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RESEARCH ARTICLE

Partition regularity of Pythagorean pairs

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Abstract
We address a core partition regularity problem in Ramsey theory by proving that every finite coloring of the positive integers contains monochromatic Pythagorean pairs (i.e., $x, y \in \mathbb{N}$ such that $x^2 \pm y^2 = z^2$ for some $z \in \mathbb{N}$). We also show that partitions generated by level sets of multiplicative functions taking finitely many values always contain Pythagorean triples. Our proofs combine known Gowers uniformity properties of aperiodic multiplicative functions with a novel and rather flexible approach based on concentration estimates of multiplicative functions.

Contents

1. Introduction and main results

1.1. Introduction

A fundamental problem in Ramsey theory is to determine which patterns must appear in a single cell for every partition of $\mathbb{N} = \{1, 2, \ldots\}$ into finitely many cells. A famous example is provided by an early theorem of Schur [44], which states that every finite partition of $\mathbb{N}$ has a solution to the equation $x + y = z$ where all variables x, y, z belong to the same cell. Equations (and systems of equations) that satisfy this property are called *partition regular*.

In 1933, Rado significantly extended Schur's theorem by characterizing all systems of linear equations that are partition regular [41]. Polynomial equations, however, have proven to be much more difficult to tackle. In particular, the following notorious problem of Erdős and Graham [23, 24] remains unsolved.

Problem. *Determine whether the equation $x^2 + y^2 = z^2$ is partition regular.*

Integer solutions to the equation $x^2 + y^2 = z^2$ are known as *Pythagorean triples*, so the problem is colloquially referred to as the partition regularity problem for Pythagorean triples. Graham in [23] places the origin of the problem in the late 70's and offered \$250 for its solution, noting that 'There is actually very little data (in either direction) to know which way to guess'. While this was perhaps true a decade ago, in the last few years there have been some positive developments. The case where one allows only partitions of $\mathbb{N}$ into two sets was verified in 2016 with the help of a computer search [31]; this endeavor was hailed as the 'longest mathematical proof' at the time, occupying 200 terabytes of data [36].

Pioneering work in nonlinear partition regularity goes back to the famous theorems of Furstenberg [22] and Sárközy [42], culminating in the influential polynomial Szemerédi theorem of Bergelson and Leibman [6]. While these results apply only to shift-invariant configurations, there are now also several non-shift invariant configurations that are known to be or not to be partition regular. Bergelson showed in [3] that the equation $x^2 + y = z$ is partition regular, and the equation $x^2 + y = z^2$ was shown to be partition regular by the third author in [39]. However, the equation $x + y = z^2$ was shown not to be partition regular by Csikvári, Gyarmati and Sárközy in [12] (however, it is partition regular if we restrict to 2-colorings [28, 40]). Resolving an old conjecture, Khalfalah and Szemerédi [33] showed that the equation $x + y = z^2$ is partition regular if we only require x, y to be of the same color and allow any $z \in \mathbb{N}$. Other partition regularity results of similar flavor can be found in [1, 2, 5, 14, 15, 16, 37]. Lastly, we remark that in the case of more variables, a result by Chow, Lindqvist and Prendiville [11] establishes that the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$ is partition regular (see also [8, 10] for related results).

Despite these developments, even the question of whether in any finite partition of $\mathbb{N}$ there is a Pythagorean triple with two terms in the same cell was still open. We will say informally that $(x, y) \in \mathbb{N}^2$ is a *Pythagorean pair* if there exists $z \in \mathbb{N}$ such that either

$$x^2 + y^2 = z^2 \quad \text{or} \quad x^2 + z^2 = y^2.$$

An attempt to address the question of whether Pythagorean pairs are partition regular was made by the first author and Host in [21], where an approach using Gowers uniformity properties and related decomposition results of multiplicative functions was proposed. This approach covered pairs (x, y) satisfying, say, the equations $16x^2 + 9y^2 = z^2$ or $x^2 + y^2 - xy = z^2$, but missed the case of Pythagorean pairs for reasons that we will explain later on. Extending these ideas, Sun in [45, 46] established partition regularity in (x, y) for the equation $x^2 - y^2 = z^2$, when $\mathbb{N}$ is replaced by the ring of integers of a larger number field, such as the Gaussian integers. However, the methods used there do not apply to $\mathbb{N}$.

In the present paper, we develop a general approach to partition regularity questions of pairs, by combining the method of [21] together with a new input related to concentration estimates of multiplicative functions. As a consequence, we show (among other things) that Pythagorean pairs (and related pairs) are partition regular (see Theorem 1.1) and density regular (see Theorem 1.2). We also show that partitions generated by level sets of multiplicative functions taking finitely many values always contain Pythagorean triples (see Theorem 1.5). The exact statements are given in the following subsections, and our proof strategy and comparison with the previous approach in [21] is described in Section 2.

1.2. Partition and density regularity of Pythagorean pairs

Our first goal is to prove partition regularity and density regularity results for Pythagorean pairs, a case covered by taking $a = b = c = 1$ in the next two results. Our results also answer the first part of Question 3 from [21] and Problem 34 from [18].

Theorem 1.1. *Let $a, b, c \in \mathbb{N}$ be squares. Then for every finite coloring of $\mathbb{N}$, there exist*

1. *distinct $x, y \in \mathbb{N}$ with the same color and $z \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$.*
2. *distinct $y, z \in \mathbb{N}$ with the same color and $x \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$.*

Remarks. ∘ In [21, Corollary 2.8], part (1) was covered under the additional restriction that $a + b$ is also a square, thus missing the case of Pythagorean pairs.

∘ In fact, Theorem 1.2 implies that all four elements x, y and y', z' in part (1) and (2), respectively, can be taken to be of the same color.

∘ We can also extend [21, Theorem 2.7], covering more general homogeneous equations of the form $p(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$, where $a, b, c, d, e, f \in \mathbb{Z}$. Our method allows to show that if $e^2 - 4ac$ and $f^2 - 4bc$ are nonzero squares, then for every finite coloring of the integers, there exist distinct monochromatic x, y and an integer z such that $p(x, y, z) = 0$.[1] In contrast, [21, Theorem 2.7] assumes in addition that $(e + f)^2 - 4c(a + b + d)$ is a nonzero square.

∘ The assumption that $a, b, c \in \mathbb{N}$ are all squares is not sufficient for partition regularity of the equation $ax^2 + by^2 = cz^2$. For example, the equation $x^2 + y^2 = 4z^2$ is not partition regular, so in this case, our result is optimal, as only pairs and not triples can be partition regular. See Section 1.6 for more details and conjectural necessary and sufficient conditions for partition regularity of such equations.

[1] Arguing as in Step 2 of [21, Appendix C], we get parametrizations for x, y of the form covered in Section 1.5.

We establish a stronger density version of these partition regularity results. It is clear that the set of odd numbers, which has additive density $1/2$, does not contain integers x, y such that $x^2 + y^2 = z^2$ for some $z \in \mathbb{N}$, ruling out a potential density version using additive density. However, since the equation $x^2 + y^2 = z^2$ is homogeneous, the set of solutions is invariant under dilations, and using a dilation-invariant notion of density turns out to be more fruitful.

To this end, we recall some standard notions. A *multiplicative Følner sequence in* $\mathbb{N}$ is a sequence $\Phi = (\Phi_K)_{K=1}^{\infty}$ of finite subsets of $\mathbb{N}$ asymptotically invariant under dilation, in the sense that

$$\forall x \in \mathbb{N}, \qquad \lim_{K\to\infty} \frac{\left|\Phi_K \cap (x \cdot \Phi_K)\right|}{|\Phi_K|} = 1.$$

An example of a multiplicative Følner sequence is given by (2.11). The *upper multiplicative density* of a set $\Lambda \subset \mathbb{N}$ with respect to a multiplicative Følner sequence $\Phi = (\Phi_K)_{K=1}^{\infty}$ is the quantity

$$\bar{d}_{\Phi}(\Lambda) := \limsup_{K\to\infty} \frac{\left|\Phi_K \cap \Lambda\right|}{|\Phi_K|},$$

and we write $d_{\Phi}(\Lambda)$ if the previous limit exists. We say that $\Lambda \subset \mathbb{N}$ has *positive multiplicative density* (or, more precisely, positive upper Banach density with respect to multiplication) if $\bar{d}_{\Phi}(\Lambda) > 0$ for some multiplicative Følner sequence Φ. A finite coloring of $\mathbb{N}$ always contains a monochromatic cell with positive multiplicative density; thus, the next result strengthens Theorem 1.1.

Theorem 1.2. *Let $a, b, c \in \mathbb{N}$ be squares. Then for every $\Lambda \subset \mathbb{N}$ with positive multiplicative density, there exist*

1. *distinct $x, y \in \Lambda$ and $z \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$.*
2. *distinct $y, z \in \Lambda$ and $x \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$.*

Remarks. ∘ In fact, we prove the following stronger property: If $\bar{d}_{\Phi}(\Lambda) > 0$, then there exist a subsequence Ψ of Φ and distinct $x, y \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$ for some $z \in \mathbb{N}$, and

$$d_{\Psi}\big((x^{-1}\Lambda) \cap (y^{-1}\Lambda)\big) > 0.$$

A similar statement also holds with the roles of x and z reversed.

∘ If $a + b \neq c$, it is not true that every $\Lambda \subset \mathbb{N}$ with positive multiplicative density contains x, y, z such that $ax^2 + by^2 = cz^2$. To see this when $a = b = c = 1$ (the argument is similar whenever $a + b \neq c$), let Φ be any multiplicative Følner sequence and α be an irrational such that the sequence $(n^2\alpha)$ is equidistributed (mod 1) with respect to a subsequence Φ' of Φ (such an α and Φ' exist by the ergodicity of the multiplicative action $T_n x = n^2 x$, $n \in \mathbb{N}$, defined on $\mathbb{T}$ with its Haar measure). Let $\Lambda := \{n \in \mathbb{N} \colon \{n^2\alpha\} \in [1/5, 2/5)\}$, which has positive upper density with respect to Φ'. If $x, y, z \in \Lambda$, then $\{(x^2 + y^2)\alpha\} \in [2/5, 4/5)$ and $\{z^2\alpha\} \in [1/5, 2/5)$; hence, we cannot have $x^2 + y^2 = z^2$. This example was shown to us by V. Bergelson.

We remark that the previous results also resolve the first part of Problem 3 in [21] and also Problem 6 in [21]. The latter implies that the starting point in Sárközy's theorem [43] (or the variant in [33] dealing with the equation $x + y = n^2$) can be taken to be a square, as the following result shows.

Corollary 1.3. *For every finite coloring of $\mathbb{N}$, there exist*

1. *distinct $m, n \in \mathbb{N}$ such that the integers m^2 and $m^2 + n^2$ have the same color.*
2. *distinct $m, n \in \mathbb{N}$ such that the integers m^2 and $n^2 - m^2$ have the same color.*

To prove part (1), let $C_1, \ldots, C_k$ be a coloring of $\mathbb{N}$. Using part (2) of Theorem 1.1 for the coloring $C_i' := \{n \in \mathbb{N} \colon n^2 \in C_i\}$, $i = 1, \ldots, k$, we deduce that there exist $i_0 \in \{1, \ldots, k\}$ and $x, z \in C_{i_0}'$ such that $x^2 + y^2 = z^2$. Then $x^2, z^2 \in C_{i_0}$. Letting $m := x$ and $n := y$, we get that $m^2, m^2 + n^2 \in C_{i_0}$. The proof of part (2) is similar and uses part (1) of Theorem 1.1.

A coloring $C_1, \ldots, C_k$ of the squares induces a coloring $C'_1, \ldots, C'_k$ of $\mathbb{N}$ in the natural way: $C'_i := \{n \in \mathbb{N} \colon n^2 \in C_i\}$, $i = 1, \ldots, k$. Applying Theorem 1.1 for the induced coloring, we deduce the following result.

Corollary 1.4. *For every finite coloring of the squares, there exist*

1. *distinct squares x, y with the same color such that $x + y$ is a square.*
2. *distinct squares x, y with the same color such that $x - y$ is a square.*

1.3. Pythagorean triples on level sets of multiplicative functions

Our second objective is to lend support to the hypothesis that Pythagorean triples are partition regular by proving that the level sets of multiplicative functions that take finitely many values always include Pythagorean triples. Since the equation $x^2 + y^2 = z^2$ is homogeneous, one might expect that a presumed counterexample to partition regularity would have 'multiplicative structure', so Theorem 1.5 below addresses the most obvious possibilities. We also remark that Rado's theorem implies that a given linear system of equations is partition regular as soon as it has monochromatic solutions in every coloring realized using a (finitely valued) completely multiplicative function; but of course this result does not apply to the Pythagorean equation.

Theorem 1.5. *Let $f \colon \mathbb{N} \to \mathbb{S}^1$ be a completely multiplicative function that takes finitely many values. Then there exist distinct $x, y, z \in \mathbb{N}$ such that*

$$x^2 + y^2 = z^2 \quad \textit{and} \quad f(x) = f(y) = f(z) = 1.$$

Remarks. ∘ There is nothing special about the value 1 in Theorem 1.5. If $\zeta \in \mathbb{S}^1$ is any other number in the range of f, then since the equation $x^2 + y^2 = z^2$ is invariant under dilations of the variables x, y, z, we get that there exist distinct $x, y, z \in \mathbb{N}$, such that

$$x^2 + y^2 = z^2 \quad \text{and} \quad f(x) = f(y) = f(z) = \zeta.$$

∘ With a bit more effort, we can extend Theorem 1.5 to cover more general equations of the form

$$ax^2 + by^2 = cz^2, \tag{1.1}$$

where $a, b, c \in \mathbb{N}$ are squares and we have either $a = c$, or $b = c$, or $a + b = c$. We outline the additional steps needed to be taken to prove such a result in Section 8.3. Note that having one of these three identities satisfied is a necessary condition for the partition regularity of (1.1). For more details and related problems, see the discussion in Section 1.6.

Related linear equations $ax + by = cz$ on the level sets of completely multiplicative functions $f \colon \mathbb{N} \to \{-1, 1\}$ have been studied in the works of Brüdern [9] and more recently by de la Bretèche and Granville [7]. One consequence of such results [7, Corollary 2] is that the number of Pythagorean triples (x, y, z) modulo any prime $p \geq 3$; that is, solutions to $x + y = z$, where $x, y, z \leq N < p$ are quadratic residues, is at least $\frac{1}{2}(k' + o_{N\to\infty}(1))N^2$, where $k' = .005044\ldots$ is a sharp constant.

1.4. Parametric reformulation of the main results

To prove our main results, it is convenient to restate them using solutions of (1.1) in parametric form.

Our assumptions give that $a = a_0^2$, $b = b_0^2$, $c = c_0^2$ for some $a_0, b_0, c_0 \in \mathbb{N}$. Then a simple computation shows that the following are solutions of $ax^2 + by^2 = cz^2$:

$$x = k\,\ell_1\,(m^2 - n^2), \quad y = k\,\ell_2\,mn, \quad z = k\,\ell_3\,(m^2 + n^2), \quad m, n \in \mathbb{N},$$

where $\ell_1 := a_0bc$, $\ell_2 := 2ab_0c$, $\ell_3 := abc_0$.

So in order to prove Theorem 1.2, it suffices to establish the following result.

Theorem 1.6. *Suppose that* $\Lambda \subset \mathbb{N}$ *satisfies* $\bar{d}_\Phi(\Lambda) > 0$ *for some multiplicative Følner sequence* Φ. *Then for every* $\ell, \ell' \in \mathbb{N}$, *there exist*

1. $m, n \in \mathbb{N}$ *with* $m > n$ *such that* $\ell\,(m^2 - n^2)$ *and* $\ell' mn$ *are distinct and*

$$\bar{d}_\Phi\big((\ell\,(m^2 - n^2))^{-1}\Lambda \cap (\ell' mn)^{-1}\Lambda\big) > 0.$$

2. $m, n \in \mathbb{N}$ *such that* $\ell\,(m^2 + n^2)$ *and* $\ell' mn$ *are distinct and*

$$\bar{d}_\Phi\big((\ell\,(m^2 + n^2))^{-1}\Lambda \cap (\ell' mn)^{-1}\Lambda\big) > 0.$$

Remark. Since $2(m^2 + n^2) = (m+n)^2 + (m-n)^2$ and $4mn = (m+n)^2 - (m-n)^2$, applying (2) with 2ℓ in place of ℓ and $4\ell'$ in place of ℓ', we can add

(iii) $m, n \in \mathbb{N}$ such that $\ell\,(m^2 + n^2)$ and $\ell'\,(m^2 - n^2)$ are distinct and

$$\bar{d}_\Phi\big((\ell\,(m^2 + n^2))^{-1}\Lambda \cap (\ell'\,(m^2 - n^2))^{-1}\Lambda\big) > 0.$$

In order to prove Theorem 1.5, it suffices to establish the following result.

Theorem 1.7. *Let* $f\colon \mathbb{N} \to \mathbb{S}^1$ *be a completely multiplicative function that takes finitely many values. Then there exist* $k, m, n \in \mathbb{N}$, *with* $m > n$, *such that the integers* $m^2 - n^2$, $2mn$, $m^2 + n^2$ *are distinct and*

$$f(k\,(m^2 - n^2)) = f(k\,2mn) = f(k\,(m^2 + n^2)) = 1. \tag{1.2}$$

1.5. Other results

Our methodology is flexible enough to allow us to handle a variety of other dilation-invariant pairs. We record a few cases next.

1.5.1. A question from [16]

The next result is related to [16, Question 7.1]. It is only here that we use logarithmic averages

$$\mathbb{E}^{\log}_{m,n\in[N]} := \frac{1}{(\log N)^2} \sum_{m,n\in[N]} \frac{1}{mn}$$

in order to have access to a result from [47].

Theorem 1.8. *Suppose that* $\Lambda \subset \mathbb{N}$ *satisfies* $\bar{d}_\Phi(\Lambda) > 0$ *for some multiplicative Følner sequence* Φ. *Then*

$$\liminf_{N\to\infty} \mathbb{E}^{\log}_{m,n\in[N]}\, \bar{d}_\Phi((n^2+n)^{-1}\Lambda \cap (m^2)^{-1}\Lambda) > 0.$$

Remark. Our method also implies the following ergodic version of the previous result, as posed in [16], using Cesàro instead of logarithmic averages: If $(T_g)_{g\in\mathbb{N}}$ is a measure-preserving action of $(\mathbb{N}, \times)$ on a probability space (X, μ) and $A \subset X$ is measurable with $\mu(A) > 0$, then

$$\liminf_{N\to\infty} \mathbb{E}^{\log}_{m,n\in[N]}\, \mu(T^{-1}_{n^2+n}A \cap T^{-1}_{m^2}A) > 0.$$

This follows from property (6.16) that we prove below.

Our argument also allows us to replace $n^2 + n$ and m^2 by $n^2 + an$ and m^r, respectively, where $r \in \mathbb{N}$ and a is a nonzero integer. The proof of Theorem 1.8 follows closely the argument used to prove part (2) of Theorem 2.2. We will outline this argument in Section 6.2.

1.5.2. General linear forms

We can also prove variants of Theorem 1.6 that cover more general patterns of the form

$$(k\,L_1(m,n)\cdot L_2(m,n), k\,L_3(m,n)\cdot L_4(m,n)),$$

where $L_i(m,n) = a_i m + b_i n$ for some $a_i \in \mathbb{N}$, $b_i \in \mathbb{Z}$, $i = 1,2,3,4$, and at least one of the forms, say $L_4(m,n)$, is not a rational multiple of the others.

Suppose we want to show, under the previous assumptions, that if $\Lambda \subset \mathbb{Z}$ satisfies $\bar{d}_\Phi(\Lambda) > 0$ for some multiplicative Følner sequence Φ, then there exist $m,n \in \mathbb{Z}$ such that $L_1(m,n)\cdot L_2(m,n)$ and $L_3(m,n)\cdot L_4(m,n)$ are distinct integers and satisfy

$$\bar{d}_\Phi\big((L_1(m,n)\cdot L_2(m,n))^{-1}\Lambda \cap (L_3(m,n)\cdot L_4(m,n))^{-1}\Lambda\big) > 0.$$

Without loss of generality, we can assume that $b_4 \neq 0$. By making the substitution $m \mapsto b_4\,m$ and $n \mapsto n - a_4\,m$ (an operation that preserves our assumptions about the forms L_i), we can assume that $a_4 = 0$. Since the form L_4 is not a rational multiple of L_i for $i = 1,2,3$, we have $a_i \neq 0$ for $i = 1,2,3$. We do another substitution $n \mapsto a_1\,a_2\,a_3\,n$. We then factor out a_i from the linear form L_i for $i = 1,2,3$. We see that it is sufficient to consider the case where the L_i are integer multiples of forms satisfying $a_1 = a_2 = a_3 = 1$ and $a_4 = 0$, $b_4 \neq 0$. Making a last substitution $m \mapsto m - b_3\,n$, we get that it suffices to prove that

$$\bar{d}_\Phi\big((\ell\,(m+an)\cdot(m+bn))^{-1}\Lambda \cap (\ell'\,m\,n)^{-1}\Lambda\big) > 0$$

whenever $\ell, \ell' \in \mathbb{N}$ and $a, b \in \mathbb{Z}$. This case can be covered by repeating the argument used to prove Theorem 1.6 (which covers the case $a = 1, b = -1$) without any essential change.

1.5.3. More general expressions and averages

The methods used to establish part (2) of Theorem 1.6, would also allow to cover patterns of the form

$$\Big(k\,(m^2+n^2)^r \prod_{i=1}^{l} L_i(m,n),\ k \prod_{i=1}^{l'} L_i'(m,n)\Big),$$

where $k \in \mathbb{N}$, $l, l', r \in \mathbb{Z}_+$ are such that $|l| + |l'| > 0$,[2] and at least one of the linear forms L_i, L_i' is not a rational multiple of the others. It should also be possible to cover variants of Theorem 2.2 below in which the averages over squares $\mathbb{E}_{m,n\in[N]}$ are replaced by averages over discs (i.e., $\mathbb{E}_{m^2+n^2\leq N}$). However, we do not pursue these directions here.

1.6. Further directions

Our approach opens the way for studying several other compelling partition regularity problems that were previously considered intractable. We note here some promising directions.

A result of Rado [41] implies that if $a,b,c \in \mathbb{N}$, then the linear equation $ax + by = cz$ is partition regular if and only if either a, b, or $a+b$ equals c, in which case we say that the triple (a,b,c) *satisfies Rado's condition*. It follows that a necessary condition for the partition regularity of the equation (1.1) is that the triple (a,b,c) satisfies Rado's condition. Perhaps this condition is also sufficient, but very little is known in this direction; in fact, there is no triple (a,b,c) for which the partition regularity of (1.1) is currently known. We state a related problem of intermediate difficulty along the lines of Theorem 1.5.

