AND C.G. WAGNER

Let $\{E_1, \dots, E_r\}$ and $\{F_1, \dots, F_s\}$ be partitions of a probability space. We exhibit a natural bijection from the set of efficient ways of verifying the independence of such partitions to the set of spanning trees of the complete bipartite graph $K_{r,s}$.

1. INTRODUCTION

In what follows, (Ω, Σ, p) is a probability space and $[n] := \{1, 2, \dots, n\}$. Partitions $\{E_i : i \in [r]\}$ and $\{F_j : j \in [s]\}$ of Ω, with $E_i, F_j \in \Sigma$, are said to be *independent* (with respect to p) if

$$p(E_i \cap F_j) = p(E_i)p(F_j) \tag{1}$$

for all $(i, j) \in [r] \times [s]$. Of course, one need not check all rs instances of (1) in order to verify independence. It is easy to see, for example, that if (1) holds for all $(i, j) \in [r-1] \times [s-1]$, then the partitions in question are independent. Let us call a subset $\mathcal{N}$ of $[r] \times [s]$ *negligible* when, if (1) holds for all $(i, j) \in \mathcal{N}^c$, then it holds for all $(i, j) \in \mathcal{N}$ as well. We show in this note that there is a natural bijection from the family of maximal negligible subsets of $[r] \times [s]$ to the family of spanning trees of the complete bipartite graph $K_{r,s}$. It follows that there are $r^{s-1}s^{r-1}$ efficient ways to verify the independence of the aforementioned partitions.

2. NEGLIGIBILITY AND LINEAR INDEPENDENCE

For all $i \in [r]$, let X_i be the $(r+s)$-dimensional unit column vector with a one in the ith position and zeros elsewhere, and for all $j \in [s]$, let Y_j be the $(r+s)$-dimensional unit column vector with a one in the $(r+j)$th position and zeros elsewhere. Then

$$\sum_{(i,j)\in[r]\times[s]} p(E_i \cap F_j)(X_i + Y_j) = \sum_{i\in[r]} p(E_i)X_i + \sum_{j\in[s]} p(F_j)Y_j, \tag{2}$$

Received 31st May, 1999
Research supported by the National Science Foundation (SBR-9528893).

as a consequence of the familiar formulas for the marginal probabilities $p(E_i)$ and $p(F_j)$.

Suppose that $\mathcal{N} \subset [r] \times [s]$, and that $p(E_i \cap F_j) = p(E_i)p(F_j)$ for all $(i,j) \in \mathcal{N}^c$. Then (2) becomes

$$(3) \qquad \sum_{(i,j)\in\mathcal{N}} p(E_i \cap F_j)(X_i + Y_j) = \sum_{i\in[r]} p(E_i)X_i + \sum_{j\in[s]} p(F_j)Y_j - \sum_{(i,j)\in\mathcal{N}^c} p(E_i)p(F_j)(X_i + Y_j).$$

Regard the quantities $p(E_i \cap F_j)$, where $(i,j) \in \mathcal{N}$, as unknowns. It is clear that $p(E_i \cap F_j) = p(E_i)p(F_j)$ for all $(i,j) \in \mathcal{N}$ furnishes a solution of (3). $\mathcal{N}$ is negligible if and only if this is the only solution of (3), and the latter condition clearly obtains if and and only if $\{X_i + Y_j : (i,j) \in \mathcal{N}\}$ is a linearly independent subset of V, the subspace of $\mathbb{R}^{r+s}$ spanned by $\{X_i + Y_j : (i,j) \in [r] \times [s]\}$. Consequently, $\mathcal{N}$ is a maximal negligible subset of $[r] \times [s]$ if and only if $\{X_i + Y_j : (i,j) \in \mathcal{N}\}$ is a basis of V.

It is easy to see that the dimension of V is $r+s-1$. In particular, the set of column vectors $\{X_1+Y_s, X_2+Y_s, \dots, X_{r-1}+Y_s, X_r+Y_1, X_r+Y_2, \dots, X_r+Y_s\}$ is a basis of V. This set spans V since $X_i+Y_j = (X_i+Y_s)+(X_r+Y_j)-(X_r+Y_s)$. It is linearly independent as a simple consequence of the linear independence of $\{X_1, \dots, X_r, Y_1, \dots, Y_s\}$. In the next section we present a graphical characterisation of bases of V consisting of vectors of the form $X_i + Y_j$.