[2]The case $l = l' = 0$ is covered in [16, Theorem 1.5].

Problem 1. *Suppose that the triple (a, b, c) satisfies Rado's condition. Then for any completely multiplicative function $f\colon \mathbb{N} \to \mathbb{S}^1$ taking finitely many values, there exist distinct $x, y, z \in \mathbb{N}$, such that*

$$ax^2 + by^2 = cz^2 \quad \textit{and} \quad f(x) = f(y) = f(z) = 1.$$

Theorem 1.5 solves this problem when $a = b = c = 1$, and as we mentioned in the second remark following the theorem, a similar argument applies to triples that satisfy Rado's condition and consist of squares. It would be interesting to solve Problem 1 for some other triples such as $(1, 1, 2)$ and $(1, 2, 1)$. The first one corresponds to the equation

$$x^2 + y^2 = 2z^2,$$

which was conjectured to be partition regular by Gyarmati and Ruzsa [29] and has parametric solutions of the form

$$x = k\,(m^2 - n^2 + 2mn),\ y = k\,(m^2 - n^2 - 2mn),\ z = k\,(m^2 + n^2).$$

The second one corresponds to the equation

$$x^2 + 2y^2 = z^2$$

with parametric solutions of the form

$$x = k\,(m^2 - 2n^2),\ y = k\,(2mn),\ z = k\,(m^2 + 2n^2).$$

Both parametrizations involve at least two quadratic forms that do not factor into products of linear forms. This is a problem for our method, since a useful variant of Proposition 2.15 is not known in this case, not even if f_1, f_2, f_3 are all equal to the Liouville function.

Another interesting problem is to relax the conditions on the coefficients a, b, c in Theorem 1.1. We mention two representative problems that seem quite challenging.

Problem 2. *Show that for every finite coloring of $\mathbb{N}$, there exist*

1. *distinct $x, y \in \mathbb{N}$ with the same color and $z \in \mathbb{N}$ such that $x^2 + y^2 = 2z^2$.*
2. *distinct $x, y \in \mathbb{N}$ with the same color and $z \in \mathbb{N}$ such that $x^2 + 2y^2 = z^2$.*

Show also similar properties with the roles of the variables y and z or x and z reversed.

Remark. More generally, we believe that if for $a, b, c \in \mathbb{N}$ at least one of the integers $ac, bc, (a+b)c$ is a square, then for every finite coloring of the integers, there exist distinct $x, y \in \mathbb{N}$ with the same color and $z \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$. Theorem 1.1 verifies this if both ac and bc are squares. We also expect that if at least one of the integers $bc, (c-a)b$ is a square, then for every finite coloring of the integers, there exist distinct $x, z \in \mathbb{N}$ with the same color and $y \in \mathbb{N}$ such that $ax^2 + by^2 = cz^2$. It may also be that stronger density regularity results hold, as in Theorem 1.2 and Theorem 1.6.

The broader issue is to find conditions for the polynomials $P, Q \in \mathbb{Z}[m, n]$ such that the following holds: If $\Lambda \subset \mathbb{N}$ satisfies $\bar{d}_\Phi(\Lambda) > 0$ for some multiplicative Følner sequence Φ, then there exist $m, n \in \mathbb{N}$ such that the integers $P(m, n)$ and $Q(m, n)$ are positive and distinct, and

$$\bar{d}_\Phi\big((P(m,n))^{-1}\Lambda \cap (Q(m,n))^{-1}\Lambda\big) > 0.$$

Equivalently, using the terminology from [16], the problem is to determine for which polynomials $P, Q \in \mathbb{Z}[m, n]$ we have that $\{P(m, n)/Q(m, n) : m, n \in \mathbb{N}\}$ is a set measurable multiplicative recurrence.

1.7. Notation

We let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{S}^1$ be the unit circle, and $\mathbb{U}$ be the closed complex unit disk. With $\mathbb{P}$ we denote the set of primes, and throughout, we use the letter p to denote primes.

For $t \in \mathbb{R}$, we let $e(t) := e^{2\pi i t}$. For $z \in \mathbb{C}$, with $\Re(z)$, $\Im(z)$, we denote the real and imaginary parts of z, respectively.

For $N \in \mathbb{N}$, we let $[N] := \{1, \ldots, N\}$. We often denote sequences $a \colon \mathbb{N} \to \mathbb{U}$ by $(a(n))$, instead of $(a(n))_{n\in\mathbb{N}}$.

If A is a finite nonempty subset of the integers and $a \colon A \to \mathbb{C}$, we let

$$\mathbb{E}_{n\in A}\, a(n) := \frac{1}{|A|} \sum_{n\in A} a(n).$$

We write $a(n) \ll b(n)$ if for some $C > 0$, we have $a(n) \le C\, b(n)$ for every $n \in \mathbb{N}$.

Throughout this article, the letter f is typically used for multiplicative functions and the letter χ for Dirichlet characters.

2. Roadmap to the proofs

This section outlines how we prove our main results in their parametric reformulation, which is given in Theorems 1.6 and 1.7.

For various facts and notions concerning multiplicative functions, we refer the reader to Section 3.3.

2.1. Reduction of Theorem 1.6 to a positivity property for multiplicative functions

We first use a version of the Furstenberg correspondence principle (see [4]) to reformulate the results in an ergodic language.

Theorem 2.1. *Let $\ell, \ell' \in \mathbb{N}$, let $T = (T_n)_{n\in\mathbb{N}}$ be a measure preserving action of $(\mathbb{N}, \times)$ on a probability space (X, μ),*[3] *and let $A \subset X$ be measurable with $\mu(A) > 0$. Then there exist*

1. *$m, n \in \mathbb{N}$ with $m > n$ such that $\ell\,(m^2 - n^2)$ and $\ell'\,mn$ are distinct and*

$$\mu(T^{-1}_{\ell(m^2-n^2)} A \cap T^{-1}_{\ell' mn} A) > 0. \tag{2.1}$$

2. *$m, n \in \mathbb{N}$ such that $\ell\,(m^2 + n^2)$ and $\ell'\,mn$ are distinct and*

$$\mu(T^{-1}_{\ell(m^2+n^2)} A \cap T^{-1}_{\ell' mn} A) > 0. \tag{2.2}$$

In fact, the set of $m, n \in \mathbb{N}$ for which (2.1) and (2.2) hold has positive lower density.

Remarks. ◦ The reduction to the previous multiple recurrence statement is merely a convenience. It facilitates the purpose of getting a further reduction to a positivity property for completely multiplicative functions that we describe in Theorem 2.2. Alternatively, one could carry out this last reduction directly, as in [21, Section 10.2].

◦ Using the terminology from [16], Theorem 2.1 can be rephrased as saying that for every $\ell, \ell' \in \mathbb{N}$, both subsets of $\mathbb{Q}^{>0}$

$$\left\{\ell(m^2 - n^2)/(\ell' mn) \colon m, n \in \mathbb{N},\ m > n\right\} \text{ and } \left\{\ell(m^2 + n^2)/(\ell' mn) \colon m, n \in \mathbb{N}\right\}$$

are sets of measurable multiplicative recurrence.

[3] Meaning, $T_n \colon X \to X$, $n \in \mathbb{N}$, are invertible measure preserving transformations such that $T_1 := \mathrm{id}$ and $T_{mn} = T_m \circ T_n$ for every $m, n \in \mathbb{N}$.

A function $f\colon \mathbb{N} \to \mathbb{U}$, where $\mathbb{U}$ is the complex unit disk, is called *multiplicative* if

$$f(mn) = f(m) \cdot f(n) \quad \text{whenever } (m,n) = 1.$$

It is called *completely multiplicative* if the previous equation holds for all $m, n \in \mathbb{N}$. Let

$$\mathcal{M} := \{f\colon \mathbb{N} \to \mathbb{S}^1 \text{ is a completely multiplicative function}\}.$$

Wherever necessary, we extend multiplicative functions to the non-positive integers in an arbitrary way. Throughout, we assume that $\mathcal{M}$ is equipped with the topology of pointwise convergence. It easily follows that $\mathcal{M}$ is a metrizable compact space with this topology. We can identify $\mathcal{M}$ with the Pontryagin dual of the (discrete) group of positive rational numbers under multiplication. Note that the map $r/s \mapsto \mu(T_r^{-1}A \cap T_s^{-1}A)$, $r, s, \in \mathbb{N}$, from $(\mathbb{Q}_+, \times)$ to $[0,1]$ is well defined and positive definite. Using a theorem of Bochner-Herglotz, we get that there exists a finite Borel measure σ on $\mathcal{M}$ such that $\sigma(\{1\}) > 0$ (in fact, $\sigma(\{1\}) \geq \delta^2$, where $\delta = \mu(A)$) and for every $r, s \in \mathbb{N}$,

$$\int_{\mathcal{M}} f(r) \cdot \overline{f(s)}\, d\sigma(f) = \mu(T_r^{-1}A \cap T_s^{-1}A).$$

In particular, we have

$$\mu(T^{-1}_{\ell(m^2-n^2)}A \cap T^{-1}_{\ell' mn}A) = \int_{\mathcal{M}} f(\ell(m^2-n^2)) \cdot \overline{f(\ell' mn)}\, d\sigma(f)$$

for every $m, n \in \mathbb{N}$ with $m > n$, and

$$\mu(T^{-1}_{\ell(m^2+n^2)}A \cap T^{-1}_{\ell' mn}A) = \int_{\mathcal{M}} f(\ell(m^2+n^2)) \cdot \overline{f(\ell' mn)}\, d\sigma(f)$$

for every $m, n \in \mathbb{N}$. Therefore, Theorem 2.1 follows from the following result.

Theorem 2.2. *Let σ be a positive bounded measure on $\mathcal{M}$ such that $\sigma(\{1\}) > 0$ and*

$$\int_{\mathcal{M}} f(r) \cdot \overline{f(s)}\, d\sigma(f) \geq 0 \quad \textit{for every } r, s \in \mathbb{N}. \tag{2.3}$$

Then for every $\ell, \ell' \in \mathbb{N}$,

1. *we have*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N], m>n} \int_{\mathcal{M}} f(\ell(m^2-n^2)) \cdot \overline{f(\ell' mn)}\, d\sigma(f) > 0. \tag{2.4}$$

2. *we have*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}} f(\ell(m^2+n^2)) \cdot \overline{f(\ell' mn)}\, d\sigma(f) > 0. \tag{2.5}$$

Remark. The limit in (2.4) exists by [20, Theorem 1.4] and the bounded convergence theorem.[4] However, the limit in (2.5) may not always exist.

The reduction up to this point is similar to that in [21]. The methods in [21] were only able to address a variant of (1) in which the expressions under the integral were products of linear factors

[4]The statement of [20, Theorem 1.4] does not have the restriction $m > n$ in the averaging, but the argument used there also covers this case without essential changes.

and were 'pairing up' when $n = 0$ and becoming nonnegative.[5] This positivity property is not shared by the expressions in (2.4) (and (2.5)), which is the main reason why it was not possible to deal with Pythagorean pairs in [21]. To overcome this obstacle, we do not use a decomposition result that covers all elements of $\mathcal{M}$ simultaneously (as was the case in [21]), but rather work separately with aperiodic and pretentious multiplicative functions (these notions are defined in Section 3.3). In particular, coupled with some measurability properties, this allows us to exploit the uniform concentration estimates of Propositions 2.5 and 2.11, which are not shared by all elements of $\mathcal{M}$. We outline our approach in the next subsections.

2.2. Proof plan for part (1) of Theorem 2.2

We prove Theorem 2.2 by taking an average over the grid

$$\{(Qm+1, Qn) : m, n \in \mathbb{N}\},$$

where $Q \in \mathbb{N}$ is chosen depending only on σ. In view of (2.3), it suffices to prove positivity in (2.4) when the average is taken along this subset of pairs. With $\ell, \ell' \in \mathbb{N}$ fixed, we introduce the following notation: for $\delta > 0$, $f \in \mathcal{M}$, and $Q, m, n \in \mathbb{N}$, let

$$A_\delta(f, Q; m, n) := w_\delta(m, n) \cdot f\big(\ell\,((Qm+1)^2 - (Qn)^2)\big) \cdot \overline{f\big(\ell'\,(Qm+1)Qn\big)}, \tag{2.6}$$

where $w_\delta : \mathbb{N}^2 \to [0, 1]$ is the weight defined in (3.2) of Lemma 3.3 for reasons that will become clear in a moment (at a first reading, the reader could just take $w_\delta = 1$). We also remark that the weight w_δ is supported on the set $\{m, n \in \mathbb{N} \colon m > n\}$, so to compute $A_\delta(f, Q; m, n)$, we only have to compute f on positive integers. Then part (1) of Theorem 2.2 follows immediately from the next result, the fact that $0 \le w_\delta(m, n) \le 1$, and the positivity property (2.3).

Theorem 2.3. *Let σ be a Borel probability measure on $\mathcal{M}$ such that $\sigma(\{1\}) > 0$. Then there exist $\delta_0 > 0$ and $Q_0 \in \mathbb{N}$ such that*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}} A_{\delta_0}(f, Q_0; m, n)\, d\sigma(f) > 0. \tag{2.7}$$

Remark. The values of $\delta_0 > 0$ and $Q_0 \in \mathbb{N}$ depend on σ but not on ℓ, ℓ'.

To analyze the limit in (2.7), we use the theory of completely multiplicative functions. When f is aperiodic, the mean values of $A_\delta(f, Q; m, n)$ vanish for every Q. This is a consequence of the following result, which in turn follows from results in [21] (see also [38] for related work). We shall explain later on how.

Proposition 2.4. *Let $f : \mathbb{N} \to \mathbb{U}$ be an aperiodic completely multiplicative function. Then for every $\delta > 0$ and $Q \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} A_\delta(f, Q; m, n) = 0. \tag{2.8}$$

Furthermore, for every completely multiplicative function $f : \mathbb{N} \to \mathbb{U}$, the previous limit exists.

Let

$$\mathcal{M}_p = \{f : \mathbb{N} \to \mathbb{S}^1 : f \text{ is a pretentious completely multiplicative function}\}; \tag{2.9}$$

[5]For instance, to establish partition regularity of pairs x, y that satisfy the equation $16x^2 + 9y^2 = z^2$, it suffices to study averages of $f\,(m(m+3n)) \cdot \overline{f\,((m+n)(m-3n))}$ for $f \in \mathcal{M}$. The key convenient property these expressions have is that they are nonnegative when $n = 0$.

we show in Lemma 3.6 that $\mathcal{M}_p$ is a Borel subset of $\mathcal{M}$. It follows from Proposition 2.4 and the bounded convergence theorem that in order to establish (2.7), it suffices to show that there exist $\delta_0 > 0$ and $Q_0 \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_p} A_{\delta_0}(f, Q_0; m, n)\, d\sigma(f) > 0. \tag{2.10}$$

If f is pretentious, then it 'pretends' to be a twisted Dirichlet character, and thus exhibits some periodicity. We exploit this periodicity by choosing a highly divisible Q for which the averages of $A_\delta(f, Q; m, n)$ take a much simpler form. More precisely, we make use of the following concentration estimate, which is an immediate consequence of [35, Lemma 2.5].

Proposition 2.5. *Let $f\colon \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi \cdot n^{it}$ for some $t \in \mathbb{R}$ and Dirichlet character χ with period q (see Section 3.3 for definitions and notation). Let also $K \in \mathbb{N}$ be large enough so that q divides all elements of the set*

$$\Phi_K := \Big\{ \prod_{p\le K} p^{a_p} : K < a_p \le 2K \Big\}. \tag{2.11}$$

Then

$$\limsup_{N\to\infty} \max_{Q\in\Phi_K} \mathbb{E}_{n\in[N]} \big| f(Qn+1) - (Qn)^{it} \cdot \exp\big(F_N(f,K)\big)\big| \ll \mathbb{D}(f, \chi\cdot n^{it}; K, \infty) + K^{-1/2},$$

where the implicit constant is absolute and

$$F_N(f,K) := \sum_{K<p\le N} \frac{1}{p} \big(f(p)\cdot \overline{\chi(p)} \cdot p^{-it} - 1\big). \tag{2.12}$$

Remarks. ◦ It is important for our argument that the implicit constant is independent of K and the quantity $F_N(f,K)$ does not depend on Q as long as $Q \in \Phi_K$ and $q \mid Q$.

◦ We will also need the following variant from [35, Lemma 2.5]: For any fixed $Q \in \mathbb{N}$ such that $q\prod_{p\le K} p \mid Q$, we have

$$\limsup_{N\to\infty} \mathbb{E}_{n\in[N]} \big| f(Qn+1) - (Qn)^{it} \cdot \exp\big(F_N(f,K)\big)\big| \ll \mathbb{D}(f, \chi\cdot n^{it}; K, \infty) + K^{-1/2}.$$

◦ If $f \sim \chi\cdot n^{it}$, then the sequence $A(N) := \sum_{1<p\le N} \frac{1}{p} \big|1 - f(p)\cdot\overline{\chi(p)}\cdot p^{-it}\big|$, $N \in \mathbb{N}$, is slowly varying, in the sense that for a fixed pretentious f, we have for every $c \in (0,1)$ that $\lim_{N\to\infty} \sup_{n\in[N^c,N]} |A(n) - A(N)| = 0$.[6] Keeping this in mind, if we use partial summation on the interval $[N^c, N]$ and then let $c \to 0^+$, we deduce that the main estimate of Proposition 2.5 still holds if we replace $\mathbb{E}_{n\in[N]}$ with $\mathbb{E}^{\log}_{n\in[N]}$.

In order to establish (2.10), we divide the integral into two parts. The first is supported on multiplicative functions other than the Archimedean characters $(n^{it})_{n\in\mathbb{N}}$, $t \in \mathbb{R}$, in which case we show using Proposition 2.5 that for a highly divisible Q_0, the contribution is essentially nonnegative. The second is supported on Archimedean characters. We show that this part is positive using our assumption $\sigma(\{1\}) > 0$ and by taking δ_0 small enough so that the weight w_{δ_0} neutralizes the effect of the Archimedean characters that are different from 1. To carry out the first part, the key idea is to average over 'multiplicatively large' values of Q. More precisely, for each $K \in \mathbb{N}$, let Φ_K be the set described in (2.11). The sequence (Φ_K) is a multiplicative Følner sequence with the property that for every $q \in \mathbb{N}$,

[6] If $a_p := 1 - f(p)\cdot\overline{\chi(p)}\cdot p^{-it}$, $p \in \mathbb{P}$, we note that $\sup_{n\in[N^c,N]} |A(n) - A(N)| \le (B_N \cdot C_N)^{1/2}$, where $B_N := \sum_{p\in[N^c,N]} \frac{|a_p|^2}{p}$, $C_N := \sum_{p\in[N^c,N]} \frac{1}{p}$, $N \in \mathbb{N}$. The sequence C_N is bounded and $\lim_{N\to\infty} B_N = 0$ because $\sum_{p\in\mathbb{P}} \frac{|a_p|^2}{p} < +\infty$.

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as soon as K is large enough, every $Q \in \Phi_K$ is divisible by q. It also has the property that for every $Q \in \Phi_K$ and a prime $p \in \mathbb{P}$, we have $p|Q$ if and only if $p \leq K$. Let also

$$\mathcal{A} := \{(n^{it})_{n\in\mathbb{N}} : t \in \mathbb{R}\}. \tag{2.13}$$

Note that $\mathcal{A}$ is a Borel subset of $\mathcal{M}$ since it is a countable union of compact sets (we caution the reader that $\mathcal{A}$ is not closed with the topology of pointwise convergence; in fact, it is dense in $\mathcal{M}$). The most important step in establishing property (2.10) is the following fact.

Lemma 2.6. *Let $f \in \mathcal{M}_p \setminus \mathcal{A}$, $\delta > 0$, $\ell, \ell' \in \mathbb{N}$, and Φ_K be as in (2.11). Then*

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K} \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, A_\delta(f, Q; m, n) = 0.$$

(Note that the inner limit exists by Proposition 2.4.)

Roughly, to prove Lemma 2.6, we use the concentration estimate of Proposition 2.5 to deduce that for $Q \in \Phi_K$, the average $\mathbb{E}_{m,n\in[N]}\, A_\delta(f, Q; m, n)$ is asymptotically equal to $C_{\ell,\ell'}(K) \cdot \overline{f(Q)} \cdot Q^{it}$ for some $C_{\ell,\ell'}(K) \in \mathbb{U}$ and $t \in \mathbb{R}$. Since $f \notin \mathcal{A}$, by Lemma 3.2, the average of the last expression, taken over $Q \in \Phi_K$, converges to 0 as $K \to \infty$.

Using the previous result, the fact that the limit $\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, A_\delta(f, Q; m, n)$ exists (by Proposition 2.4), and applying the bounded convergence theorem twice, we deduce the following vanishing property.

Corollary 2.7. *Let (Φ_K) and $\mathcal{A}$ be defined by (2.11) and (2.13), respectively. Let also σ be a Borel probability measure on $\mathcal{M}_p$. Then for every $\delta > 0$, we have*

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K} \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_p\setminus\mathcal{A}} A_\delta(f, Q; m, n)\, d\sigma(f) = 0.$$

We are left to study the part of the integral supported on $\mathcal{A}$. For such functions, the limits $\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} A_\delta(f, Q; m, n)$ do not depend on Q, and so the previous argument will not help. It is the presence of the weight w_δ that will allow us to prove the following positivity property.

Lemma 2.8. *Let σ be a Borel probability measure on $\mathcal{M}$ such that $\sigma(\{1\}) > 0$ and $\mathcal{A}$ be as in (2.13). Then there exist $\delta_0, \rho_0 > 0$, depending only on σ, such that*

$$\liminf_{N\to\infty} \inf_{Q\in\mathbb{N}} \Re\Big(\mathbb{E}_{m,n\in[N]} \int_{\mathcal{A}} A_{\delta_0}(f, Q; m, n)\, d\sigma(f)\Big) \geq \rho_0. \tag{2.14}$$

Remark. The weight $w_\delta(m, n)$ is introduced to force positivity in this case, since for some choices of ℓ, ℓ' and measures σ, the unweighted expressions have negative real parts. However, rather miraculously, if $\ell = 1$ and $\ell' = 2$ (which is the case to consider for Pythagorean pairs), we get positivity even in the unweighted case, and a somewhat simpler argument applies. We do not pursue this approach here though because it lacks generality.

Finally, we will see how the previous results allow us to reach our goal, which is to prove Theorem 2.3, thus completing the proof of part (1) of Theorems 1.2 and 2.2.

Proof of Theorem 2.3 assuming Proposition 2.4, Corollary 2.7 and Lemma 2.8. By combining Corollary 2.7 and Lemma 2.8, we deduce that there exist $\delta_0, \rho_0 > 0$, depending only on σ, such that

$$\liminf_{K\to\infty} \mathbb{E}_{Q\in\Phi_K} \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_p} A_{\delta_0}(f, Q; m, n)\, d\sigma(f) \geq \rho_0.$$

(There is no need to take the real part on this expression since it is real.) From this, we immediately deduce that (2.10) holds for some $Q_0 \in \mathbb{N}$. As we also explained before, this fact, together with Proposition 2.4, implies (2.7) via the bounded convergence theorem, completing the proof. □

To establish Theorem 2.3, it remains to prove Proposition 2.4, Lemma 2.6 (Corollary 2.7 is an immediate consequence) and Lemma 2.8. We do this in Section 4.

2.3. *Proof plan for part* (2) *of Theorem 2.2*

The general strategy is similar to that used to prove part (1) of Theorem 2.2, but there are two major differences. The first is the required concentration estimate, which is given in Proposition 2.11 below. Unlike Proposition 2.5, this result is new and of independent interest, and its proof occupies a considerable portion of the argument. The second difference is that the limit in (2.5) may not exist, which causes additional technical problems.

Arguing as before, we get that part (2) of Theorem 2.2 follows from the following positivity property.

Theorem 2.9. *Let σ be a Borel probability measure on $\mathcal{M}$ such that $\sigma(\{1\}) > 0$ and (2.3) holds. Then there exists $\delta > 0$ such that*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \tilde{w}_\delta(m,n) \cdot \int_{\mathcal{M}} f\big(\ell\big(m^2+n^2\big)\big) \cdot \overline{f\big(\ell' mn\big)}\, d\sigma(f) > 0, \tag{2.15}$$

where $\tilde{w}_\delta(m,n)$ is the weight defined in (3.3) of Lemma 3.3.

Again, to analyze the limit in (2.15), we use the theory of completely multiplicative functions. We introduce the following notation: for $\delta > 0$, $f \in \mathcal{M}$, and $Q, m, n \in \mathbb{N}$, let

$$B_\delta(f,Q;m,n) := \tilde{w}_\delta(m,n) \cdot f\big(\ell\big((Qm+1)^2+(Qn)^2\big)\big) \cdot \overline{f\big(\ell'\,(Qm+1).(Qn)\big)}. \tag{2.16}$$

If f is aperiodic, we have the following result, which we will deduce from the results in [21].

Proposition 2.10. *Let $f\colon \mathbb{N} \to \mathbb{U}$ be an aperiodic multiplicative function. Then for every $\delta > 0$ and $Q \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, B_\delta(f,Q;m,n) = 0. \tag{2.17}$$

Remark. It follows that (2.17) also holds even if Q depends on N, but its values are taken from a finite subset of $\mathbb{N}$.

If f is pretentious, we will crucially use the following concentration estimate (which is a direct consequence of a more general result proved in Section 5) to analyze the average (2.15). It features a version of the pretentious distance that only considers primes[7] congruent to 1 mod 4:

$$\mathbb{D}_1(f,\chi\cdot n^{it};K,\infty)^2 := \sum_{\substack{K<p,\\ p\equiv 1 \pmod 4}} \frac{1}{p}\,\big(1-\Re\big(f(p)\cdot\overline{\chi(p)}\cdot p^{-it}\big)\big).$$

Proposition 2.11. *Let $f\colon \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f \sim \chi\cdot n^{it}$ for some $t \in \mathbb{R}$ and Dirichlet character χ with period q. Let also Φ_K be as in (2.11) and suppose that K is large enough so that, say, $\mathbb{D}_1(f,\chi\cdot n^{it};K,\infty) \le 1$ and q divides all elements of Φ_K. Then*

[7]The reason we only need primes ≡ 1 mod 4 is that these are the primes that split in the splitting field of m^2+n^2. In a subsequent work, we extended these techniques to obtain concentration estimates to general binary quadratic forms.

$$\limsup_{N\to\infty} \max_{Q\in\Phi_K} \mathbb{E}_{m,n\in[N]} \left|f\big((Qm+1)^2+(Qn)^2\big) - Q^{2it}\cdot(m^2+n^2)^{it}\cdot \exp\big(G_N(f,K)\big)\right| \ll \mathbb{D}_1(f,\chi\cdot n^{it};K,\infty)+K^{-1/2},$$

where the implicit constant is absolute and

$$G_N(f,K) := 2 \sum_{\substack{K<p\leq N,\\ p\equiv 1 \pmod 4}} \frac{1}{p}\,(f(p)\cdot\overline{\chi(p)}\cdot p^{-it}-1). \tag{2.18}$$

Remarks. ∘ It is important for our argument that the implicit constant does not depend on K and that $\exp\big(G_N(f,K)\big)$ is the same for all $Q\in\Phi_K$ that are divisible by q. It is also important for our applications that we get some uniformity over the $Q\in\Phi_K$.

∘ For the future applications in mind, we prove a somewhat more general and quantitatively more explicit variant; see Proposition 5.1 below.

As in the proof of Theorem 2.3, in order to prove Theorem 2.9, we split the integral into two parts, one that is supported on Archimedean characters and the other on its complement. To handle the second part, we use the following result, which is proved using Proposition 2.11 and can be compared to Corollary 2.7. Again, taking multiplicative averages over the variable Q is a key maneuver, but the non-convergence of the averages $\mathbb{E}_{m,n\in[N]}\, B_\delta(f,Q;m,n)$ causes considerable technical difficulties in our proofs.

Proposition 2.12. *Let* (Φ_K), $\mathcal{A}$, $B_\delta(f,Q;m,n)$ *be defined by (2.11), (2.13), (2.16), respectively, and* $\delta>0$. *Let also* σ *be a Borel probability measure on* $\mathcal{M}_p$. *Then*

$$\lim_{K\to\infty}\limsup_{N\to\infty}\left|\mathbb{E}_{Q\in\Phi_K}\mathbb{E}_{m,n\in[N]}\int_{\mathcal{M}_p\setminus\mathcal{A}} B_\delta(f,Q;m,n)\,d\sigma(f)\right| = 0.$$

Remark. Unlike the case of Corollary 2.7, we cannot pass the limit over N inside the average over Q. This will cause some minor problems in our later analysis, which we will overcome by using the positivity property (2.3) of the measure σ (this is why this positivity property is used in the statement of Theorem 2.9 but not in Theorem 2.3).

We are left to study the contribution of the set $\mathcal{A}$ of Archimedean characters in which case the presence of the weight $\tilde{w}_\delta$ allows us to establish positivity by taking δ small enough.

Lemma 2.13. *Let* σ *be a Borel probability measure on* $\mathcal{M}_p$ *such that* $\sigma(\{1\})>0$ *and* $\mathcal{A}$ *be as in (2.13). Then there exist* $\delta_0,\rho_0>0$, *depending only on* σ, *such that*

$$\liminf_{N\to\infty}\inf_{Q\in\mathbb{N}}\Re\Big(\mathbb{E}_{m,n\in[N]}\int_{\mathcal{A}} B_{\delta_0}(f,Q;m,n)\,d\sigma(f)\Big)\geq \rho_0.$$

We conclude this section by noting how the previous results allow us to reach our goal, which is to prove Theorem 2.9, thus completing the proof of part (2) of Theorems 1.2 and 2.2.

Proof of Theorem 2.9 assuming Proposition 2.10, Proposition 2.12 and Lemma 2.13. We start by combining Proposition 2.12 and Lemma 2.13. We deduce that there exist $\delta_0,\rho_0>0$, depending only on σ, such that

$$\liminf_{K\to\infty}\liminf_{N\to\infty}\mathbb{E}_{Q\in\Phi_K}\Re\Big(\mathbb{E}_{m,n\in[N]}\int_{\mathcal{M}_p} B_{\delta_0}(f,Q;m,n)\,d\sigma(f)\Big)\geq\rho_0.$$

In this case, it is a little bit tricky to deduce that (2.15) holds. We do it as follows. The last estimate implies that there exist $K_0\in\mathbb{N}$ and $Q_N\in\Phi_{K_0}$, $N\in\mathbb{N}$, such that

$$\liminf_{N\to\infty} \Re\Big(\mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_p} B_{\delta_0}(f, Q_N; m,n)\, d\sigma(f)\Big) \geq \rho_0/2.$$

Note that since Q_N belongs to a finite set, Proposition 2.10 implies that in the last expression we can replace $\mathcal{M}_p$ with $\mathcal{M}$. Hence,

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}} B_{\delta_0}(f, Q_N; m,n)\, d\sigma(f) \geq \rho_0/2. \tag{2.19}$$

(The real part is no longer needed since the last expression is known to be real by (2.3).) It is easy to verify (following the line of reasoning at the beginning of Proposition 4.1 below) that

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]} \sup_{Q\in\mathbb{N}} |\tilde{w}_\delta(Qm+1, Qn) - \tilde{w}_\delta(m,n)| = 0.$$

We deduce that if in the definition of $B_{\delta_0}(f, Q_N; m,n)$ given in (2.16) we replace $\tilde{w}_\delta(m,n)$ with $\tilde{w}_\delta(Q_N m+1, Q_N n)$, the limit on the left side of (2.19) remains unchanged. Keeping this in mind, and since Q_N takes values in a finite set with upper bound, say Q_0, and by the positivity property (2.3), we have

$$\tilde{w}_\delta(m,n) \cdot \int_{\mathcal{M}} f\big(\ell(m^2+n^2)\big) \cdot \overline{f(\ell' mn)}\, d\sigma(f) \geq 0$$

for every $m, n \in \mathbb{N}$, and we deduce that

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \tilde{w}_\delta(m,n) \cdot \int_{\mathcal{M}} f\big(\ell(m^2+n^2)\big) \cdot \overline{f(\ell' mn)}\, d\sigma(f) \geq \rho_0/(2Q_0^2).$$

This establishes (2.15) and ends the proof. □

In order to establish Theorem 2.9, it remains to prove Proposition 2.10, Proposition 2.12 and Lemma 2.13. We do this in Section 6, after having established Proposition 2.11 in Section 5, which is crucially used in the proof of Proposition 2.12.

2.4. *Proof plan of Theorem 1.7*

For notational convenience, when we write $\mathbb{E}^*_{k\in\mathbb{N}}$ in the following statements, we mean the limit $\lim_{K\to\infty} \mathbb{E}_{k\in\Phi_K}$, where (Φ_K) is an arbitrary multiplicative Følner sequence, chosen so that all the limits in the following statements exist. Since our setting will always involve a countable collection of limits, such a Følner sequence always exists and can be taken as a subsequence of any given multiplicative Følner sequence.

Our argument is divided into two parts. In the first part, we reduce the problem to a positivity property of pretentious multiplicative functions, and in the second part, we verify this positivity property. To carry out the first part, we note that to prove Theorem 1.7, it is only necessary to establish the subsequent averaged version.

Theorem 2.14. *Suppose that the completely multiplicative function $f\colon \mathbb{N} \to \mathbb{S}^1$ takes finitely many values and $F := \mathbf{1}_{\{1\}}$. Then*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N], m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, F(f(k\,(m^2-n^2))) \cdot F(f(k\,2mn)) \cdot F(f(k\,(m^2+n^2))) > 0. \tag{2.20}$$

Remark. The 'multiplicative average' $\mathbb{E}^*_{k\in\mathbb{N}}$ is needed in our analysis to 'clear out' some unwanted terms.

We write $f = gh$, where g has aperiodicity properties and h is pretentious (see Lemma 7.3 for the exact statement). Since f is finite-valued, it follows that g takes values in d-roots of unity for some $d \in \mathbb{N}$. Hence, we have

$$F \circ g = \mathbf{1}_{g=1} = \mathbb{E}_{0 \le j < d}\, g^j.$$

We use the previous facts to analyze the average in (2.20). The aperiodic part is covered by the next result, which is a direct consequence of [21, Theorem 9.7].

Proposition 2.15. *Let $f_1, f_2, f_3 \colon \mathbb{N} \to \mathbb{U}$ be completely multiplicative functions and suppose that either f_1 or f_2 is aperiodic. Then*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N], m>n}\, f_1(m^2 - n^2) \cdot f_2(mn) \cdot f_3(m^2 + n^2) = 0.$$

Combining the above and some technical maneuvering, we get the following reduction, which completes the first part needed to prove Theorem 2.14.

Proposition 2.16. *Suppose that for every finite-valued completely multiplicative function $h \colon \mathbb{N} \to \mathbb{S}^1$, with $h \sim 1$, and modified Dirichlet character $\tilde{\chi} \colon \mathbb{N} \to \mathbb{S}^1$ (see Section 3.3 for the definitions), we have*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N], m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, A(k\,(m^2 - n^2)) \cdot A(k\, 2mn) \cdot A(k\,(m^2 + n^2)) > 0,$$

where

$$A(n) := F(h(n)) \cdot F(\tilde{\chi}(n)), \quad n \in \mathbb{N}, \quad F := \mathbf{1}_{\{1\}}.$$

Then for every finite-valued completely multiplicative function $f \colon \mathbb{N} \to \mathbb{S}^1$, we have

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N], m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, F(f(k\,(m^2 - n^2))) \cdot F(f(k\, 2mn)) \cdot F(f(k\,(m^2 + n^2))) > 0.$$

Therefore, it remains to verify the assumption of this result. For this purpose, we will make crucial use of the following concentration estimates, which easily follow from Propositions 2.5 and 2.11, as we will see later.

Corollary 2.17. *Let $f \colon \mathbb{N} \to \mathbb{U}$ be a finite-valued multiplicative function such that $f \sim \chi$ for some Dirichlet character χ with period q. Then for every $\varepsilon > 0$, there exists $Q_0 = Q_0(f, \varepsilon) \in \mathbb{N}$ such that the following holds:*

1. *For all $Q \in \mathbb{N}$ such that $Q_0 \mid Q$, we have*

$$\limsup_{N\to\infty} \mathbb{E}_{n\in[N]} \big| f(Qn + 1) - 1 \big| \ll \varepsilon,$$

where the implicit constant is absolute.

2. *For all $Q \in \mathbb{N}$ such that $Q_0 \mid Q$, we have*

$$\limsup_{N\to\infty} \mathbb{E}_{m,n\in[N]} \big| f\big((Qm + 1)^2 + (Qn)^2\big) - 1 \big| \ll \varepsilon,$$

where the implicit constant is absolute.

Finally, using the previous concentration estimates and the key maneuver of taking multiplicative averages over $Q \in \mathbb{N}$, which was also a crucial element in the proof of Theorem 2.2, we verify the assumptions of Proposition 2.16.

Proposition 2.18. *Let $f\colon \mathbb{N}\to\mathbb{S}^1$ be a finite-valued pretentious multiplicative function and $\tilde{\chi}\colon \mathbb{N}\to\mathbb{S}^1$ be a modified Dirichlet character. Then*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, A(k\,(m^2-n^2))\cdot A(k\,2mn)\cdot A(k\,(m^2+n^2))>0,$$

where

$$A(n):=F(f(n))\cdot F(\tilde{\chi}(n)),\quad n\in\mathbb{N},\quad F:=\mathbf{1}_{\{1\}}.$$

Thus, to prove Theorem 2.14, it remains to verify Propositions 2.16 and 2.18. We do this in Sections 7 and 8 (the other results mentioned in this subsection are needed in the proofs of these two results and will also be verified).

3. Background and preparation

3.1. Some elementary facts

We will use the following elementary property.

Lemma 3.1. *Let $a\colon\mathbb{Z}\to\mathbb{U}$ be an even sequence and $l_1,l_2\in\mathbb{Z}$, not both of them 0. Suppose that for some $\varepsilon>0$ and for some sequence $L_N\colon\mathbb{N}\to\mathbb{U}$, we have*

$$\limsup_{N\to\infty}\mathbb{E}_{n\in[N]}|a(n)-L_N|\le\varepsilon.$$

Then

$$\limsup_{N\to\infty}\mathbb{E}_{m,n\in[N]}|a(l_1m+l_2n)-L_{lN}|\le 2l\varepsilon,$$

where $l:=|l_1|+|l_2|$.

Proof. We have

$$\mathbb{E}_{m,n\in[N]}|a(l_1m+l_2n)-L_{lN}|\le\frac{1}{N^2}\sum_{|k|\le lN} w_N(k)\,|a(k)-L_{lN}|, \tag{3.1}$$

where for $k\in\mathbb{Z}$, we let

$$w_N(k):=|\{(m,n)\in[N]^2\colon l_1m+l_2n=k\}|.$$

For every $k\in\mathbb{Z}$ and $m\in[N]$, there exists at most one $n\in[N]$ for which $l_1m+l_2n=k$. Hence, $|w_N(k)|\le N$ for every $k\in\mathbb{Z}$. Since a is even, we deduce that the right-hand side in (3.1) is bounded by

$$2\,l\cdot\mathbb{E}_{k\in[lN]}|a(k)-L_{lN}|.$$

The asserted estimate now follows from this and our assumption. □

The next well-known property of multiplicative functions will also be used several times.

Lemma 3.2. *Let (Φ_K) be a multiplicative Følner sequence. If $f\colon\mathbb{N}\to\mathbb{U}$ is a completely multiplicative function and $f\ne 1$, then*

$$\lim_{K\to\infty}\mathbb{E}_{n\in\Phi_K} f(n)=0.$$

Proof. Since $f \neq 1$, there exists $p \in \mathbb{P}$ such that $f(p) \neq 1$. By the definition of Φ_K, we have

$$\lim_{K\to\infty} \frac{|\Phi_K \cap (p \cdot \Phi_K)|}{|\Phi_K|} = 1.$$

From this and the fact that $f(pn) = f(p) \cdot f(n)$, we get

$$\mathbb{E}_{n\in\Phi_K} f(n) = \mathbb{E}_{n\in p\cdot\Phi_K} f(n) + o_{K\to\infty}(1) = f(p) \cdot \mathbb{E}_{n\in\Phi_K} f(n) + o_{K\to\infty}(1).$$

Since $f(p) \neq 1$, we deduce that $\mathbb{E}_{n\in\Phi_K} f(n) = o_{K\to\infty}(1)$. □

3.2. Some useful weights

In the proof of Theorems 1.1 and 1.2, we will utilize weighted averages. The weights are employed to ensure that the averages $\mathbb{E}_{m,n\in[N]} A_\delta(f, Q, m, n)$ and $\mathbb{E}_{m,n\in[N]} B_\delta(f, Q, m, n)$, where A_δ, B_δ are as in (2.6), (2.16), respectively, have a positive real part if f is an Archimedean character and δ is sufficiently small.

We will now define these weights. If $\delta \in (0, 1/2)$, we consider the circular arc with center 1 given by

$$I_\delta := \{e(\phi) \colon \phi \in (-\delta, \delta)\}.$$

Lemma 3.3. *For every $\delta \in (0, 1/2)$, let $F_\delta \colon \mathbb{S}^1 \to [0, 1]$ be the trapezoid function that is equal to 1 on the arc $I_{\delta/2}$ and 0 outside the arc I_δ. Let also*

$$w_\delta(m, n) := F_\delta\big((\ell(m^2 - n^2))^i \cdot (\ell' mn)^{-i}\big) \cdot \mathbf{1}_{m>n}, \quad m, n \in \mathbb{N}, \tag{3.2}$$

and

$$\tilde{w}_\delta(m, n) := F_\delta\big((\ell(m^2 + n^2))^i \cdot (\ell' mn)^{-i}\big), \quad m, n \in \mathbb{N}. \tag{3.3}$$

Then

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, w_\delta(m, n) > 0 \ \text{ and } \ \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \tilde{w}_\delta(m, n) > 0.$$

Remark. We opted for a continuous function for F_δ instead of an indicator function, to make it easier to prove Propositions 4.1 and 6.1 later on.

Proof. We first cover the weight in (3.2). Note that the limit we want to evaluate is equal to

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, F_\delta\big((\ell((m/N)^2 - (n/N)^2))^i \cdot (\ell'(m/N) \cdot (n/N))^{-i}\big) \cdot \mathbf{1}_{m/N>n/N}.$$

Let $\tilde{F}_\delta \colon [0, 1] \times [0, 1] \to [0, 1]$ be given by

$$\tilde{F}_\delta(x, y) := F_\delta\big((\ell(x^2 - y^2))^i \cdot (\ell' xy)^{-i}\big) \cdot \mathbf{1}_{x>y}, \quad x, y \in [0, 1].$$

Then $\tilde{F}_\delta$ is Riemann integrable on $[0, 1] \times [0, 1]$ as it is bounded and continuous except for a set of Lebesgue measure 0. Hence, the limit we aim to compute exists and is equal to the Riemann integral

$$\int_0^1 \int_0^1 \tilde{F}_\delta(x, y)\, dx\, dy.$$

It remains to show that this integral is positive, and since $\tilde{F}_\delta$ is nonnegative, it suffices to show that $\tilde{F}_\delta$ does not vanish almost everywhere.

To verify the nonvanishing property, note that if $x = ay$ where $a := \frac{\ell'+\sqrt{(\ell')^2+4\ell^2}}{2\ell} > 1$, then $x > y$ and $\ell(x^2 - y^2) = \ell' xy$, and as a consequence, $(\ell(x^2 - y^2))^i \cdot (\ell' xy)^{-i} = 1$. Hence, $\tilde{F}_\delta(x, y) = F_\delta(1) = 1$ on the line $x = ay$. Since $\tilde{F}_\delta$ is continuous in the region $x > y$, this proves that it stays bounded away from zero in a neighborhood of the line $x = ay$ in that region, and hence, it does not vanish almost everywhere. This completes the proof for the weight (3.2).

The argument for the second weight (3.3) is very similar, so we only summarize it. Let $\tilde{F}_\delta\colon [0, 1] \times [0, 1] \to [0, 1]$ be given by

$$\tilde{F}_\delta(x, y) := F_\delta\big((\ell(x^2 + y^2))^i \cdot (\ell' xy)^{-i}\big) \cdot \mathbf{1}_{(0,1]\times(0,1]}(x, y).$$

Then the limit we want to evaluate exists and is equal to the Riemann integral

$$\int_0^1 \int_0^1 \tilde{F}_\delta(x, y)\, dx\, dy.$$

The integral is positive because $\tilde{F}_\delta$ is nonnegative and does not vanish almost everywhere. To verify the nonvanishing property, we argue as follows. Pick $k \in \mathbb{Z}_+$ such that $b := \ell'/\ell \cdot e^{2k\pi} > 2$ and let $a := \frac{b+\sqrt{b^2-4}}{2}$. If $x, y \in [0, 1]$ are such that $x = ay$, then $x > y$ and $\ell(x^2 + y^2) = e^{2k\pi}\, \ell' xy$, which implies $(\ell(x^2 + y^2))^i \cdot (\ell' xy)^{-i} = e^{2k\pi i} = 1$. □

3.3. *Multiplicative functions*

We record here some basic notions and facts about multiplicative functions that will be used throughout the article.