3. A Natural Bijection

We assume in this section familiarity with the basic terminology and elementary results of graph theory, as described, for example, in [**3**]. In particular, we use the fact that if all vertices of a graph have degree at least two, then the graph contains a cycle [**3,** Lemma 1.2.18], and the fact that a graph with n vertices is a tree if and only if it is acyclic and has $n-1$ edges [**3,** Theorem 2.13].

Consider the complete bipartite graph $K_{r,s}$ with vertex set $\mathcal{V} = \{X_1, \dots, X_r, Y_1, \dots, Y_s\}$ and edge set $\mathcal{E} = \{\{X_i, Y_j\} : (i,j) \in [r] \times [s]\}$. To each $\mathcal{S} \subset [r] \times [s]$ we associate the subgraph of $K_{r,s}$ having vertex set $\mathcal{V}(\mathcal{S}) = \bigcup_{(i,j)\in\mathcal{S}} \{X_i, Y_j\}$ and edge set $\mathcal{E}(\mathcal{S}) = \{\{X_i, Y_j\}; (i,j) \in \mathcal{S}\}$. The map $\mathcal{S} \mapsto (\mathcal{V}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$ is clearly an injection from $2^{[r]\times[s]}$ into the set of all subgraphs of $K_{r,s}$.

Lemma. *The set of column vectors $S(\mathcal{S}) = \{X_i + Y_j : (i,j) \in \mathcal{S}\}$ is linearly dependent if and only if $(\mathcal{V}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$ contains a cycle.*

Proof: Sufficiency. Suppose, with no loss of generality, that $(\mathcal{V}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$ contains the cycle $X_{i_1}, Y_{i_1}, X_{i_2}, Y_{i_2}, \dots, X_{i_n}, Y_{i_n}, X_{i_1}$. Since then $X_{i_1}+Y_{i_1}, X_{i_2}+Y_{i_1}, X_{i_2}+$

https://doi.org/10.1017/S0004972700022255 Published online by Cambridge University Press

$Y_{i_2}, \ldots, X_{i_n} + Y_{i_n}$ and $X_{i_1} + Y_{i_n} \in S(\mathcal{S})$ and $(X_{i_1} + Y_{i_1}) - (X_{i_2} + Y_{i_1}) + (X_{i_2} + Y_{i_2}) - \cdots + (X_{i_n} + Y_{i_n}) - (X_{i_1} + Y_{i_n}) = 0$, it follows that $S(\mathcal{S})$ is linearly dependent.

Necessity. Suppose that $S(\mathcal{S})$ is linearly dependent. Then there exists a nonempty subset $\mathcal{S}^+$ of $\mathcal{S}$ and, for each $(i,j) \in \mathcal{S}^+$, a nonzero real number α_{ij} such that

$$\sum_{(i,j)\in\mathcal{S}^+} \alpha_{ij}(X_i + Y_j) = 0. \tag{4}$$

Consider the graph $(\mathcal{V}(\mathcal{S}^+), \mathcal{E}(\mathcal{S}^+))$. Clearly, every vertex in $\mathcal{V}(\mathcal{S}^+)$ has degree at least one. Suppose some vertex, say X_{i^*}, has degree one, belonging only to the edge $\{X_{i^*}, Y_{j^*}\}$. Then X_i^* occurs just once in (4), with the nonzero coefficient $\alpha_{i^*j^*}$. This implies that X_{i^*} is a linear combination of $\{X_1, \ldots, X_r, Y_1, \ldots, Y_s\}\backslash\{X_{i^*}\}$, contradicting the linear independence of $\{X_1, \ldots, X_r, Y_1, \ldots, Y_s\}$. Hence every vertex in $\mathcal{V}(\mathcal{S}^+)$ has degree at least two, and so $(\mathcal{V}(\mathcal{S}^+), \mathcal{E}(\mathcal{S}^+))$, and thus $(\mathcal{V}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$, contains a cycle. ∎

Students of matroid theory will not be surprised by the above lemma. Indeed, it establishes a special case of a much more general result, namely the fact that the cycle matroid of every graph has a vectorial representation [**2**, Section 9.5]. We may now establish the main result of this note.

THEOREM. *The map* $\mathcal{N} \mapsto (\mathcal{V}(\mathcal{N}), \mathcal{E}(\mathcal{N}))$ *is a bijection from the family of all maximal negligible subsets of* $[r] \times [s]$ *to the set of all spanning trees of* $K_{r,s}$.