3.3.1. Dirichlet characters

A *Dirichlet character* χ is a periodic completely multiplicative function and is often thought of as a multiplicative function on $\mathbb{Z}_m$ for some $m \in \mathbb{N}$. In this case, χ takes the value 0 on integers that are not coprime to m and takes values on $\phi(m)$-roots of unity on all other integers, where ϕ is the Euler totient function. If χ is a Dirichlet character, we define the *modified Dirichlet character* $\tilde{\chi}\colon \mathbb{N} \to \mathbb{S}^1$ to be the completely multiplicative function satisfying

$$\tilde{\chi}(p) := \begin{cases} \chi(p), & \text{if } \chi(p) \neq 0 \\ 1, & \text{if } \chi(p) = 0. \end{cases}$$

We note in passing that the level sets of modified Dirichlet characters $\tilde{\chi}$, which can be seen as finite colorings of $\mathbb{N}$, are precisely the colorings that appear in Rado's theorem when showing that certain systems of linear equations are not partition regular. In particular, a system of linear equations is partition regular if and only if it has a monochromatic solution in any coloring realized by a modified Dirichlet character.

3.3.2. Distance between multiplicative functions

Following Granville and Soundararajan [25, 27], in this and the next subsection, we define a distance and a related notion of pretentiousness between multiplicative functions. If $f, g\colon \mathbb{N} \to \mathbb{U}$ are multiplicative functions and $x, y \in \mathbb{R}_+$ with $x < y$, we let

$$\mathbb{D}(f, g; x, y)^2 := \sum_{x<p\leq y} \frac{1}{p}\,(1 - \Re(f(p) \cdot \overline{g(p)})). \tag{3.4}$$

We also let

$$\mathbb{D}(f,g)^2 := \sum_{p\in\mathbb{P}} \frac{1}{p}\,(1 - \Re(f(p)\cdot\overline{g(p)})). \tag{3.5}$$

Note that if $|f| = |g| = 1$, then

$$\mathbb{D}(f,g)^2 = \frac{1}{2}\cdot\sum_{p\in\mathbb{P}} \frac{1}{p}\,|f(p) - g(p)|^2.$$

It can be shown (see [26] or [27, Section 2.1.1]) that $\mathbb{D}$ satisfies the triangle inequality

$$\mathbb{D}(f,g) \le \mathbb{D}(f,h) + \mathbb{D}(h,g)$$

for all $f,g,h\colon \mathbb{P}\to\mathbb{U}$. Also, for all $f_1, f_2, g_1, g_2\colon \mathbb{P}\to\mathbb{U}$, we have (see [25, Lemma 3.1])

$$\mathbb{D}(f_1 f_2, g_1 g_2) \le \mathbb{D}(f_1,g_1) + \mathbb{D}(f_2,g_2). \tag{3.6}$$

3.3.3. Pretentious multiplicative functions

If $f,g\colon \mathbb{N}\to\mathbb{U}$ are multiplicative functions, we say that f *pretends to be* g, and write $f \sim g$, if $\mathbb{D}(f,g) < +\infty$. It follows from (3.6) that if $f_1 \sim g_1$ and $f_2 \sim g_2$, then $f_1 f_2 \sim g_1 g_2$. We say that f is *pretentious* if $f \sim \chi\cdot n^{it}$ for some $t\in\mathbb{R}$ and Dirichlet character χ, in which case

$$\sum_{p\in\mathbb{P}} \frac{1}{p}\,(1 - \Re(f(p)\cdot\overline{\chi(p)}\cdot p^{-it})) < +\infty.$$

The value of t is uniquely determined; this follows from (3.6) and the fact that $n^{it} \nsim \chi$ for every nonzero $t\in\mathbb{R}$ and Dirichlet character χ (see, for example, [27, Corollary 11.4] or [26, Proposition 7]).

Although real-valued or finite-valued multiplicative functions always have a mean value, we caution the reader that this is not the case for general multiplicative functions with values on the unit circle. For example, we have

$$\mathbb{E}_{n\in[N]}\, n^{it} = N^{it}/(1+it) + o_N(1),$$

so we have non-convergent means when $t \neq 0$. But even multiplicative functions satisfying $f \sim 1$ can have non-convergent means. In particular, if $f \sim 1$ is a completely multiplicative function, then it is known (see, for example, [17, Theorems 6.2]) that there exists $c \neq 0$ such that

$$\mathbb{E}_{n\in[N]}\, f(n) = c\cdot e(A(N)) + o_N(1),$$

where $A(N) := \sum_{p\le N} \frac{1}{p}\,\Im(f(p))$, $N\in\mathbb{N}$. Hence, we have non-convergent means when, for example,

$$\sum_{p\in\mathbb{P}} \frac{1}{p}\,\Im(f(p)) = +\infty,$$

which is the case if $f(p) := e(1/\log\log p)$, $p\in\mathbb{P}$. This oscillatory behavior of the mean values of some complex-valued multiplicative functions has to be taken into account and will cause problems in the proofs of some of our main results.

Finally, we record an observation that will only be used in the proof of Theorem 1.5.

Lemma 3.4. *Let $f\colon \mathbb{N}\to\mathbb{U}$ be a pretentious finite-valued multiplicative function. Then $f\sim\chi$ for some Dirichlet character χ and*

$$\sum_{p\in\mathbb{P}}\frac{1}{p}|1-f(p)\cdot\overline{\chi(p)}|<+\infty. \tag{3.7}$$

Remark. It can be shown using (3.7) that finite-valued pretentious multiplicative functions always have convergent means.

Proof. Since f is pretentious, we have $f\sim\chi\cdot n^{it}$ for some $t\in\mathbb{R}$ and Dirichlet character χ. Then $\mathbb{D}(n^{it},g)<+\infty$, where $g:=f\cdot\overline{\chi}$ is a finite-valued multiplicative function. In particular, there exists $d\in\mathbb{N}$ for which g^d is the constant 1, so from (3.6), it follows that $\mathbb{D}(n^{idt},1)<+\infty$, which in turn implies that $t=0$ (hence, $f\sim\chi$) and

$$\sum_{p\in\mathbb{P}\colon f(p)\cdot\overline{\chi(p)}\neq 1}\frac{1}{p}<+\infty.$$

Hence,

$$\sum_{p\in\mathbb{P}}\frac{1}{p}|\Im(f(p)\cdot\overline{\chi(p)})|<+\infty.$$

If we combine this with $\mathbb{D}(f,\chi)<+\infty$, we deduce that (3.7) holds. □

3.3.4. Aperiodic multiplicative functions

We say that a multiplicative function $f\colon\mathbb{N}\to\mathbb{U}$ is *aperiodic* if for every $a,b\in\mathbb{N}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(an+b)=0.$$

The following well-known result of Daboussi-Delange [13, Corollary 1] states that a multiplicative function is aperiodic if and only if it is non-pretentious.

Lemma 3.5. *Let $f\in\mathcal{M}$. Then either $f\sim\chi\cdot n^{it}$ for some Dirichlet character χ and $t\in\mathbb{R}$, or f is aperiodic.*

In our arguments, we typically distinguish two cases – one where a multiplicative function is aperiodic; then we show that the expressions we are interested in vanish. The complementary one where the multiplicative function is pretentious is treated using concentration estimates.

3.4. Some Borel measurability results

Recall that $\mathcal{M}$ is equipped with the topology of pointwise convergence. In the proof of Theorem 2.2, we require certain Borel measurability properties of subsets of $\mathcal{M}$ and related maps. The second property proved below will only be used in the proof of part (2) of Theorem 2.2.

Recall that if f is pretentious, then there exist a unique $t=t_f\in\mathbb{R}$ and a Dirichlet character χ such that $f\sim\chi\cdot n^{it}$.

Lemma 3.6.

1. *The set $\mathcal{M}_p$ of pretentious completely multiplicative functions is Borel.*
2. *The map $f\mapsto t_f$ from $\mathcal{M}_p$ to $\mathbb{R}$ is Borel measurable.*

Proof. We prove (1). For $a, b \in \mathbb{N}$, we let $M_{a,b}$ be the set of $f \in \mathcal{M}$ such that

$$\limsup_{N\to\infty} |\mathbb{E}_{n\in[N]} f(an+b)| > 0.$$

Clearly, $M_{a,b}$ is a Borel subset of $\mathcal{M}$. By Lemma 3.5, we have $\mathcal{M}_p = \bigcup_{a,b\in\mathbb{N}} M_{a,b}$, and the result follows.

We prove (2). By [32, Theorem 14.12], it suffices to show that the graph

$$\Gamma := \{(f, t_f) \in \mathcal{M}_p \times \mathbb{R}\}$$

is a Borel subset of $\mathcal{M}_p \times \mathbb{R}$. If χ_k, $k \in \mathbb{N}$, is an enumeration of all Dirichlet characters, and

$$\Gamma_k := \{(f, t_f) \in \mathcal{M}_p \times \mathbb{R} \colon f \sim \chi_k \cdot n^{it_f}\},$$

then

$$\Gamma = \bigcup_{k\in\mathbb{N}} \Gamma_k.$$

Hence, it suffices to show that for every $k \in \mathbb{N}$, the set Γ_k is Borel. Note that

$$\Gamma_k := \{(f, t) \in \mathcal{M}_p \times \mathbb{R} \colon \mathbb{D}(f, \chi_k \cdot n^{it}) < \infty\}.$$

Since for $k \in \mathbb{N}$ the map $(f, t) \mapsto \mathbb{D}(f, \chi_k \cdot n^{it})$ is clearly Borel, the set Γ_k is Borel. This completes the proof. □

4. Type I Pythagorean pairs

As explained in Section 2.2, in order to complete the proof of Theorem 2.3 (and thus of part (1) of Theorem 2.2), it remains to prove Proposition 2.4, Lemma 2.6 and Lemma 2.8. We do this in this section.

We start with Proposition 2.4, which we state here in an equivalent form.

Proposition 4.1. *Let $f : \mathbb{N} \to \mathbb{U}$ be an aperiodic completely multiplicative function, let $\ell, \ell', Q \in \mathbb{N}$ and $\delta > 0$. Then, with $w_\delta : \mathbb{N}^2 \to [0, 1]$ described by (3.2), we have*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, w_\delta(m, n) \cdot f(\ell(Qm + 1)^2 - (Qn)^2) \cdot \overline{f(\ell'(Qm + 1)(Qn))} = 0. \tag{4.1}$$

Furthermore, the limit in (4.1) exists for all multiplicative functions $f : \mathbb{N} \to \mathbb{U}$.

Proof. Recall that

$$w_\delta(m, n) := F_\delta\big((\ell(m^2 - n^2))^i \cdot (\ell' mn)^{-i}\big) \cdot \mathbf{1}_{m>n}, \quad m, n \in \mathbb{N},$$

where $F_\delta : \mathbb{S}^1 \to [0, 1]$ is the continuous function defined in Lemma 3.3. Since F_δ can be approximated uniformly by polynomials, using linearity, we deduce that it suffices to verify (4.1) with $w_\delta(m, n)$ replaced by $(m^2 - n^2)^{ki} \cdot (mn)^{-ki} \cdot \mathbf{1}_{m>n}$ for arbitrary $k \in \mathbb{Z}$. Furthermore, since $\lim_{n\to\infty}(\log(Qn + 1) - \log(Qn)) = 0$, the limit in (4.1) remains unchanged if we replace $(m^2 - n^2)^{ki} \cdot (mn)^{-ki} \cdot \mathbf{1}_{m>n}$ with $((Qm+1)^2 - (Qn)^2)^{ki} \cdot ((Qm+1)(Qn))^{-ki} \cdot \mathbf{1}_{m>n}$ (after we omit $+1$, the Q's are going to cancel because of the conjugate). Hence, in order to establish (4.1), it suffices to show that for every $k \in \mathbb{Z}$, we have

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \mathbf{1}_{Qm+1>Qn} \cdot f_k((Qm + 1)^2 - (Qn)^2) \cdot \overline{f_k((Qm + 1)(Qn))} = 0, \tag{4.2}$$

where $f_k(n) := f(n) \cdot n^{ki}$, $n \in \mathbb{N}$. Note that since the indicator function of an arithmetic progression is a linear combination of Dirichlet characters, in order to establish (4.2), it suffices to show that for all Dirichlet characters χ, χ', we have

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \mathbf{1}_{m>n} \cdot \chi(m) \cdot \chi'(n) \cdot f_k(m^2 - n^2) \cdot \overline{f_k(mn)} = 0,$$

or, equivalently, that

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \mathbf{1}_{m>n} \cdot f_k(m^2 - n^2) \cdot (\overline{f_k} \cdot \chi)(m) \cdot (\overline{f_k} \cdot \chi')(n) = 0. \tag{4.3}$$

Since f is aperiodic, so is f_k. Since f is aperiodic, so is $\overline{f_k} \cdot \chi$ (and $\overline{f_k} \cdot \chi'$). Combining [21, Theorem 2.5] and [21, Lemma 9.6], we deduce that (4.3) holds, completing the proof.

Finally, to prove convergence for all multiplicative functions, we argue as before, using the fact that convergence in the case $w_\delta = 1$ follows from [20, Theorem 1.4]. We note that although [20, Theorem 1.4] only covers the case without the weight $\mathbf{1}_{m>n}$, exactly the same argument can be used to cover this weighted variant. □

Next, we restate and prove Lemma 2.6. Recall that A_δ, $\mathcal{M}_p$ and $\mathcal{A}$ were defined in (2.6), (2.9) and (2.13), respectively.

Lemma 2.6 . *Let $f \in \mathcal{M}_p \setminus \mathcal{A}$, $\delta > 0$, $\ell, \ell' \in \mathbb{N}$, and let $(\Phi_K)_{K\in\mathbb{N}}$ be the Følner sequence described in (2.11). Then*

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K} \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, A_\delta(f, Q; m, n) = 0. \tag{4.4}$$

Proof. Let $\delta > 0$ and $f \in \mathcal{M}_p \setminus \mathcal{A}$. Then for some $t \in \mathbb{R}$ and Dirichlet character χ, we have

$$f(n) = n^{it} \cdot g(n), \quad \text{where } g \sim \chi,\ g \neq 1. \tag{4.5}$$

For reasons that will become clear later, for $\delta > 0$ and $Q \in \mathbb{N}$, let

$$\tilde{L}_\delta(f, Q) := f(Q) \cdot Q^{-it} \cdot \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, A_\delta(f, Q; m, n). \tag{4.6}$$

Note that the limit in the definition of $\tilde{L}_\delta(f, Q)$ exists by the second part of Proposition 4.1. The idea to prove (4.4) is to show that $\tilde{L}_\delta(f, Q)$ does not depend strongly on Q (it depends only on the prime factors of Q), so that, as a function of Q, it is orthogonal to any nontrivial completely multiplicative function with respect to multiplicative averages. Since the left-hand side of (4.4) is the correlation between $\tilde{L}(f, Q)$ and the completely multiplicative function $Q \mapsto f(Q) \cdot Q^{-it}$, which is nontrivial by (4.5), the conclusion will follow.

Fix $\varepsilon > 0$ and take $K_0 = K_0(\varepsilon, f)$ so that

$$\sum_{p \geq K_0} \frac{1}{p}\big(1 - \Re(f(p) \cdot \overline{\chi(p)} \cdot p^{-it})\big) + K_0^{-1/2} \leq \varepsilon.$$

Using Proposition 2.5 (and noting that the function $K \mapsto \mathbb{D}(f, \chi \cdot n^{it}; K, N)$ is decreasing for any fixed f and N), it follows that for every $N > K > K_0$ and $Q \in \Phi_K$,

$$\mathbb{E}_{n\in[N]} \big| f(Qn+1) - (Qn)^{it} \cdot \exp\big(F_N(f, K)\big)\big| \ll \varepsilon. \tag{4.7}$$

Using this identity and Lemma 3.1 with $a(n) := f(Qn+1) \cdot (Qn)^{-it}$ and $l_1 = 1$, $l_2 = -1$, it follows that

$$\limsup_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n} \Big| f(Q(m-n)+1) - \big(Q(m-n)\big)^{it} \exp\big(F_{2N}(f, K)\big)\Big| \ll \varepsilon. \tag{4.8}$$

Using (4.7) and Lemma 3.1 with $a(n) := f(Qn+1)\cdot(Qn)^{-it}$ and $l_1 = l_2 = 1$, it follows that

$$\limsup_{N\to\infty} \mathbb{E}_{m,n\in[N]} \Big| f(Q(m+n)+1) - \big(Q(m+n)\big)^{it} \exp\big(F_{2N}(f,K)\big)\Big| \ll \varepsilon. \tag{4.9}$$

Combining (4.7), (4.8), (4.9), and since all terms involved are 1-bounded, we deduce that for every $K > K_0$ and $Q \in \Phi_K$,

$$\limsup_{N\to\infty} \ \mathbb{E}_{m,n\in[N],m>n} \Big| f\big((Qm+1)^2 - (Qn)^2\big) \cdot \overline{f(Qm+1)} \\ - Q^{it}\cdot (m^2-n^2)^{it}\cdot m^{-it}\cdot \exp(2F_{2N}(f,K))\cdot \overline{\exp(F_N(f,K))}\Big| \ll \varepsilon.$$

Multiplying by $c_{\ell,\ell'}\cdot w_\delta(m,n)\cdot \overline{f(Qn)}\cdot Q^{-it}\cdot f(Q) = c_{\ell,\ell'}\cdot w_\delta(m,n)\cdot\overline{f(n)}\cdot Q^{-it}$, where $c_{\ell,\ell'} := f(\ell)\cdot\overline{f(\ell')}$, we deduce that

$$\limsup_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n} \Big| A_\delta(f,Q;m,n)\cdot Q^{-it}\cdot f(Q) \\ - c_{\ell,\ell'}\cdot w_\delta(m,n)\cdot (m^2-n^2)^{it}\cdot m^{-it}\cdot \overline{f(n)}\cdot \exp(2F_{2N}(f,K))\cdot\overline{\exp(F_N(f,K))}\Big| \ll \varepsilon.$$

This implies that for every $K > K_0$,

$$\limsup_{N\to\infty}\ \sup_{Q\in\Phi_K} \Big| \tilde{L}_\delta(f,Q) - \\ c_{\ell,\ell'}\cdot \mathbb{E}_{m,n\in[N]}\, w_\delta(m,n)\cdot (m^2-n^2)^{it}\cdot m^{-it}\cdot\overline{f(n)}\cdot\exp(2F_{2N}(f,K))\cdot\overline{\exp(F_N(f,K))}\Big| \ll \varepsilon.$$

Since the second term does not depend on Q, we conclude that for every $K > K_0$ and $Q, Q' \in \Phi_K$, $\big|\tilde{L}_\delta(f,Q) - \tilde{L}_\delta(f,Q')\big| \ll \varepsilon$. We can choose ε arbitrarily small by sending $K\to\infty$, so it follows that

$$\lim_{K\to\infty}\ \max_{Q,Q'\in\Phi_K} \big|\tilde{L}_\delta(f,Q) - \tilde{L}_\delta(f,Q')\big| = 0.$$

For $K\in\mathbb{N}$, let Q_K be any element of Φ_K. From the last identity and (4.6), it follows that

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K} \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, A_\delta(f,Q;m,n) = \lim_{K\to\infty} \tilde{L}_\delta(f,Q_K)\cdot \mathbb{E}_{Q\in\Phi_K}\overline{f(Q)}\cdot Q^{it}.$$

By (4.5), we have that $Q\mapsto f(Q)\cdot Q^{-it}$ is a nontrivial multiplicative function; hence, the last limit is zero by Lemma 3.2. This establishes (4.4) and completes the proof. □

Lastly, we restate and prove Lemma 2.8.

Lemma 2.8. *Let σ be a Borel probability measure on $\mathcal{M}_p$ such that $\sigma(\{1\}) > 0$ and let $\mathcal{A}$ be as in (2.13). Then there exist $\delta_0, \rho_0 > 0$, depending only on σ, such that*

$$\liminf_{N\to\infty}\ \inf_{Q\in\mathbb{N}} \Re\Big(\mathbb{E}_{m,n\in[N]} \int_{\mathcal{A}} A_{\delta_0}(f,Q;m,n)\, d\sigma(f)\Big) \geq \rho_0. \tag{4.10}$$

Proof. Let $a := \sigma(\{1\}) > 0$ and for $\delta > 0$, let

$$\mu_\delta := \lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, w_\delta(m,n).$$

Note that by Lemma 3.3, we have $\mu_\delta > 0$. For $T\in\mathbb{R}_+$, we consider the sets

$$\mathcal{A}_T := \{(n^{it})_{n\in\mathbb{N}} \colon t\in[-T,T]\}.$$

These sets are closed and, as a consequence, Borel. Since $\mathcal{A}_N$ increases to $\mathcal{A}$ as $N \to \infty$, and the Borel measure σ is finite, there exists $T_0 = T_0(\sigma) > 0$ such that

$$\sigma(\mathcal{A} \setminus \mathcal{A}_{T_0}) \leq \frac{a}{2}. \tag{4.11}$$

Note also that since $\lim_{n\to\infty} \sup_{Q\in\mathbb{N}} |\log(Qn+1) - \log(Qn))| = 0$, we have

$$\lim_{N\to\infty} \sup_{f\in\mathcal{A}_{T_0}, Q\in\mathbb{N}} \mathbb{E}_{m,n\in[N],m>n} \Big| f(\ell((Qm+1)^2 - (Qn)^2)) \cdot \overline{f(\ell'(Qm+1)(Qn))} - f(\ell(m^2-n^2)) \cdot \overline{f(\ell' mn)} \Big| = 0,$$

and by the definition of w_δ given in Lemma 3.3, we have

$$\lim_{\delta\to 0^+} \limsup_{N\to\infty} \sup_{f\in\mathcal{A}_{T_0}} \Big| \mathbb{E}_{m,n\in[N]}\, w_\delta(m,n) \cdot f(\ell(m^2-n^2)) \cdot \overline{f(\ell' mn)} - \mathbb{E}_{m,n\in[N]}\, w_\delta(m,n) \Big| = 0.$$

We deduce from the last two identities that if δ_0 is small enough (depending only on T_0 and hence only on σ), then for every $Q \in \mathbb{N}$, we have

$$\liminf_{N\to\infty} \inf_{Q\in\mathbb{N}} \Re\Big(\mathbb{E}_{m,n\in[N]} \int_{\mathcal{A}_{T_0}} A_{\delta_0}(f,Q;m,n)\, d\sigma(f) \Big) \geq \frac{\sigma(\mathcal{A}_{T_0}) \cdot \mu_{\delta_0}}{2} \geq \frac{a\cdot\mu_{\delta_0}}{2},$$

where we used that $1 \in \mathcal{M}_{T_0}$; hence, $\sigma(\mathcal{A}_{T_0}) \geq \sigma(\{1\}) = a$. However, using (4.11) and the triangle inequality, we get

$$\limsup_{N\to\infty} \sup_{Q\in\mathbb{N}} \Big| \mathbb{E}_{m,n\in[N]} \int_{\mathcal{A}\setminus\mathcal{A}_{T_0}} A_{\delta_0}(f,Q;m,n)\, d\sigma(f) \Big| \leq \frac{a\cdot\mu_{\delta_0}}{4}.$$

Combining the last two estimates, we deduce that (4.10) holds with $\rho_0 := \frac{a\cdot\mu_{\delta_0}}{4}$. □

5. Nonlinear concentration estimates

Our goal is to prove the concentration estimate of Proposition 2.11, which is a crucial ingredient in the proof of part (2) of Theorem 1.2 and in the proof of Theorem 1.5. In fact, we will prove a more general and quantitatively more explicit statement with further applications in mind.

Let $f, g\colon \mathbb{N} \to \mathbb{U}$ be multiplicative functions and let χ be a Dirichlet character and $t \in \mathbb{R}$. For every $K_0 \in \mathbb{N}$, we let

$$G_N(f, K_0) := 2 \sum_{\substack{K_0 < p \leq N, \\ p\equiv 1 \pmod 4}} \frac{1}{p} \left(f(p) \cdot \overline{\chi(p)} \cdot n^{-it} - 1 \right) \tag{5.1}$$

and

$$\mathbb{D}_1(f,g;x,y)^2 := \sum_{\substack{x<p\leq y, \\ p\equiv 1 \pmod 4}} \frac{1}{p} \left(1 - \Re(f(p)\cdot\overline{g(p)})\right). \tag{5.2}$$

Proposition 5.1. *Let $K_0, N \in \mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{U}$ be a multiplicative function. Let also $t \in \mathbb{R}$, χ be a Dirichlet character with period q, $Q = \prod_{p\leq K_0} p^{a_p}$ for some $a_p \in \mathbb{N}$, and suppose that $q \mid Q$. If N is large enough, depending only on Q and t, then for all $a, b \in \mathbb{Z}$ with $-Q \leq a, b \leq Q$ and $(a^2+b^2, Q) = 1$, we have*

$$\mathbb{E}_{m,n\in[N]}\left|f\left((Qm+a)^2+(Qn+b)^2\right)-\chi(a^2+b^2)\cdot Q^{2it}\cdot(m^2+n^2)^{it}\cdot\exp\left(G_N(f,K_0)\right)\right|\ll$$
$$(\mathbb{D}_1+\mathbb{D}_1^2)(f,\chi\cdot n^{it};K_0,\sqrt{N})+Q^2\cdot\mathbb{D}_1(f,\chi\cdot n^{it};N,3Q^2N^2)+Q\cdot\mathbb{D}_1(f,\chi\cdot n^{it};\sqrt{N},N)+K_0^{-1/2},\tag{5.3}$$

where $G_N, \mathbb{D}_1$ are as in (5.1), (5.2), and the implicit constant is absolute.

Remarks. ∘ Note that if $f\sim\chi\cdot n^{it}$, we have $\lim_{N\to\infty}\mathbb{D}_1(f,\chi\cdot n^{it};N,3Q^2N^2)=0$ and $\lim_{N\to\infty}\mathbb{D}_1(f,\chi\cdot n^{it};\sqrt{N},N)=0$. Hence, renaming K_0 as K, taking the max over all $Q\in\Phi_K$, and then letting $N\to\infty$ in (5.3) gives the estimate in Proposition 2.11.

∘ The averaging over both variables $m,n\in\mathbb{N}$ is crucial for our argument and allows to overcome issues with large primes. In fact, by slightly modifying the example of [34, Lemma 2.1], one can construct completely multiplicative functions (both pretentious and aperiodic) $f:\mathbb{N}\to\{-1,1\}$, such that for every $a\in\mathbb{Z}_+$, the averages

$$\mathbb{E}_{n\in[N]}\,f((Qn+a)^2+1)$$

behave rather 'erratically', and a similar concentration estimate fails. However, in [48], Teräväinen proved a version of the concentration estimates for values of $f(P(Qn+a))$, where $P\in\mathbb{Z}[x]$ is arbitrary and the multiplicative functions f satisfies $f(p)=p^{it}\chi(p)$ for $p>N$ (with a somewhat more particular choice of Q). In our setting, however, we cannot afford to make such assumptions on f.

The proof is carried out in several steps, covering progressively more general settings. Throughout the argument, we write $p\ ||\ n$ if $p\mid n$ but $p^2\nmid n$.

5.1. *Preparatory counting arguments*

The following lemma will be used multiple times subsequently.

Lemma 5.2. *For $Q,N\in\mathbb{N}$, $a,b\in\mathbb{Z}$, and primes p,q such that $p,q\equiv 1 \pmod 4$ and $(pq,Q)=1$, let*

$$w_{N,Q}(p,q):=\frac{1}{N^2}\sum_{\substack{m,n\in[N],\\ p,q\,||\,(Qm+a)^2+(Qn+b)^2}}1.\tag{5.4}$$

Then

$$w_{N,Q}(p,p)=\frac{2}{p}\Big(1-\frac{1}{p}\Big)^2+O\Big(\frac{1}{N}\Big),\tag{5.5}$$

and if $p\neq q$, we have

$$w_{N,Q}(p,q)=\frac{4}{pq}\Big(1-\frac{1}{p}\Big)^2\Big(1-\frac{1}{q}\Big)^2+O\Big(\frac{1}{N}\Big),\tag{5.6}$$

where the implicit constants are absolute.

Remark. We deduce the approximate identity

$$w_{N,Q}(p,q)=w_{N,Q}(p,p)\cdot w_{N,Q}(q,q)+O\Big(\frac{1}{N}\Big),$$

which is crucial for the proof of the concentration estimates. However, because of the $O(\frac{1}{N})$ errors, these approximate identities will only be useful to us for sums that contain $o(N)$ terms.

Proof. Throught the discussion, we use ϵ to designate a number in $\{0,1,2,3,4\}$.

We first establish (5.5). Let p satisfy the assumptions. Note first that if $p \mid Qn+b$ and $p \mid (Qm+a)^2+(Qn+b)^2$, then also $p^2 \mid (Qm+a)^2+(Qn+b)^2$; hence, we get no contribution to the sum (5.4) in this case. So we can assume that $p \nmid Qn+b$. Since $p \equiv 1 \pmod 4$, the number -1 is a quadratic residue mod p, and we have exactly two solutions $m \pmod p$ to the congruence

$$(Qm+a)^2+(Qn+b)^2 \equiv 0 \pmod p. \tag{5.7}$$

Hence, for those $n \in [N]$, we have $2[N/p]+\epsilon$ solutions in the variable $m \in [N]$ to (5.7). Since there are $N-[N/p]+\epsilon$ integers $n \in [N]$ with $p \nmid Qn+b$ (we used that $(p,Q)=1$ here), we get a total of

$$2[N/p]\,(N-[N/p])+O(N) = 2N^2/p - 2N^2/p^2 + O(N)$$

solutions of $m,n \in [N]$ to the congruence (5.7). Similarly, we get that if $p \nmid Qn+b$, then the number of solutions $m,n \in [N]$ to the congruence $(Qm+a)^2+(Qn+b)^2 \equiv 0 \pmod{p^2}$ is

$$2[N/p^2]\,(N-[N/p])+O(N) = 2N^2/p^2 - 2N^2/p^3 + O(N).$$

(We used that -1 is also a quadratic residuemod p^2.) These solutions should be subtracted from the previous solutions of (5.7) in order to count the number of solutions of $m,n \in [N]$ for which $p \mid\mid (Qm+a)^2+(Qn+b)^2$. We deduce that

$$\frac{1}{N^2}\sum_{\substack{m,n\in[N],\\ p\,\mid\mid\,(Qm+a)^2+(Qn+b)^2}} 1 = \frac{2}{p}-\frac{4}{p^2}+\frac{2}{p^3}+O\Big(\frac{1}{N}\Big) = \frac{2}{p}\Big(1-\frac{1}{p}\Big)^2+O\Big(\frac{1}{N}\Big), \tag{5.8}$$

which proves (5.5).

Next, we establish (5.6). Let p,q satisfy the assumptions. As explained in the previous case, those $n \in [N]$ for which $p \mid Qn+b$ or $q \mid Qn+b$ do not contribute to the sum (5.4) defining $w_{N,Q}(p,q)$; hence, we can assume that $(pq, Qn+b)=1$. Let

$$A_{r,s} := \frac{1}{N^2}\sum_{\substack{m,n\in[N],\\ r,s\mid(Qm+a)^2+(Qn+b)^2,\ (rs,Qn+b)=1}} 1$$

and note that

$$w_{N,Q}(p,q) = A_{p,q} - A_{p^2,q} - A_{p,q^2} + A_{p^2,q^2}. \tag{5.9}$$

We first compute $A_{p,q}$. Since $p \equiv q \equiv 1 \pmod 4$, the number -1 is a quadratic residuemod p and mod q, and we get by the Chinese remainder theorem, that for each $n \in [N]$ with $p,q \nmid Qn+b$, we have 4 solutions $m \pmod{pq}$ to the congruence

$$(Qm+a)^2+(Qn+b)^2 \equiv 0 \pmod{pq}.^{8} \tag{5.10}$$

We deduce that for each $n \in [N]$ with $(pq, Qn+b)=1$, we have $4[N/(pq)]+\epsilon$ solutions in the variable $m \in [N]$ to the congruence (5.10). Since the number of $n \in [N]$ for which $(pq,Qn+b)=1$ is $N-[N/p]-[N/q]+[N/pq]$, we get that the total number of solutions to the congruence (5.10) with $m,n \in [N]$ and $(pq,n)=1$ is

$$4[N/(pq)]\,(N-[N/p]-[N/q]+[N/(pq)])+O(N) = N^2\cdot(4/(pq))\cdot(1-1/p-1/q+1/(pq))+O(N). \tag{5.11}$$

[8] If $pq > N$, these may translate to no solutions in $m \in [N]$, but this is also going to be reflected in our computation below since in this case, $4[N/(pq)]+\epsilon = \epsilon$ could very well be 0.

Hence,

$$A_{p,q} := \frac{4}{pq}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)+O\Big(\frac{1}{N}\Big).$$

Similarly, using that -1 is also a quadratic residue mod p^k and mod q^k for $k = 1, 2$, we find that

$$A_{p^2,q} = \frac{4}{p^2q}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)+O\Big(\frac{1}{N}\Big),$$

and

$$A_{p,q^2} = \frac{4}{pq^2}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)+O\Big(\frac{1}{N}\Big).$$

Also,

$$A_{p^2,q^2} = \frac{4}{p^2q^2}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big)+O\Big(\frac{1}{N}\Big).$$

Using the last four identities and (5.9), we deduce that (5.6) holds. This completes the proof. □

We will also need to give upper bounds for $w_{N,Q}(p, q)$ when p, q are not necessarily primes, and also give upper bounds that do not involve the error terms $O(1/N)$ that cause us problems in some cases (this is only relevant when $pq \geq N$). The next lemma is crucial for us and gives an upper bound that is good enough for our purposes.

Lemma 5.3. *For $l, Q, N \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ with $-Q \leq a, b \leq Q$, let*

$$w_{N,Q}(l) := \frac{1}{N^2}\sum_{\substack{m,n\in[N],\\ l\mid(Qm+a)^2+(Qn+b)^2}} 1.$$

If l is a sum of two squares, then

$$w_{N,Q}(l) \ll \frac{Q^2}{l}, \tag{5.12}$$

where the implicit constant is absolute. In particular, if $w_{N,Q}(p, q)$ is as in (5.4), taking $l = p$ and $l = pq$ where p, q are distinct primes of the form 1 (mod 4), *we get*

$$w_{N,Q}(p,p) \ll \frac{Q^2}{p}, \quad w_{N,Q}(p,q) \ll \frac{Q^2}{pq}. \tag{5.13}$$

Remark. These estimates will allow us to show later that the contribution of the $m, n \in [N]$ for which $(Qm + a)^2 + (Qn + b)^2$ have large prime divisors (say $\geq \sqrt{N}$) is negligible for our purposes. In contrast, we could not have done the same for the $n \in [N]$ for which $n^2 + 1$ have large prime divisors.

Proof. Recall that an integer is a sum of two squares if and only if in its factorization as a product of primes, all prime factors congruent to 3 (mod 4) occur with even multiplicity. It follows that if l is a sum of two squares and $l \mid (Qm + a)^2 + (Qn + b)^2$, then the ratio $\big((Qm + a)^2 + (Qn + b)^2\big)/l$ is also a sum of two squares. We deduce from this and our assumption $|a|, |b| \leq Q$ that if l is a sum of two squares, then

$$w_{N,Q}(l) \leq \frac{1}{N^2}\sum_{k\leq 3Q^2N^2/l} r_2(k) \ll \frac{Q^2}{l},$$

where $r_2(k)$ denotes the number of representations of k as a sum of two squares, and to get the second estimate, we used the well-known fact $\sum_{k\le n} r_2(k) \ll n$. This completes the proof. □

5.2. Concentration estimate for additive functions

We start with a concentration estimate for additive functions that will eventually get lifted to a concentration estimate for multiplicative functions.

Definition 5.1. We say that $h\colon \mathbb{N} \to \mathbb{C}$ is *additive* if it satisfies $h(mn) = h(m) + h(n)$ whenever $(m,n) = 1$.

Lemma 5.4 (Turán-Kubilius inequality for sums of squares). *Let $K_0, N \in \mathbb{N}$, $a, b \in \mathbb{Z}$ with $-Q \le a, b, \le Q$, and $h\colon \mathbb{N} \to \mathbb{C}$ be an additive function that is bounded by 1 on primes and such that*

1. $h(p) = 0$ *for all primes* $p \le K_0$ *and* $p > N$;
2. $h(p) = 0$ *for all primes* $p \equiv 3 \pmod 4$;
3. $h(p^k) = 0$ *for all primes* p *and* $k \ge 2$.

Let also $Q = \prod_{p\le K_0} p^{a_p}$ for some $a_p \in \mathbb{N}$. Then for all large enough N, depending only on K_0, we have

$$\mathbb{E}_{m,n\in[N]} \left|h\big((Qm+a)^2 + (Qn+b)^2\big) - H_N(h, K_0)\right|^2 \ll \mathbb{D}^2(h; K_0, \sqrt{N}) + Q^2 \cdot \mathbb{D}^2(h; \sqrt{N}, N) + K_0^{-1}, \tag{5.14}$$

where the implicit constant is absolute,

$$H_N(h, K_0) := 2 \sum_{K_0 < p \le N} \frac{h(p)}{p} \tag{5.15}$$

and

$$\mathbb{D}^2(h; K_0, N) := \sum_{K_0 < p \le N} \frac{|h(p)|^2}{p}.$$

Proof. We consider the additive functions h_1, h_2, which are the restrictions of h to the primes $K_0 < p \le \sqrt{N}$ and $\sqrt{N} < p \le N$.[9] More precisely,

$$h_1(p^k) := \begin{cases} h(p), & \text{if } k = 1 \text{ and } K_0 < p \le \sqrt{N} \\ 0, & \text{otherwise} \end{cases}$$

and

$$h_2(p^k) := \begin{cases} h(p), & \text{if } k = 1 \text{ and } \sqrt{N} < p \le N \\ 0, & \text{otherwise} \end{cases}.$$

We also define

$$H_{i,N}(h_i, K_0) := 2 \sum_{K_0 < p \le N} \frac{h_i(p)}{p}, \quad i = 1, 2, \tag{5.16}$$

[9] If we worked with h only, we would run into trouble establsihing (5.25) below, since a non-acceptable term of the form $O(\sum_{p,q\le N} N^{-1})$ would appear in our estimates. For h_1, this term becomes $O(\sum_{p,q\le\sqrt{N}} N^{-1}) = O((\log N)^{-2})$, which is acceptable. We could have also worked with the restriction to the interval $[K_0, N^a]$ for any $a \le 1/2$. In the case of linear concentration estimates, this splitting is not needed since the error that appears in this case is $O(\sum_{pq\le N} N^{-1}) = O(\log\log N/\log N)$.

and the technical variant

$$H'_{1,N}(h_i, Q, K_0) := \sum_{K_0<p\le N} w_{N,Q}(p) \cdot h_1(p), \tag{5.17}$$

where

$$w_{N,Q}(p) := \frac{1}{N^2} \sum_{\substack{m,n\in[N],\\ p\,||\,(Qm+a)^2+(Qn+b)^2}} 1. \tag{5.18}$$

(Note that $w_{N,Q}(p) = w_{N,Q}(p,p)$, where $w_{N,Q}(p,q)$ is as in (5.4).) The reason for introducing this variant is because it gives the mean value of h_1 along sums of squares. Indeed, using properties (1)–(3), we have

$$\begin{aligned}\mathbb{E}_{m,n\in[N]}\, h_1((Qm+a)^2+(Qn+b)^2) &= \mathbb{E}_{m,n\in[N]} \sum_{p\,||\,(Qm+a)^2+(Qn+b)^2} h_1(p) \\ &= \frac{1}{N^2}\sum_{K_0<p\le N} h_1(p) \sum_{\substack{m,n\in[N],\\ p\,||\,(Qm+a)^2+(Qn+b)^2}} 1 = H'_{1,N}(h_1,Q,K_0).\end{aligned} \tag{5.19}$$

Using (5.5) of Lemma 5.2 and that $h_1(p) = 0$ for $p > \sqrt{N}$ and $h_1(p)$ is bounded by 1, we get

$$|H_{1,N}(h_1,K_0) - H'_{1,N}(h_1,Q,K_0)| \ll \sum_{K_0<p\le\sqrt{N}} \frac{1}{p^2} + \frac{1}{\sqrt{N}} \le \frac{1}{K_0} + \frac{1}{\sqrt{N}}. \tag{5.20}$$

Hence, in order to prove (5.14), it suffices to estimate

$$\mathbb{E}_{m,n\in[N]}\left|h_1((Qm+a)^2+(Qn+b)^2) - H'_{1,N}(h_1,Q,K_0)\right|^2 \tag{5.21}$$

and

$$\mathbb{E}_{m,n\in[N]}\left|h_2((Qm+a)^2+(Qn+b)^2)\right|^2 + |H_{2,N}(h_1,K_0)|^2. \tag{5.22}$$

We first deal with the expression (5.21). Using (5.19) and expanding the square below, we get

$$\begin{aligned}&\mathbb{E}_{m,n\in[N]}\left|h_1((Qm+a)^2+(Qn+b)^2) - H'_{1,N}(h_1,Q,K_0)\right|^2 = \\ &\qquad \mathbb{E}_{m,n\in[N]}\left|h_1((Qm+a)^2+(Qn+b)^2)\right|^2 - |H'_{1,N}(h_1,Q,K_0)|^2.\end{aligned} \tag{5.23}$$

To estimate this expression, first note that since h_1 is additive and $h_1(p^k) = 0$ for $k \ge 2$, we have

$$\mathbb{E}_{m,n\in[N]}\left|h_1((Qm+a)^2+(Qn+b)^2)\right|^2 = \mathbb{E}_{m,n\in[N]}\left|\sum_{p\,||\,(Qm+a)^2+(Qn+b)^2} h_1(p)\right|^2. \tag{5.24}$$

Expanding the square, using the fact that $h_1(p) = 0$ unless $K_0 < p \le \sqrt{N}$, and the definition of $w_{N,Q}(p,q)$ given in (5.4), we get that the right-hand side is equal to

$$\sum_{K_0<p\le\sqrt{N}} |h_1(p)|^2 \cdot w_{N,Q}(p,p) + \sum_{K_0<p,q\le\sqrt{N},\ p\ne q} h_1(p)\cdot\overline{h_1(q)}\cdot w_{N,Q}(p,q).$$

Using equation (5.5) of Lemma 5.2, we get that the first term is at most

$$2 \cdot \sum_{K_0 < p \leq \sqrt{N}} \frac{|h_1(p)|^2}{p} + O(N^{-1/2}).$$

Using equations (5.5) and (5.6) of Lemma 5.2, we get that the second term is equal to (we crucially use the bound $p, q \leq \sqrt{N}$ here and the prime number theorem)

$$\sum_{K_0 < p,q \leq \sqrt{N},\, p \neq q} h_1(p) \cdot \overline{h_1(q)} \cdot w_{N,Q}(p,p) \cdot w_{N,Q}(q,q) + O((\log N)^{-2}) \leq$$
$$(H'_{1,N}(h_1, Q, K_0))^2 + O((\log N)^{-2}),$$

where to get the last estimate, we added to the sum the contribution of the diagonal terms $p = q$ (which is nonnegative), and used (5.17) and the fact that $h_1(p) = 0$ for $p > \sqrt{N}$. Combining (5.23) with the previous estimates, we are led to the bound

$$\mathbb{E}_{m,n \in [N]} \left| h_1((Qm+a)^2 + (Qn+b)^2) - H'_{1,N}(h_1, Q, K_0) \right|^2 \ll$$
$$\mathbb{D}^2(h_1; K_0, \sqrt{N}) + O((\log N)^{-2}). \quad (5.25)$$

Next, we estimate the expression (5.22). Since h_2 is additive and satisfies properties (1)–(3), we get using (5.24) (with h_2 in place of h_1) and expanding the square

$$\mathbb{E}_{m,n \in [N]} \left| h_2((Qm+a)^2 + (Qn+b)^2) \right|^2 = \sum_{\sqrt{N} < p,q \leq N} h_2(p)\, \overline{h_2(q)}\, w_{N,Q}(p,q).$$

Since $h_2(p) \neq 0$ only when $p \equiv 1 \pmod 4$, using (5.13) of Lemma 5.3, we get that the right-hand side is bounded by

$$\ll Q^2 \cdot \Big(\sum_{\sqrt{N} < p,q \leq N} \frac{|h_2(p)|\,|h_2(q)|}{pq} + \sum_{\sqrt{N} < p \leq N} \frac{|h_2(p)|^2}{p} \Big) =$$
$$Q^2 \cdot \Big(\Big(\sum_{\sqrt{N} < p \leq N} \frac{|h_2(p)|}{p} \Big)^2 + \mathbb{D}^2(h_2; \sqrt{N}, N) \Big) \leq$$
$$Q^2 \cdot \Big(\sum_{\sqrt{N} < p \leq N} \frac{|h_2(p)|^2}{p} \cdot \sum_{\sqrt{N} < p \leq N} \frac{1}{p} + \mathbb{D}^2(h_2; \sqrt{N}, N) \Big) \ll Q^2 \cdot \mathbb{D}^2(h_2; \sqrt{N}, N),$$

where we crucially used the estimate

$$\sum_{\sqrt{N} < p \leq N} \frac{1}{p} \ll 1.$$

Similarly, we find

$$(H_{2,N}(h_2, K_0))^2 = 4 \Big(\sum_{\sqrt{N} < p \leq N} \frac{|h_2(p)|}{p} \Big)^2 \ll \mathbb{D}^2(h_2; \sqrt{N}, N).$$

Combining the previous estimates, we get the following bound for the expression in (5.22):

$$\mathbb{E}_{m,n\in[N]}\big(h_2((Qm+a)^2+(Qn+b)^2)\big)^2+(H_{2,N}(h_1,K_0))^2 \ll Q^2\cdot\mathbb{D}^2(h_2;\sqrt{N},N). \tag{5.26}$$

Combining the bounds (5.20), (5.25), (5.26), we get the asserted bound (5.14), completing the proof. □

5.3. Concentration estimates for multiplicative functions

Next we use Lemma 5.4 to get a variant that deals with multiplicative functions.