PROOF: If $\mathcal{N}$ is a maximal negligible subset of $[r]\times[s]$, then, as shown in Section 2 above, $S(\mathcal{N}) = \{X_i + Y_j : (i,j) \in \mathcal{N}\}$ is a basis of V, the $(r+s-1)$-dimensional subspace of $\mathbb{R}^{r+s}$ spanned by $\{X_i + Y_j : (i,j) \in [r] \times [s]\}$. By the preceding lemma, $(\mathcal{V}(\mathcal{N}), \mathcal{E}(\mathcal{N}))$ is acyclic. Clearly, $|\mathcal{E}(\mathcal{N})| = |S(\mathcal{N})| = r + s - 1$. Also, $\mathcal{V}(\mathcal{N}) = \{X_1, \ldots, X_r, Y_1, \ldots, Y_s\}$, for if not, $S(\mathcal{N})$ would not span V. Hence, $|\mathcal{V}(\mathcal{N})| = r + s$. It follows that $(\mathcal{V}(\mathcal{N}), \mathcal{E}(\mathcal{N}))$ is a tree with the same vertex set as $K_{r,s}$, and edge set contained in the edge set of $K_{r,s}$, that is, a spanning tree of $K_{r,s}$.

As a restriction of the injective map $\mathcal{S} \mapsto (\mathcal{V}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$, the map $\mathcal{N} \mapsto (\mathcal{V}(\mathcal{N}), \mathcal{E}(\mathcal{N}))$ is injective. It remains only to show that this map is surjective. Let $(\mathcal{V}, \mathcal{E})$ be a spanning tree of $K_{r,s}$, so that $\mathcal{V} = \{X_1, \ldots, X_r, Y_1, \ldots, Y_s\}$ and $\mathcal{E} \subset \{\{X_i, Y_j\} : (i,j) \in [r] \times [s]\}$. Then $|\mathcal{E}| = r + s - 1$. Suppose that $\mathcal{E} = \{\{X_i, Y_j\} : (i,j) \in \mathcal{N}\}$ where $\mathcal{N} \subset [r] \times [s]$. Clearly, $\mathcal{V}(\mathcal{N}) = \mathcal{V}$ and $\mathcal{E}(\mathcal{N}) = \mathcal{E}$. By the lemma, $S(\mathcal{N}) = \{X_i + Y_j : (i,j) \in \mathcal{N}\}$ is linearly independent since $(\mathcal{V}(\mathcal{N}), \mathcal{E}(\mathcal{N}))$ is acyclic. Since $|S(\mathcal{N})| = |\mathcal{N}| = |\mathcal{E}| = r + s - 1$, $S(\mathcal{N})$ is a basis of V. Hence by the results of Section 2 above, $\mathcal{N}$ is a maximal negligible subset of $[r] \times [s]$, which completes the proof of surjectivity. ∎

Since the complete bipartite graph $K_{r,s}$ has $r^{s-1}s^{r-1}$ spanning trees [**1**], it follows that there are $r^{s-1}s^{r-1}$ efficient ways to verify the independence of partitions

$\{E_1, \dots, E_r\}$ and $\{F_1, \dots, F_s\}$.

REMARK 1. The foregoing analysis could have be carried out with $\{X_1, \dots, X_r, Y_1, \dots, Y_s\}$ being any set of distinct indeterminates over $\mathbb{R}$.

REMARK 2. Our characterisation in Section 2 above of the efficient ways of verifying the independence of two partitions of a probability space may be generalised to the case of three or more partitions. In the case of partitions $\{E_1, \dots, E_r\}$, $\{F_1, \dots, F_s\}$, and $\{G_1, \dots, G_t\}$, for example, maximal negligible subsets of $[r] \times [s] \times [t]$ correspond to bases of the vector space generated by $\{X_i + Y_j + Z_k : (i,j,k) \in [r] \times [s] \times [t]\}$ comprised of vectors of the form $X_i + Y_j + Z_k$. The problem of enumerating bases of this type has, as far as we know, not been solved. The vectors comprising such bases correspond in a natural way to edges of a hypergraph, but it is not clear what sorts of hypergraphs arise in this way, or whether they facilitate the enumeration in question.

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Department of Mathematics
The University of Tennessee
Knoxville, TN 37996-1300
United States of America