Lemma 5.5. *Let $K_0, N \in \mathbb{N}$, $a, b \in \mathbb{Z}$ with $-Q \le a, b \le Q$, and $f:\mathbb{N}\to\mathbb{U}$ be a multiplicative function that satisfies*

1. *$f(p) = 1$ for all primes $p \le K_0$ and $p > N$;*
2. *$f(p) = 1$ for all primes $p \equiv 3 \pmod 4$;*
3. *$f(p^k) = 1$ for all primes p and $k \ge 2$.*

Let also $Q = \prod_{p\le K_0} p^{a_p}$ with $a_p \in \mathbb{N}$. If N is large enough, depending only on K_0, then

$$\mathbb{E}_{m,n\in[N]}\big|f\big((Qm+a)^2+(Qn+b)^2\big)-\exp\big(G_N(f,K_0)\big)\big| \ll \\ (\mathbb{D}+\mathbb{D}^2)(f,1;K_0,\sqrt{N})+Q\cdot\mathbb{D}(f,1;\sqrt{N},N)+K_0^{-\frac{1}{2}}, \tag{5.27}$$

where the implicit constant is absolute and

$$G_N(f,K_0) := 2\sum_{K_0<p\le N}\frac{1}{p}(f(p)-1). \tag{5.28}$$

Proof. Let $h:\mathbb{N}\to\mathbb{C}$ be the additive function given on prime powers by

$$h(p^k) := f(p^k)-1.$$

We note that due to our assumptions on f, properties (1)–(3) of Lemma 5.4 are satisfied for $h/2$, which is bounded by 1 on primes.

Using that $z = e^{z-1}+O(|z-1|^2)$ for $|z|\le 1$ and property (3), we have

$$f(m^2+n^2) = \prod_{p^k\,||\,m^2+n^2} f(p^k) = \prod_{p\,||\,m^2+n^2}\big(\exp(h(p))+O(|h(p)|^2)\big).$$

Applying the estimate $|\prod_{i\le k} z_i - \prod_{i\le k} w_i| \le \sum_{i\le k}|z_i-w_i|$, we deduce that for all $m, n \in \mathbb{N}$, we have

$$f(m^2+n^2) = \exp(h(m^2+n^2))+O\Big(\sum_{p\,||\,m^2+n^2}|h(p)|^2\Big).$$

Using this and since $G_N(f,K_0) = H_N(h,K_0)$, where $H_N(h,K_0)$ is given by (5.15), we get

$$\mathbb{E}_{m,n\in[N]}\big|f\big((Qm+a)^2+(Qn+b)^2\big)-\exp\big(G_N(f,K_0)\big)\big| \ll \\ \mathbb{E}_{m,n\in[N]}\big|\exp\big(h((Qm+a)^2+(Qn+b)^2)\big)-\exp(H_N(h,K_0))\big| + \\ \mathbb{E}_{m,n\in[N]}\sum_{p\,||\,(Qm+a)^2+(Qn+b)^2}|h(p)|^2. \tag{5.29}$$

Next we use the inequality $|e^{z_1} - e^{z_2}| \le |z_1 - z_2|$, which is valid for $\Re z_1, \Re z_2 \le 0$, to bound the last expression by

$$\mathbb{E}_{m,n\in[N]} |h((Qm+a)^2+(Qn+b)^2) - H_N(h,K_0)| + \mathbb{E}_{m,n\in[N]} \sum_{p \,||\, (Qm+a)^2+(Qn+b)^2} |h(p)|^2. \quad (5.30)$$

To bound the first term, we use Lemma 5.4. It gives that for all large enough N, depending on K_0 only, we have

$$\mathbb{E}_{m,n\in[N]} \left|h\big((Qm+a)^2+(Qn+b)^2\big) - H_N(h,K_0)\right| \ll \\ \mathbb{D}(h;K_0,\sqrt{N}) + Q\cdot\mathbb{D}(h;\sqrt{N},N) + K_0^{-\frac{1}{2}} \ll \mathbb{D}(f,1;K_0,\sqrt{N}) + Q\cdot\mathbb{D}(f,1;\sqrt{N},N) + K_0^{-\frac{1}{2}}, \quad (5.31)$$

where to get the last bound, we used that $|h(p)|^2 \le 2 - 2\,\Re(f(p))$, which holds since $|f(p)| \le 1$. To bound the second term in (5.30), we note that using properties (1)–(3) of Lemma 5.4, we have

$$\mathbb{E}_{m,n\in[N]} \sum_{p \,||\, (Qm+a)^2+(Qn+b)^2} |h(p)|^2 = \sum_{K_0<p\le N} |h(p)|^2\, w_{N,Q}(p) \ll \\ \sum_{K_0<p\le N} \frac{|h(p)|^2}{p} + O((\log N)^{-1}) \ll \mathbb{D}^2(f,1;K_0,N) + O((\log N)^{-1}), \quad (5.32)$$

where $w_{N,P}(p)$ is as in (5.18) and we used equation (5.5) of Lemma 5.2 and the prime number theorem to get the first bound. Combining (5.29)–(5.32), we get the asserted bound. □

We use the previous result to deduce the following improved version.

Lemma 5.6. *Let $K_0, N \in \mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f(p) = 1$ for all primes $p > N$ with $p \equiv 1 \pmod 4$. Let also $Q = \prod_{p\le K_0} p^{a_p}$ with $a_p \in \mathbb{N}$. If N is large enough, depending only on K_0, then for all $a, b \in \mathbb{Z}$ with $-Q \le a, b \le Q$ and $(a^2+b^2, Q) = 1$, we have*

$$\mathbb{E}_{m,n\in[N]} \left|f\big((Qm+a)^2+(Qn+b)^2\big) - \exp\big(G_N(f,K_0)\big)\right| \ll \\ (\mathbb{D}_1+\mathbb{D}_1^2)(f,1;K_0,\sqrt{N}) + Q\cdot\mathbb{D}_1(f,1;\sqrt{N},N) + K_0^{-\frac{1}{2}}, \quad (5.33)$$

where the implicit constant is absolute and

$$G_N(f,K_0) := 2 \sum_{\substack{K_0<p\le N;\\ p\equiv 1 \pmod 4}} \frac{1}{p}\,(f(p)-1), \quad (5.34)$$

$$\mathbb{D}_1(f,1;x,y)^2 := \sum_{\substack{x<p\le y;\\ p\equiv 1 \pmod 4}} \frac{1}{p}\,(1-\Re(f(p)))$$

for $x < y$.

Proof. We first define the multiplicative function $\tilde{f}\colon \mathbb{N} \to \mathbb{U}$ on prime powers as follows

$$\tilde{f}(p^k) := \begin{cases} f(p^k), & \text{if } p > K_0 \\ 1, & \text{otherwise} \end{cases}.$$

Since $p \le K_0$ implies $p \mid Q$ and $(a^2+b^2, Q) = 1$, we get that $p \nmid (Qm+a)^2+(Qn+b)^2$ for every $p \le K_0$; hence,

$$f((Qm+a)^2+(Qn+b)^2) = \tilde{f}((Qm+a)^2+(Qn+b)^2) \quad \text{for every } m,n \in \mathbb{N}.$$

Note also that $G_N(\tilde{f}, Q) = G_N(f, Q)$ and $\mathbb{D}_1(\tilde{f}, 1; K_0, N) = \mathbb{D}_1(f, 1; K_0, N)$. It follows that in order to establish (5.33), it is enough to show that for all large enough N, depending only on K_0, we have

$$\mathbb{E}_{m,n\in[N]}|\tilde{f}((Qm+a)^2 + (Qn+b)^2) - \exp(G_N(\tilde{f}, Q))| \ll (\mathbb{D}_1 + \mathbb{D}_1^2)(\tilde{f}, 1; K_0, \sqrt{N}) + Q \cdot \mathbb{D}_1(\tilde{f}, 1; \sqrt{N}, N) + K_0^{-\frac{1}{2}}. \quad (5.35)$$

In order to establish (5.35), we make a series of further reductions that will eventually allow us to apply Lemma 5.5. For every $p \equiv 3 \pmod 4$, we have that $p \mid m^2 + n^2$ implies that $p \mid m$ and $p \mid n$. Consequently, the contribution to the average of those $m, n \in [N]$ for which $(Qm + a)^2 + (Qn + b)^2$ is divisible by some prime $p \equiv 3 \pmod 4$ is (note that $(Qm + a)^2 + (Qn + b)^2$ is only divisible by primes $p > K_0$)

$$\ll \frac{1}{N^2} \sum_{K_0 < p \le N} \left[\frac{N}{p}\right]^2 \ll \frac{1}{K_0},$$

which is acceptable.

Next, we show that the contribution to the average in (5.35) of those $m, n \in [N]$ for which $(Qm + a)^2 + (Qn + b)^2$ is divisible by p^2 for some prime $p \equiv 1 \pmod 4$ with $p > P_0$ (hence $p \nmid Q$) is also acceptable. Indeed, for fixed $n \in [N]$ such that $p \nmid Qn + b$, there exist at most $2[N/p^2] + 2$ values of $m \in [N]$ such that $p^2 \mid (Qm + a)^2 + (Qn + b)^2$. However, if $p \mid Qn + b$ and $p \mid (Qm + a)^2 + (Qn + b)^2$, then also $p \mid Qm + a$. Hence, the contribution to the average in (5.35) of those $m, n \in [N]$ for which $(Qm + a)^2 + (Qn + b)^2$ is divisible by p^2 for some prime $p \equiv 1 \pmod 4$ is bounded by (note again that $(Qm + a)^2 + (Qn + b)^2$ is only divisible by primes $p > K_0$)

$$\ll \frac{1}{N^2}\Big(\sum_{K_0 < p \le N} \Big(\left[\frac{N}{p^2}\right] + 1\Big) N + \sum_{K_0 < p \le N} \left[\frac{N}{p}\right]^2 \Big) \ll \frac{1}{K_0} + \frac{1}{\log N},$$

where we used the prime number theorem to bound $\frac{1}{N} \sum_{K_0 < p \le N} 1$.

Combining the above reductions, we deduce that in order to establish the estimate (5.35), we may further assume that

$$\tilde{f}(p^k) = 1 \text{ for all } p \in \mathbb{P},\ k \ge 2, \text{ and } \tilde{f}(p^k) = 1 \text{ for all } p \equiv 3 \pmod 4,\ k \in \mathbb{N}. \quad (5.36)$$

We are now in a situation where Lemma 5.5 is applicable and gives that for all large enough N, depending only on K_0, if $\mathbb{D}_1(f, 1; K_0, N) \le 1$, we have (note that (5.36) implies that $\mathbb{D}_1(\tilde{f}, 1; K_0, N) = \mathbb{D}(\tilde{f}, 1; K_0, N)$)

$$\mathbb{E}_{m,n\in[N]}|\tilde{f}((Qm+a)^2 + (Qn+b)^2) - \exp(G_N(\tilde{f}, K_0))| \ll (\mathbb{D}_1 + \mathbb{D}_1^2)(\tilde{f}, 1; K_0, \sqrt{N}) + Q \cdot \mathbb{D}_1(\tilde{f}, 1; \sqrt{N}, N) + K_0^{-\frac{1}{2}}.$$

Combining this bound with the bounds we got in order to arrive to this reduction, we get that (5.35) is satisfied. This completes the proof. □

5.4. *Proof of Proposition 5.1*

We start with some reductions. Suppose that the statement holds when $\chi = 1$ and $t = 0$. We will show that it holds for arbitrary χ and t. Let $\tilde{f} := f \cdot \overline{\chi} \cdot n^{-it}$, and apply the conclusion for $\chi = 1, t = 0$. We get the following bound for $\tilde{f}$:

$$\mathbb{E}_{m,n\in[N]} \left|\tilde{f}\big((Qm+a)^2 + (Qn+b)^2\big) - \exp\big(G_N(\tilde{f}, K_0)\big)\right| \ll (\mathbb{D}_1 + \mathbb{D}_1^2)(f, 1; K_0, \sqrt{N}) + Q^2 \cdot \mathbb{D}_1(f, 1; N, 3Q^2N^2) + Q \cdot \mathbb{D}_1(f, 1; \sqrt{N}, N) + K_0^{-1/2}. \quad (5.37)$$

Note that since χ is periodic with period q and $q \mid Q$, we have $\chi((Qm+a)^2+(Qn+b)^2) = \chi(a^2+b^2)$ for every $m, n \in \mathbb{N}$. Furthermore, since by assumption $(a^2+b^2, Q) = 1$ and $q \mid Q$, we have $(a^2+b^2, q) = 1$; hence, $|\chi(a^2+b^2)| = 1$. Also, $\lim_{m,n\to\infty}((Qm+a)^2+(Qn+b)^2)^{it} - Q^{2it} \cdot (m^2+n^2)^{it} = 0$ and $\mathbb{D}_1(\tilde{f}, 1; x, y) = \mathbb{D}_1(f, \chi \cdot n^{it}; x, y)$. Lastly, note that

$$G_N(\tilde{f}, K_0) = 2 \sum_{\substack{K_0 < p \leq N, \\ p \equiv 1 \pmod 4}} \frac{1}{p}(\tilde{f}(p) - 1) = 2 \sum_{\substack{K_0 < p \leq N, \\ p \equiv 1 \pmod 4}} \frac{1}{p}(f(p) \cdot \overline{\chi(p)} \cdot n^{-it} - 1) = G_N(f, K_0).$$

After inserting this information in (5.37), we get that (5.3) is satisfied.

So it suffices to show that if $Q = \prod_{p \leq K_0} p^{a_p}$ for some $a_p \in \mathbb{N}$, then if N is large enough, depending only on Q, and $\mathbb{D}_1(f, 1; K_0, N) \leq 1$, we have

$$\begin{aligned} &\mathbb{E}_{m,n\in[N]} \left| f\left((Qm+a)^2 + (Qn+b)^2\right) - \exp\left(G_N(f, K_0)\right) \right| \ll \\ &\quad (\mathbb{D}_1 + \mathbb{D}_1^2)(f, 1; K_0, \sqrt{N}) + Q^2 \cdot \mathbb{D}_1(f, 1; N, 3Q^2N^2) + Q \cdot \mathbb{D}_1(f, 1; \sqrt{N}, N) + K_0^{-1/2}. \end{aligned} \tag{5.38}$$

For every $N \in \mathbb{N}$, we decompose f as $f = f_{N,1} \cdot f_{N,2}$, where the multiplicative functions $f_{N,1}, f_{N,2} \colon \mathbb{N} \to \mathbb{U}$ are defined on prime powers as follows:

$$f_{N,1}(p^k) := \begin{cases} f(p), & \text{if } k = 1 \text{ and } p > N,\ p \equiv 1 \pmod 4 \\ 1, & \text{otherwise} \end{cases},$$

$$f_{N,2}(p^k) := \begin{cases} 1, & \text{if } k = 1 \text{ and } p > N,\ p \equiv 1 \pmod 4 \\ f(p^k), & \text{otherwise} \end{cases}.$$

We first study the function $f_{N,1}$. Following the notation of Lemma 5.3 for $l, Q, N \in \mathbb{N}$, we let

$$w_{N,Q}(l) := \frac{1}{N^2} \sum_{\substack{m,n\in[N], \\ l \mid (Qm+a)^2+(Qn+b)^2}} 1.$$

Lemma 5.3 implies that if l is a sum of two squares, then

$$w_{N,Q}(l) \ll \frac{Q^2}{l}. \tag{5.39}$$

Since for $N \gg Q$ we have $f_{N,1}((Qm+a)^2 + (Qn+b)^2) - 1 \neq 0$ only if $(Qm+a)^2 + (Qn+b)^2$ is divisible by one or two primes $p > N$,[10] we get

$$\begin{aligned} &\mathbb{E}_{m,n\in[N]} |f_{N,1}\left((Qm+a)^2 + (Qn+b)^2\right) - 1| \leq \\ &\quad \sum_{\substack{N < p \leq 3Q^2N^2, \\ p \equiv 1 \pmod 4}} |f(p) - 1|\, w_{N,Q}(p) + \sum_{\substack{N < p,q \leq 3Q^2N^2,\ p \neq q \\ p,q \equiv 1 \pmod 4}} |f(pq) - 1|\, w_{N,Q}(pq), \end{aligned} \tag{5.40}$$

where we used that $f_{N,1}(p) = f(p)$ for all $p > N$, and in the second sum, we have ignored the contribution of the diagonal terms $p = q$ since, by construction, $f_{N,1}(p^2) = 1$ for all primes p. Using (5.39) for $l := p$, which is a sum of two squares since $p \equiv 1 \pmod 4$, we estimate the first term as follows:[11]

[10]For $m, n \in [N]$ and $-Q \leq a, b \leq Q$, we have $(Qm+a)^2 + (Qn+b)^2 \ll Q^2N^2$, so if $(Qm+a)^2 + (Qn+b)^2$ was divisible by three or more primes greater than N, we would have $N^3 \ll Q^2N^2$, which fails if $Q \ll N$.

[11]Bounding $w_{N,Q}(p)$ using (5.5) would lead to non-acceptable errors here, because the range of summation is much larger than N.

$$\sum_{\substack{N<p\le 3Q^2N^2,\\ p\equiv 1 \pmod 4}} |f(p)-1|\, w_{N,Q}(p) \ll Q^2 \sum_{\substack{N<p\le 3Q^2N^2,\\ p\equiv 1 \pmod 4}} \frac{|f(p)-1|}{p} \le$$

$$Q^2\cdot\Big(\sum_{\substack{N<p\le 3Q^2N^2,\\ p\equiv 1 \pmod 4}} \frac{|f(p)-1|^2}{p}\Big)^{\frac12}\cdot\Big(\sum_{\substack{N<p\le 3Q^2N^2,\\ p\equiv 1 \pmod 4}} \frac{1}{p}\Big)^{\frac12} \ll Q^2\cdot \mathbb{D}_1(f,1;N,3Q^2N^2),$$

where we used that $\sum_{N\le p\le 3Q^2N^2}\frac{1}{p}\ll 1$ for $N\ge Q$. Similarly, using (5.39) for $l := pq$, which is a sum of two squares since $pq\equiv 1 \pmod 4$, we estimate the second term in (5.40) as follows (note that since $p\neq q$, we have $f(pq)=f(p)f(q)$):

$$\sum_{\substack{N<p,q\le 3Q^2N^2,\, p\neq q,\\ p,q\equiv 1 \pmod 4}} |f(pq)-1|\, w_{N,Q}(pq) \ll Q^2 \sum_{\substack{N<p,q\le 3Q^2N^2,\\ p,q\equiv 1 \pmod 4}} \frac{|f(p)-1|+|f(q)-1|}{pq} \le$$

$$2\,Q^2\cdot\Big(\sum_{\substack{N<p,q\le 3Q^2N^2,\\ p,q\equiv 1 \pmod 4}} \frac{|f(p)-1|^2}{pq}\Big)^{\frac12}\cdot\Big(\sum_{\substack{N<p,q\le 3Q^2N^2,\\ p,q\equiv 1 \pmod 4}} \frac{1}{pq}\Big)^{\frac12} \ll Q^2\cdot \mathbb{D}_1(f,1;N,3Q^2N^2),$$

where we used that $\sum_{N\le p\le 3Q^2N^2}\frac{1}{p}\ll 1$ for $N\ge Q$. Combining the above estimates and (5.40), we deduce that for $N\gg Q$, we have

$$\mathbb{E}_{m,n\in[N]}\big|f_{N,1}\big((Qm+a)^2+(Qn+b)^2\big)-1\big| \ll Q^2\cdot\mathbb{D}_1(f,1;N,3Q^2N^2). \tag{5.41}$$

Next, we move to the function f_2. Since $f_2(p)=1$ for all primes $p\ge N$, Lemma 5.6 is applicable. We get that if N is large enough, depending on K_0, we have

$$\mathbb{E}_{m,n\in[N]}\big|f_{N,2}\big((Qm+a)^2+(Qn+b)^2\big)-\exp(G_N(f,K_0))\big| \ll$$
$$(\mathbb{D}_1+\mathbb{D}_1^2)(f,1;K_0,\sqrt{N})+Q\cdot\mathbb{D}_1(f,1;\sqrt{N},N)+K_0^{-\frac12}, \tag{5.42}$$

where we used that $f_{N,2}(p)=f(p)$ for all primes $p\equiv 1 \pmod 4$ with $p\le N$; hence, $G_N(f_{N,2},K_0)=G_N(f,K_0)$ and $\mathbb{D}_1(f_{N,2},1;K_0,N)=\mathbb{D}_1(f,1;K_0,N)$.

Finally, we use the triangle inequality and combine (5.41) and (5.42) to obtain that the left-hand side in (5.38) is bounded by

$$\mathbb{E}_{m,n\in[N]}\Big(\big|f_{N,1}\big((Qm+a)^2+(Qn+b)^2\big)-1\big|+\big|f_{N,2}\big((Qm+a)^2+(Qn+b)^2\big)-\exp(G_N(f,K_0))\big|\Big)$$
$$\ll (\mathbb{D}_1+\mathbb{D}_1^2)(f,1;K_0,\sqrt{N})+Q^2\cdot\mathbb{D}_1(f,1;N,3Q^2N^2)+Q\cdot\mathbb{D}_1(f,1;\sqrt{N},N)+K_0^{-1/2}.$$

Thus, (5.38) holds, completing the proof.

6. Type II Pythagorean pairs and more

6.1. Proof of Theorem 2.9

As explained in Section 2.3, in order to complete the proof of Theorem 2.9 (and hence of part (2) of Theorem 2.2), it remains to prove Proposition 2.10, Proposition 2.12 and Lemma 2.13. We do this in this section.

We repeat the statement of Proposition 2.10 and explain how it can be derived from results in [21].

Proposition 6.1. *Let $f : \mathbb{N} \to \mathbb{U}$ be an aperiodic completely multiplicative function. Then for every $\delta > 0$ and $\ell, \ell', Q \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \tilde{w}_\delta(m,n) \cdot f(\ell((Qm+1)^2 + (Qn)^2)) \cdot \overline{f(\ell'(Qm+1)(Qn))} = 0, \tag{6.1}$$

where $\tilde{w}_\delta(m,n)$ is as in (3.3).

Proof. Recall that

$$\tilde{w}_\delta(m,n) := F_\delta\big((\ell(m^2+n^2))^i \cdot (\ell' mn)^{-i}\big), \quad m,n \in \mathbb{N},$$

where $F_\delta : \mathbb{S}^1 \to [0,1]$ is the continuous function defined in Lemma 3.3. Using uniform approximation of F_δ by trigonometric polynomials and linearity, we get that it suffices to verify (6.1) when $\tilde{w}_\delta(m,n)$ is replaced by $(m^2+n^2)^{ki} \cdot (mn)^{-ki}$ for arbitrary $k \in \mathbb{Z}$. Furthermore, the limit remains unchanged if we replace $(m^2+n^2)^{ki} \cdot (mn)^{-ki}$ with $((Qm+1)^2 + (Qn)^2)^{ki} \cdot ((Qm+1)(Qn))^{-ki}$ (the $+1$ makes no difference asymptotically, so we can omit it, and then Q^2 is going to cancel because of the conjugate). Hence, it suffices to establish that for every $k \in \mathbb{Z}$, we have

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, f_k((Qm+1)^2 + (Qn)^2) \cdot \overline{f_k((Qm+1)(Qn))} = 0, \tag{6.2}$$

where $f_k(n) := f(n) \cdot n^{ki}$, $n \in \mathbb{N}$. Note that since the indicator function of an arithmetic progression is a linear combination of Dirichlet characters, in order to establish (6.2), it suffices to show that for all Dirichlet characters χ, χ', we have

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, \chi(m) \cdot \chi'(n) \cdot f_k(m^2+n^2) \cdot \overline{f_k(mn)} = 0,$$

or, equivalently, that

$$\lim_{N\to\infty} \mathbb{E}_{m,n\in[N]}\, f_k(m^2+n^2) \cdot (\overline{f_k} \cdot \chi)(m) \cdot (\overline{f_k} \cdot \chi')(n) = 0. \tag{6.3}$$

Since f is aperiodic, so is $\overline{f_k} \cdot \chi$ (and $\overline{f_k} \cdot \chi'$). By [21, Theorem 9.7] (applied to $Q(m,n) := m^2+n^2$), we deduce that (6.3) holds, completing the proof. □

Recall that in (2.11), we defined the multiplicative Følner sequence (Φ_K) by

$$\Phi_K := \Big\{ \prod_{p\le K} p^{a_p} : K < a_p \le 2K \Big\}, \quad K \in \mathbb{N}.$$

Note that every $q \in \mathbb{N}$ divides all elements of Φ_K when $K \in \mathbb{N}$ is large enough depending on q.

The next result is a key ingredient in the proof of Proposition 2.12 below.

Lemma 6.2. *Let $f : \mathbb{N} \to \mathbb{S}^1$ be a completely multiplicative function such that $f \sim \chi \cdot n^{it}$ for some $t \in \mathbb{R}$ and Dirichlet character χ. Let also $\delta > 0$ be fixed, $\tilde{w}_\delta$ be as in (3.3), and (Φ_K) be as in (2.11). For $Q, N \in \mathbb{N}$, we let*

$$L_{\delta,N}(f,Q) := \mathbb{E}_{m,n\in[N]}\, \tilde{w}_\delta(m,n) \cdot f(\ell((Qm+1)^2 + (Qn)^2)) \cdot \overline{f(\ell'(Qm+1)\,n)} \tag{6.4}$$

and

$$\tilde{L}_{\delta,N}(f,Q) := Q^{-it} \cdot L_{\delta,N}(f,Q). \tag{6.5}$$

Then

$$\lim_{K\to\infty} \limsup_{N\to\infty} \max_{Q,Q'\in\Phi_K} |\tilde{L}_{\delta,N}(f,Q) - \tilde{L}_{\delta,N}(f,Q')| = 0. \tag{6.6}$$

Proof. For $K \in \mathbb{N}$, let $F_N(f,K)$ and $G_N(f,K)$ be defined as in (2.12) and (2.18), respectively.

We apply the concentration inequalities of Proposition 2.5 and Proposition 2.11. Since $f \sim \chi \cdot n^{it}$ for some $t \in \mathbb{R}$ and Dirichlet character χ, we get that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \max_{Q\in\Phi_K} \mathbb{E}_{m\in[N]} |f(Qm+1) - (Qm)^{it} \exp\big(F_N(f,K)\big)| = 0$$

and

$$\lim_{K\to\infty} \limsup_{N\to\infty} \max_{Q\in\Phi_K} \mathbb{E}_{m,n\in[N]} \big|f\big((Qm+1)^2 + (Qn)^2\big) - Q^{2it} \cdot (m^2+n^2)^{it} \cdot \exp\big(G_N(f,K)\big)\big| = 0.$$

We deduce that if

$$M_{\delta,N}(f) := f(\ell) \cdot \overline{f(\ell')} \cdot \mathbb{E}_{m,n\in[N]}\, \tilde{w}_\delta(m,n) \cdot (m^2+n^2)^{it} \cdot m^{-it} \cdot \overline{f(n)},$$

then

$$\lim_{K\to\infty} \limsup_{N\to\infty} \max_{Q\in\Phi_K} |\tilde{L}_{\delta,N}(f,Q) - M_{\delta,N}(f) \cdot \exp(G_N(f,K)) \cdot \overline{\exp(F_N(f,K))}| = 0.$$

Using this identity and the triangle inequality, we deduce that (6.6) holds. □

Recall that $\mathcal{M}_p$ and $\mathcal{A}$ were defined in (2.9) and (2.13), respectively. The next result follows easily from Lemma 3.2 and the continuity of finite Borel measures.

Lemma 6.3. *Let σ be a Borel probability measure on $\mathcal{M}_p$. Then for every $\varepsilon > 0$, there exist a Borel subset $\mathcal{M}_\varepsilon$ of $\mathcal{M}_p \setminus \mathcal{A}$ and $K_0 \in \mathbb{N}$, such that*

$$\sigma((\mathcal{M}_p \setminus \mathcal{A}) \setminus \mathcal{M}_\varepsilon) \leq \varepsilon \tag{6.7}$$

and

$$\sup_{f\in\mathcal{M}_\varepsilon} |\mathbb{E}_{Q\in\Phi_K} f(Q) \cdot Q^{-it_f}| \leq \varepsilon \text{ for all } K \geq K_0, \tag{6.8}$$

where t_f is the unique real for which $f \sim \chi \cdot n^{it_f}$ for some Dirichlet character χ.

Remark. The important point is that K_0 does not depend on f as long as $f \in \mathcal{M}_\varepsilon$.

Proof. Let $\varepsilon > 0$. For $m \in \mathbb{N}$, we let

$$\mathcal{M}_{\varepsilon,m} := \{f \in \mathcal{M}_p \setminus \mathcal{A} \colon |\mathbb{E}_{Q\in\Phi_K} f(Q) \cdot Q^{-it_f}| \leq \varepsilon \text{ for all } K \geq m\}.$$

Note that by Lemma 3.6, the map $f \mapsto t_f$ from $\mathcal{M}_p$ to $\mathbb{R}$ is Borel; hence, for every $\varepsilon > 0$, the sets $\mathcal{M}_{\varepsilon,m}$ form an increasing family of Borel sets. Since for $f \notin \mathcal{A}$ we have $f \cdot n^{-it_f} \neq 1$, we get by Lemma 3.2 that for every $f \in \mathcal{M}_p \setminus \mathcal{A}$, we have

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K} f(Q) \cdot Q^{-it_f} = 0.$$

Hence,

$$(\mathcal{M}_p \setminus \mathcal{A}) := \bigcup_{m\in\mathbb{N}} \mathcal{M}_{\varepsilon,m}.$$

It follows that there exists $m_0 \in \mathbb{N}$ such that

$$\sigma((\mathcal{M}_p \setminus \mathcal{A}) \setminus \mathcal{M}_{\varepsilon,m_0}) \le \varepsilon.$$

Renaming $\mathcal{M}_{\varepsilon,m_0}$ as $\mathcal{M}_\varepsilon$ and letting $K_0 := m_0$ gives the asserted statement. □

Using the previous two results, we are going to prove Proposition 2.12, which we formulate again for convenience.

Proposition 2.12 . *Let (Φ_K), $\mathcal{A}$, $B_\delta(f,Q;m,n)$ be defined by (2.11), (2.13), (2.16), respectively, and $\delta>0$. Let also σ be a Borel probability measure on $\mathcal{M}_p$. Then*

$$\lim_{K\to\infty} \limsup_{N\to\infty} \Big|\mathbb{E}_{Q\in\Phi_K} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_p\setminus\mathcal{A}} B_\delta(f,Q;m,n)\, d\sigma(f)\Big| = 0.$$

Proof. Let $\delta, \varepsilon > 0$. By Lemma 6.3, there exists $K_0 = K_0(\sigma) \in \mathbb{N}$ and a Borel subset $\mathcal{M}_\varepsilon$ of $\mathcal{M} \setminus \mathcal{A}$, such that

$$\sigma((\mathcal{M}_p \setminus \mathcal{A}) \setminus \mathcal{M}_\varepsilon) \le \varepsilon/4 \tag{6.9}$$

and

$$\sup_{f\in\mathcal{M}_\varepsilon} |\mathbb{E}_{Q\in\Phi_K} f(Q)\cdot Q^{-it_f}| \le \varepsilon/2 \text{ for all } K \ge K_0. \tag{6.10}$$

Because of (6.9), and since $|B_\delta(f,Q;m,n)| \le 1$, it suffices to show that

$$\limsup_{K\to\infty} \limsup_{N\to\infty} \Big|\mathbb{E}_{Q\in\Phi_K} \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_\varepsilon} B_\delta(f,Q;m,n)\, d\sigma(f)\Big| \le \varepsilon. \tag{6.11}$$

As in Lemma 6.2, for $Q, N \in \mathbb{N}$, we let

$$\tilde{L}_{\delta,N}(f,Q) := f(Q)\cdot Q^{-it_f} \cdot \mathbb{E}_{m,n\in[N]} B_\delta(f,Q;m,n).$$

We also let for $Q, N \in \mathbb{N}$

$$I(Q,N) := \mathbb{E}_{m,n\in[N]} \int_{\mathcal{M}_\varepsilon} B_\delta(f,Q;m,n)\, d\sigma(f) = \int_{\mathcal{M}_\varepsilon} \overline{f(Q)\cdot Q^{-it_f}} \cdot \tilde{L}_{\delta,N}(f,Q)\, d\sigma(f). \tag{6.12}$$

Finally, for $K \in \mathbb{N}$, we let Q_K be an arbitrary element of Φ_K, and define

$$I_1(Q,N) := \int_{\mathcal{M}_\varepsilon} \overline{f(Q)\cdot Q^{-it_f}} \cdot \tilde{L}_{\delta,N}(f,Q_K)\, d\sigma(f), \quad Q \in \Phi_K, N \in \mathbb{N}. \tag{6.13}$$

Recall that by part (2) of Lemma 3.6, the map $f \mapsto t_f$ from $\mathcal{M}_p$ to $\mathbb{R}$ is Borel, so the integral defining $I_1(Q,N)$ is well-defined. Using (6.12) and (6.13), we get that

$$\max_{Q\in\Phi_K} |I(Q,N) - I_1(Q,N)| \le \max_{Q\in\Phi_K} |\tilde{L}_{\delta,N}(f,Q) - \tilde{L}_{\delta,N}(f,Q_K)|, \quad K \in \mathbb{N}.$$

We deduce from this and equation (6.6) of Lemma 6.2 that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \max_{Q\in\Phi_K} |I(Q,N) - I_1(Q,N)| = 0.$$

It follows from the above facts that in order to show that (6.11) holds, it suffices to show that

$$\limsup_{K\to\infty} \limsup_{N\to\infty} |\mathbb{E}_{Q\in\Phi_K} I_1(Q,N)| \le \varepsilon. \tag{6.14}$$

Using the definition of $I_1(Q,N)$ in (6.13) and the estimate (6.10), we get that for every $K \geq K_0$, we have

$$\sup_{N\in\mathbb{N}} |\mathbb{E}_{Q\in\Phi_K} I_1(Q,N)| \leq \sup_{f\in\mathcal{M}_\varepsilon} |\mathbb{E}_{Q\in\Phi_K} f(Q)\cdot Q^{-it_f}| \leq \varepsilon.$$

Hence,

$$\limsup_{K\to\infty} \limsup_{N\to\infty} |\mathbb{E}_{Q\in\Phi_K} I_1(Q,N)| \leq \varepsilon,$$

establishing (6.14) and completing the proof. □

Finally, we restate and prove Lemma 2.13.

Lemma 2.13. *Let σ be a Borel probability measure on $\mathcal{M}_p$ such that $\sigma(\{1\}) > 0$ and $\mathcal{A}$ as in (2.13). Then there exist $\delta_0, \rho_0 > 0$, depending only on σ, such that*

$$\liminf_{N\to\infty} \inf_{Q\in\mathbb{N}} \Re\Big(\mathbb{E}_{m,n\in[N]} \int_{\mathcal{A}} B_{\delta_0}(f,Q;m,n)\,d\sigma(f)\Big) \geq \rho_0. \tag{6.15}$$

Proof. Using the positiveness property of the weight $\tilde{w}_\delta(m,n)$ in Lemma 3.3, the proof is identical to the one used to establish Lemma 2.8, and so we omit it. □

6.2. *Proof of Theorem 1.8*

We sketch the proof of Theorem 1.8. Following the reduction in Section 2.1, we need to show that under the assumptions of Theorem 2.2, we have

$$\liminf_{N\to\infty} \mathbb{E}^{\log}_{m,n\in[N]} \int_{\mathcal{M}} f(n(n+1))\cdot\overline{f(m^2)}\,d\sigma(f) > 0. \tag{6.16}$$

To prove this, we follow the argument used in the proof of part (2) of Theorem 2.2.[12] We will restrict our average to the grid $\{(Qn,m)\colon m,n\in\mathbb{N}\}$. This is why for $f\in\mathcal{M}$ and $Q,m,n\in\mathbb{N}$, we let

$$B(f,Q;m,n) := f((Qn)(Qn+1))\cdot\overline{f(m^2)}.$$

(For reasons that will become clear shortly, in this case, we do not have to introduce any kind of weight w_δ.)

We first claim that if $f\in\mathcal{M}$ is aperiodic, then for every $Q\in\mathbb{N}$, we have

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{m,n\in[N]} B(f,Q;m,n) = 0.$$

(This corresponds to Proposition 2.10.) Since f is completely multiplicative, it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N]} f(n)\cdot f(Qn+1) = 0 \quad \text{or} \quad \lim_{N\to\infty} \mathbb{E}^{\log}_{m\in[N]} f^2(m) = 0. \tag{6.17}$$

Suppose that f^2 does not have logarithmic mean value 0. Then by a consequence of a result of Halász [30], we have $f^2 \sim 1$.[13] Combining this with the following consequence of a result of Tao in [47], we deduce that (6.17) holds.

[12] We follow part (2) of Theorem 2.2 and not part (1) because we do not know that the limit of the averages $\mathbb{E}^{\log}_{m,n\in[N]} f(n(n+1))\cdot\overline{f(m^2)}$ exists for every $f\in\mathcal{M}$.

[13] Halász's theorem gives that $f^2 \sim n^{it}$ for some $t\in\mathbb{R}$, but for logarithmic averages, we have that if $g\sim n^{it}$ for some $t\neq 0$, then g has mean 0.

Lemma 6.4. *Suppose that $f \in \mathcal{M}$ is aperiodic and satisfies $f^2 \sim 1$. Then for every $Q \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \mathbb{E}^{\log}_{n\in[N]} f(n) \cdot f(Qn+1) = 0. \tag{6.18}$$

Proof. We say that $f \in \mathcal{M}$ is strongly aperiodic if for every Dirichlet character χ and $A \geq 1$, we have $\lim_{N\to\infty} \min_{|t|\leq AN} \mathbb{D}(f, \chi \cdot n^{it}; 1, N) = +\infty$. It was shown in [47, Corollary 1.5] that if f is strongly aperiodic, then (6.18) holds for every $Q \in \mathbb{N}$. Thus, it remains to show that if f is aperiodic and $f^2 \sim 1$, then f is strongly aperiodic. This can be shown exactly as in the proof of [19, Proposition 6.1]; the assumption $f^2 \sim 1$ in our setting replaces the assumption $f^k = 1$ for some $k \in \mathbb{N}$ that was used in [19]. □

Using the previous claim and the bounded convergence theorem, we get that it suffices to establish (6.16) when the range of integration $\mathcal{M}$ is replaced by the subset $\mathcal{M}_p$ of pretentious multiplicative functions.

Next, we claim that if (Φ_K) is as in (2.11) and σ is a Borel probability measure on $\mathcal{M}_p$, then

$$\lim_{K\to\infty} \limsup_{N\to\infty} \Big| \mathbb{E}_{Q\in\Phi_K} \mathbb{E}^{\log}_{m,n\in[N]} \int_{\mathcal{M}_p\setminus\{1\}} B(f, Q; m, n)\, d\sigma(f) \Big| = 0. \tag{6.19}$$

(This corresponds to Proposition 2.12. Note that $\mathcal{A}$ can be replaced by $\{1\}$ in this case, which is the reason why the weight $\tilde{w}_\delta$ is not needed for this argument.) To prove this, we argue as in the proof of Proposition 2.12. If $f \sim \chi \cdot n^{it_f}$ for some $t_f \in \mathbb{R}$ and Dirichlet character χ, for $Q, N \in \mathbb{N}$, we let

$$\tilde{L}_N(f, Q) := Q^{-it_f} \cdot \mathbb{E}^{\log}_{m,n\in[N]} f(n(Qn+1)) \cdot \overline{f(m^2)}$$

and show that

$$\lim_{K\to\infty} \limsup_{N\to\infty} \max_{Q,Q'\in\Phi_K} |\tilde{L}_{\delta,N}(f, Q) - \tilde{L}_{\delta,N}(f, Q')| = 0.$$

We do this exactly as in the proof of Lemma 6.2, using in this case the concentration estimate of Proposition 2.5 for logarithmic averages (see the third remark following Proposition 2.5). Then (6.19) follows exactly as in the proof of Proposition 2.12. The reason why we only have to exclude the multiplicative function $\{1\}$ in the integral in (6.19) (versus the set $\mathcal{A}$ of all Archimedean characters) is because in our current setting, we have

$$\mathbb{E}_{m,n\in[N]} B(f, Q; m, n) = f(Q) \cdot Q^{it_f} \cdot \tilde{L}_N(f, Q),$$

and $Q \mapsto f(Q) \cdot Q^{it_f}$ is the trivial multiplicative function only when $f = 1$. Note also that the variant of Lemma 2.13 is trivial in our case, since $\mathcal{A}$ is replaced by $\{1\}$. With the above information, we can complete the proof of (6.16) exactly as we did at the end of Section 2.3.

7. Pythagorean triples on level sets - Reduction to the pretentious case

First, let us recall a convention made in Section 2.4, which we will continue to follow in this and the next section. When we write $\mathbb{E}^*_{k\in\mathbb{N}}$, we mean the limit $\lim_{K\to\infty} \mathbb{E}_{k\in\Phi_K}$, where (Φ_K) is a multiplicative Følner sequence chosen so that all the limits in the following statements exist. Since it will always be the case in our arguments that only a countable collection of limits needs to be considered, such a Følner sequence can be taken as a subsequence of any given multiplicative Følner sequence.

As explained in Section 2.4, the proof of Theorem 2.14 splits in two parts, Propositions 2.16 and 2.18. Our goal in this section is to establish the first part, which we now state in a more general form (we do not assume that f takes finitely many values).

Proposition 7.1. *Suppose that for every completely multiplicative function $h\colon \mathbb{N} \to \mathbb{S}^1$, with $h \sim n^{it}$ for some $t \in \mathbb{R}$, modified Dirichlet character $\tilde{\chi}\colon \mathbb{N} \to \mathbb{S}^1$, and open arc I on $\mathbb{S}^1$ around 1, we have*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, A(k\,(m^2-n^2)) \cdot A(k\cdot 2mn) \cdot A(k\,(m^2+n^2)) > 0,$$

where

$$A(n) := F(h(n)) \cdot F(\tilde{\chi}(n)), \quad n \in \mathbb{N}, \quad F := \mathbf{1}_I.$$

Then for every completely multiplicative function $f\colon \mathbb{N} \to \mathbb{S}^1$ and open arc I around 1, we have

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, F(f(k\,(m^2-n^2))) \cdot F(f(k\,2mn)) \cdot F(f(k\,(m^2+n^2))) > 0,$$

where F is as before. Furthermore, if our assumptions hold for all finite-valued completely multiplicative functions h, then the conclusion holds for all finite-valued completely multiplicative functions f.

7.1. Preparation

Recall that we write $f \sim g$ if $\mathbb{D}(f,g) < +\infty$, where $\mathbb{D}(f,g)$ is as in (3.5).

Lemma 7.2. *Let $f\colon \mathbb{N} \to \mathbb{S}^1$ be a completely multiplicative function such that $f \sim n^{it}$ for some $t \in \mathbb{R}$. Then for every $d \in \mathbb{N}$, there exists a completely multiplicative function $g\colon \mathbb{N} \to \mathbb{S}^1$, such that $g \sim n^{it/d}$ and $g^d = f$. Furthermore, if f takes finitely many values, then so does g.*

Proof. Suppose first that $f \sim 1$. Then $f(p) = e(\theta_p)$, $p \in \mathbb{P}$, for some $\theta_p \in [-1/2, 1/2)$ with $\sum_{p\in\mathbb{P}} \frac{1-\cos(\theta_p)}{p} < +\infty$. Hence, $\sum_{p\in\mathbb{P}} \frac{\theta_p^2}{p} < +\infty$. We define the completely multiplicative function $g\colon \mathbb{N} \to \mathbb{S}^1$ by

$$g(p) := e(\tilde{\theta}_p), \ \text{ where } \tilde{\theta}_p := \theta_p/d, \ p \in \mathbb{P}.$$

We have $g^d = f$. Also, $\sum_{p\in\mathbb{P}} \frac{\tilde{\theta}_p^2}{p} < +\infty$; hence, $g \sim 1$.

Now suppose that $f \sim n^{it}$, and let $d \in \mathbb{N}$. Then $f \cdot n^{-it} \sim 1$, and the previous argument gives that there exists $h\colon \mathbb{N} \to \mathbb{S}^1$ with $h \sim 1$ such that $h^d = f \cdot n^{-it}$. Let $g := h \cdot n^{it/d}$. Then $g^d = f$ and $g \sim n^{it/d}$. □

A similar statement is not always true when $f \sim \chi$ where χ is a Dirichlet character (not even when $f = \chi$).

We remind the reader that modified Dirichlet characters $\tilde{\chi}$ were defined in Section 3.3. If a completely multiplicative function $f\colon \mathbb{N} \to \mathbb{S}^1$ is such that f^l is aperiodic for every $l \in \mathbb{N}$, then things are easier for us. If this is not the case (for example, it is never the case when f is finite-valued), then the next lemma gives a useful decomposition to work with.

Lemma 7.3. *Let $f\colon \mathbb{N} \to \mathbb{S}^1$ be an aperiodic completely multiplicative function such that f^d is pretentious for some $d \in \mathbb{N}$, and suppose that $d \geq 2$ is the smallest such d. Then there exist completely multiplicative functions $g, h\colon \mathbb{N} \to \mathbb{S}^1$ and a Dirichlet character χ, such that*

1. $f = g \cdot h$
2. $g, \ldots, g^{d-1}$ *are aperiodic and* $g^d = \tilde{\chi}$.
3. $h \sim n^{it}$ *for some* $t \in \mathbb{R}$.

Furthermore, if f takes finitely many values, then so does h and $h \sim 1$.

Proof. By our assumption, we have that $f, \dots, f^{d-1}$ are aperiodic and $f^d \sim \chi \cdot n^{it}$ for some $t \in \mathbb{R}$ and Dirichlet character χ. Then $f^d \cdot \overline{\tilde{\chi}} \sim n^{it}$, and Lemma 7.2 gives that there exists a completely multiplicative function $h\colon \mathbb{N} \to \mathbb{S}^1$ such that

$$h \sim n^{it/d} \quad \text{and} \quad h^d = f^d \cdot \overline{\tilde{\chi}}.$$

Let $g := f \cdot \overline{h}$. Then obviously $f = g \cdot h$. Also, for $j = 1, \dots, d-1$, we have $g^j = f^j \cdot h^j$ is aperiodic, since by assumption, f^j is aperiodic and h^j is pretentious. Moreover,

$$g^d = f^d \cdot \overline{h}^d = \tilde{\chi}.$$

Lastly, suppose that f takes finitely many values. Since g also takes finitely many values, and $h := f \cdot \overline{g}$, we have that h takes finitely many values. Also, since h takes finitely many values and $h \sim n^{it}$ for some $t \in \mathbb{R}$, we have that $t = 0$. This completes the proof. □

Since χ is a Dirichlet character, there exists $r \in \mathbb{N}$ such that $\tilde{\chi}^r = 1$. We gather some facts about g that we shall use in the proof of Proposition 7.1:

- $g^{rd} = \tilde{\chi}^r = 1$; hence, g takes values in (rd)-roots of unity and the sequence $(g^j)_{j \in \mathbb{N}}$ is periodic with period rd.
- $g^d = \tilde{\chi}$, $g^{2d} = \tilde{\chi}^2, \dots, g^{(r-1)d} = \tilde{\chi}^{r-1}$, $g^{rd} = 1$.
- g^j is aperiodic if $j \not\equiv 0 \pmod{d}$.

7.2. *Proof of Proposition 7.1*

In this subsection, we prove Proposition 7.1. For convenience, we use the following notation.

Definition 7.1. If I is a circular arc around 1 and $d \in \mathbb{N}$, we let

$$I/d := \{e(t/d)\colon e(t) \in I, t \in [-1/2, 1/2)\}.$$

Let $f\colon \mathbb{N} \to \mathbb{S}^1$ be a completely multiplicative function and I be an open arc around 1. Let also $F\colon \mathbb{S}^1 \to [0,1]$ be a continuous function such that

$$\mathbf{1}_{I/4} \le F \le \mathbf{1}_{I/2}.$$

It suffices to show that under the assumption of Proposition 7.1, we have

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, F(f(k(m^2-n^2))) \cdot F(f(k\,2mn)) \cdot F(f(k(m^2+n^2))) > 0. \tag{7.1}$$

We consider three cases.

Case 1. If f is pretentious, then $f = h \cdot \tilde{\chi}$, where $h \sim n^{it}$ for some $t \in \mathbb{R}$, and $\tilde{\chi}$ is a modified Dirichlet character, and the conclusion follows from our assumption.

Case 2. Suppose that f is aperiodic and f^d is pretentious for some $d \ge 2$. We use Lemma 7.3 to get a decomposition $f = gh$, where g takes values on rd roots of unity for some $r \in \mathbb{N}$, $g, \dots, g^{d-1}$ are aperiodic and $g^d = \tilde{\chi}$ for some modified Dirichlet character $\tilde{\chi}$, and $h \sim n^{it}$ for some $t \in \mathbb{R}$. Note first that in order to establish (7.1), it suffices to show that

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, c_{k,m,n} \cdot F(g(k(m^2-n^2))) \cdot F(g(k\,2mn)) \cdot F(g(k(m^2+n^2))) > 0, \tag{7.2}$$

where

$$c_{k,m,n} := F(h(k(m^2-n^2))) \cdot F(h(k\,2mn)) \cdot F(h(k(m^2+n^2))), \quad k,m,n \in \mathbb{N}. \tag{7.3}$$

This is so, since if $g(n), h(n) \in I/2$, then $f(n) = g(n) \cdot h(n) \in I$.

Main Claim. If for $G := \mathbf{1}_{\{1\}}$ and $c_{k,m,n}$ as in (7.3) we have

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, c_{k,m,n}\cdot G(\tilde{\chi}(k(m^2-n^2)))\cdot G(\tilde{\chi}(k\,2mn))\cdot G(\tilde{\chi}(k(m^2+n^2)))>0, \tag{7.4}$$

then (7.2) holds.

Note that (7.4) is satisfied from the hypothesis of Proposition 7.1. So to finish the proof of Proposition 7.1 in Case 2, it remains to verify the above claim.

We start with a simple identity. Since g takes values in rd roots of unity, we have

$$\mathbf{1}_{g=1} = \mathbb{E}_{0\le j<rd}\, g^j.$$

Since $F \ge \mathbf{1}_{\{1\}}$, it suffices to verify (7.2) with $\sum_{j=0}^{rd-1} g^j$ in place of $F\circ g$. Let

$$J := \{0\le j<rd\colon\, j\not\equiv 0 \pmod d\}.$$

Recall that g^j is aperiodic for $j\in J$. Also, $g^d=\tilde{\chi}$ and $\tilde{\chi}$ takes values on r-th roots of unity; hence,

$$\sum_{j=0}^{rd-1} g^j = \sum_{j=0}^{r-1}\tilde{\chi}^j + \sum_{j\in J} g^j = r\cdot \mathbf{1}_{\tilde{\chi}=1} + \sum_{j\in J} g^j.$$

Hence, in order to verify (7.2), it suffices to show that

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, c_{k,m,n}\cdot H(k(m^2-n^2))\cdot H(k\,mn)\cdot H(k(m^2+n^2))>0, \tag{7.5}$$

where

$$H := r\cdot \mathbf{1}_{\tilde{\chi}=1} + \sum_{j\in J} g^j.$$

After expanding the product, we get a finite sum of expressions of the form

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, c_{k,m,n}\cdot H_1(k(m^2-n^2))\cdot H_2(k\,mn)\cdot H_3(k(m^2+n^2)), \tag{7.6}$$

where each H_1, H_2, H_3 is either of the form $r\cdot \mathbf{1}_{\tilde{\chi}=1}$, or of the form g^j for some $j\in J$.

With this in mind, we see that the positiveness property (7.5) would follow once we establish the following three claims:

1. If $H_1=H_2=H_3=r\cdot \mathbf{1}_{\tilde{\chi}=1}$, then the limit in (7.6) is positive.
2. If $H_1=H_2=r\cdot \mathbf{1}_{\tilde{\chi}=1}$ and $H_3=g^j$ for some $j\in J$, then the limit in (7.6) is 0.
3. If $H_1=g^j$ or $H_2=g^j$ for some $j\in J$, then the limit in (7.6) is 0.

(We do not combine the last two cases because the argument we use is different.)

We prove (1). This follows immediately from the assumption (7.4) of the Main Claim.

We prove (2). We will show that for every $m,n\in\mathbb{N}$ with $m>n$, we have

$$\mathbb{E}^*_{k\in\mathbb{N}}\, c_{k,m,n}\cdot H_1(k(m^2-n^2))\cdot H_2(k\,mn)\cdot H_3(k(m^2+n^2))=0.$$

Using the definition of $c_{k,m,n}$ in (7.3) and uniform approximation of F, it suffices to show that for every $m,n\in\mathbb{N}$ with $m>n$, we have

$$\mathbb{E}^*_{k\in\mathbb{N}}\, H_1'(k(m^2-n^2))\cdot H_2'(k\,mn)\cdot H_3'(k(m^2+n^2))=0,$$

where $H_1' := \tilde{\chi}^{j_1}\cdot h^{j_2}$, $H_2' := \tilde{\chi}^{j_3}\cdot h^{j_4}$, and $H_3' := g^{j_5}\cdot h^{j_6}$, for some $j_1, j_2, j_3, j_4, j_6 \in \mathbb{Z}$ and $j_5 := j \in J$. Factoring out the multiplicative average $\mathbb{E}^*_{k\in\mathbb{N}}$, we get that it suffices to show that

$$\mathbb{E}^*_{k\in\mathbb{N}}\, \tilde{H}(k)=0 \quad \text{where} \quad \tilde{H} := \tilde{\chi}^{j_1+j_3}\cdot h^{j_2+j_4+j_6}\cdot g^{j_5}.$$

Since g^{j_5} is aperiodic and $\tilde{\chi}^{j_1+j_3}\cdot h^{j_2+j_4+j_6}$ is pretentious, we get that $H\neq 1$; hence, $\mathbb{E}^*_{k\in\mathbb{N}}\, H(k)=0$.

We prove (3). Suppose that $H_1 = g^{j_1}$ for some $j_1\in J$; the argument is similar for j_2. Using the definition of $c_{k,m,n}$ from (7.3) and uniform approximation of F, it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}_{m,n\in[N],m>n}\,\mathbb{E}^*_{k\in\mathbb{N}}\, H_1'(k(m^2-n^2))\cdot H_2'(k\,mn)\cdot H_3'(k(m^2+n^2))=0,$$

where $H_1' := g^{j_1}\cdot h^{j_2}$, $H_2' := \tilde{\chi}^{j_3}\cdot h^{j_4}$ or $g^{j_5}\cdot h^{j_6}$, $H_3' := \tilde{\chi}\cdot h^{j_7}$ or $g^{j_8}\cdot h^{j_9}$, for some $j_2,\dots,j_9\in\mathbb{Z}$. Factoring out the multiplicative average $\mathbb{E}^*_{k\in\mathbb{N}}\,(H_1'\cdot H_2'\cdot H_3')(k)$, we get that it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}_{m,n\in[N],m>n}\, H_1'((m^2-n^2))\cdot H_2'(mn)\cdot H_3'(m^2+n^2)=0, \tag{7.7}$$

where H_1' is an aperiodic completely multiplicative function (since g^{j_1} is aperiodic and h^{j_2} is pretentious), and H_2', H_3' are completely multiplicative functions. The hypothesis of Proposition 2.15 is satisfied, and we deduce that (7.7) holds.

This finishes the proof of the Main Claim and the proof of Case 2.

Case 3. Suppose that f^l is aperiodic for every $l\in\mathbb{N}$. In this case, we claim that the following identity holds:

$$\liminf_{N\to\infty}\mathbb{E}_{m,n\in[N],m>n}\,\mathbb{E}^*_{k\in\mathbb{N}}\, F(f(k(m^2-n^2)))\cdot F(f(k\,mn))\cdot F(f(k(m^2+n^2)))=\Big(\int F\, dm_{\mathbb{S}^1}\Big)^3.$$

If we prove this, then (7.1) holds, since $\int F\, dm_{\mathbb{S}^1}\geq m_{\mathbb{S}^1}(I/4)>0$.

Using uniform approximation of F, it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}_{m,n\in[N],m>n}\,\mathbb{E}^*_{k\in\mathbb{N}}\, F_1(k(m^2-n^2))\cdot F_2(k\,mn)\cdot F_3(k(m^2+n^2))=0$$

when for $i=1,2,3$, we have $F_i = f^{j_i}$, $j_i\in\mathbb{Z}$, and at least one of the j_1, j_2, j_3 is nonzero.

We consider two cases. Suppose first that $j_1=j_2=0$. Then $j_3\neq 0$. After factoring out the multiplicative average $\mathbb{E}^*_{k\in\mathbb{N}}$, it suffices to show that

$$\mathbb{E}^*_{k\in\mathbb{N}}\, f^{j_3}(k)=0.$$

This is the case since f^{j_3} is a nontrivial completely multiplicative function.

Suppose now that $j_1\neq 0$; the argument is similar if $j_2\neq 0$. After factoring out the multiplicative average $\mathbb{E}^*_{k\in\mathbb{N}}$, it suffices to show that

$$\lim_{N\to\infty}\mathbb{E}_{m,n\in[N],m>n}\, F_1((m^2-n^2))\cdot F_2(2mn)\cdot F_3(m^2+n^2)=0. \tag{7.8}$$

By our assumption, $F_1 = f^{j_1}$ is aperiodic. Note also that all F_1, F_2, F_3 are completely multiplicative function. The asserted identity then follows again from Proposition 2.15.[14]

[14]It is crucial for this part of the argument that we avoided working with an aperiodicity assumption on F_3, since such an assumption does not imply that (7.8) holds (but it does hold if F_1 or F_2 are aperiodic completely multiplicative functions).

8. Pythagorean triples on level sets - The pretentious case

Our goal in this section is to prove Proposition 2.18, which combined with Proposition 7.1 (Proposition 2.16 is a direct consequence) implies Theorem 1.5. We first restate Proposition 2.18 in a slightly more convenient form. Let $f\colon \mathbb{N} \to \mathbb{S}^1$ be a pretentious completely multiplicative function taking finitely many values. Then for some $d \in \mathbb{N}$, it takes values on d-th roots of unity. We can assume that d is minimal with this property, in which case, we have $f^j \neq 1$ for $j = 1, \dots, d-1$. In this case, we will show the following.

Proposition 8.1. *Let $d \in \mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{S}^1$ be pretentious multiplicative function taking values on d-th roots of unity and $\tilde{\chi}\colon \mathbb{N} \to \mathbb{S}^1$ be a modified Dirichlet character. Then*

$$\liminf_{N\to\infty} \mathbb{E}_{m,n\in[N],m>n}\, \mathbb{E}^*_{k\in\mathbb{N}}\, A(k\,(m^2-n^2)) \cdot A(k \cdot 2mn) \cdot A(k\,(m^2+n^2)) > 0,$$

where

$$A(n) := F(f(n)) \cdot F(\tilde{\chi}(n)), \quad n \in \mathbb{N}, \quad F := \mathbf{1}_{\{1\}}. \tag{8.1}$$

Remark. Note that in the argument that follows, we only deal with countably many choices of multiplicative functions and other choices of parameters, so we can choose a subsequence of positive integers (N_l) along which all the limits (as $l \to \infty$) that appear below exist. We make this implicit assumption throughout.

Before giving the proof of Proposition 8.1, we show how the concentration estimates of Corollary 2.17 follow from Propositions 2.5 and 2.11.

8.1. *Proof of Corollary 2.17*

We will deduce part (1) from Proposition 2.5. In a similar fashion, we can deduce part (2) from Proposition 2.11.

Let $\varepsilon > 0$ and $\varepsilon < 1$. Since f is a finite-valued pretentious multiplicative function, we have by Lemma 3.4 that $f \sim \chi$ for some Dirichlet character χ with period q and

$$\sum_{p\in\mathbb{P}} \frac{1}{p}\, |1 - f(p) \cdot \overline{\chi(p)}| < \infty.$$

Hence, there exists $K_0 \in \mathbb{N}$ such that

$$\sum_{p\geq K_0} \frac{1}{p}\, |1 - f(p) \cdot \overline{\chi(p)}| + K_0^{-1/2} \leq \varepsilon.$$

This implies that

$$\mathbb{D}(f,\chi;K_0,\infty) \leq \varepsilon \quad \text{and} \quad \left|\exp\left(F_N(f,K_0)\right) - 1\right| \ll \varepsilon,$$

where $F_N(f,K_0) = \sum_{K_0<p\leq N} \frac{1}{p}\left(f(p)\cdot\overline{\chi(p)} - 1\right)$.

We let $Q_0 := q \cdot \prod_{p\leq K_0} p$. If $Q \in \mathbb{N}$ is such that $Q_0 \mid Q$, then using the second remark following Proposition 2.5 with $t = 0$, we get that

$$\limsup_{N\to\infty} \mathbb{E}_{n\in[N]} |f(Qn+1) - \exp\left(F_N(f,K_0)\right)| \ll \varepsilon.$$

Since $|\exp\left(F_N(f,K_0)\right) - 1| \ll \varepsilon$, this completes the proof.

8.2. Proof of Proposition 8.1

Recall that $A(n)$ is given by (8.1). Since $A(n) \geq 0$ for every $n \in \mathbb{N}$, it suffices to show that there exist $Q \in \mathbb{N}$ and $N_l \to \infty$ (which can be taken to be a subsequence of any given $M_l \to \infty$) such that all limits appearing below as $l \to \infty$ exist and

$$\lim_{l\to\infty} \mathbb{E}_{m,n\in[N_l],m>n} \mathbb{E}^*_{k\in\mathbb{N}} A(k((Qm+1)^2-(Qn)^2)\cdot A(k\,2(Qm+1)(Qn))\cdot A(k((Qm+1)^2+(Qn)^2)) > 0.$$

Since f takes values on d-th roots of unity and $\tilde{\chi}$ takes values on d'-th roots of unity for some $d, d' \in \mathbb{N}$, we have

$$F(f) = \mathbf{1}_{f=1} = \mathbb{E}_{0\leq j<d}\, f^j, \quad F(\tilde{\chi}) = \mathbf{1}_{\tilde{\chi}=1} = \mathbb{E}_{0\leq j<d'}\, \tilde{\chi}^j. \tag{8.2}$$

Let $m, n \in \mathbb{N}$ with $m > n$ be fixed. In order to compute

$$\mathbb{E}^*_{k\in\mathbb{N}} A(k(m^2-n^2))\cdot A(k\,2mn)\cdot A(k(m^2+n^2)),$$

we use (8.2), expand, and use that by Lemma 3.2, we have $\mathbb{E}^*_{k\in\mathbb{N}}\, g(k) = 0$ for all completely multiplicative functions $g\colon \mathbb{N} \to \mathbb{U}$ with $g \neq 1$ (in particular, this holds if $g := f^k \cdot \tilde{\chi}^{k'} \neq 1$). We see that the previous expression is equal to $1/(dd')^3$ times

$$\sum_{k_i,k_i'\in\mathcal{K}} (f^{k_1}\cdot\tilde{\chi}^{k_1'})(m^2-n^2)\cdot(f^{k_2}\cdot\tilde{\chi}^{k_2'})(2mn)\cdot(f^{k_3}\cdot\tilde{\chi}^{k_3'})(m^2+n^2),$$

where

$$\mathcal{K} := \{0\leq k_1,k_2,k_3<d,\ 0\leq k_1',k_2',k_3'<d'\colon f^{k_1+k_2+k_3}\cdot\tilde{\chi}^{k_1'+k_2'+k_3'} = 1\}.$$

In what follows, we implicitly assume that all k_i, k_i' belong to $\mathcal{K}$.

Let q be the period of χ, then $\tilde{\chi}(qn+1) = 1$ for every $n \in \mathbb{N}$. Taking the previous facts in mind, we see that in order to establish the needed positiveness, it suffices to show that there exists $Q \in \mathbb{N}$ such that $q \mid Q$ and

$$L(Q) := \sum_{k_i,k_i'\in\mathcal{K}} \Re(L_{k_1,k_2,k_3,k_2'}(Q)) > 0, \tag{8.3}$$

where

$$L_{k_1,k_2,k_3,k_2'}(Q) := \lim_{l\to\infty} \mathbb{E}_{m,n\in[N_l],m>n} f^{k_1}((Qm+1)^2-(Qn)^2)\cdot f^{k_2}(2(Qm+1)(Qn))\cdot f^{k_3}((Qm+1)^2+(Qn)^2)\cdot\tilde{\chi}^{k_2'}(2(Qn)). \tag{8.4}$$

(We used that $\tilde{\chi}(j) = 1$ for $j \in Q\mathbb{Z}+1$.)

Claim 1 ($f^{k_2}\cdot\tilde{\chi}^{k_2'} = 1$). *For every $\varepsilon > 0$, there exists $Q_0 = Q_0(f,\tilde{\chi},\varepsilon) \in \mathbb{N}$ with $q \mid Q_0$, such that the following holds: If $Q \in \mathbb{N}$ satisfies $Q_0 \mid Q$, then for all $k_1, k_2, k_3, k_2' \in \mathcal{K}$ with $f^{k_2}\cdot\tilde{\chi}^{k_2'} = 1$, we have*

$$\Re(L_{k_1,k_2,k_3,k_2'}(Q)) \geq 1-\varepsilon. \tag{8.5}$$

As a consequence, there exists $Q_0 := Q_0(f,\tilde{\chi})$, such that if $Q \in \mathbb{N}$ satisfies $Q_0 \mid Q$, then

$$\sum_{k_1,k_2,k_3,k_2'\colon f^{k_2}\cdot\tilde{\chi}^{k_2'}=1} \Re(L_{k_1,k_2,k_3,k_2'}(Q)) \geq 1. \tag{8.6}$$

We prove the claim. Let $\varepsilon > 0$. Note first that since $f^{k_2} \cdot \tilde{\chi}^{k_2'} = 1$, we get using (8.4) that

$$L_{k_1,k_2,k_3,k_2'}(Q) := \lim_{l\to\infty} \mathbb{E}_{m,n\in[N_l],m>n}\, f^{k_1}(Q(m-n)+1) \cdot f^{k_1}(Q(m+n)+1)$$
$$f^{k_2}(Qm+1) \cdot f^{k_3}((Qm+1)^2+(Qn)^2). \quad (8.7)$$

Using this identity, Corollary 2.17 and Lemma 3.1, we deduce that there exists $Q_0 = Q_0(f,\varepsilon)$, with $q \mid Q_0$, such that if $Q \in \mathbb{N}$ satisfies $Q_0 \mid Q$, then for all $k_1, k_2, k_3, k_2' \in \mathcal{K}$ such that $f^{k_2} \cdot \tilde{\chi}^{k_2'} = 1$, we have

$$|L_{k_1,k_2,k_3,k_2'}(Q) - 1| \ll_d \varepsilon.$$

This proves (8.5). Since $L_{0,0,0,0} = 1$, using (8.5) for $\varepsilon = 1/2$, we deduce (8.6). This completes the proof of Claim 1.

Claim 2 ($\mathrm{p}f^{k_2} \cdot \tilde{\chi}^{k_2'} \neq 1$). *Let $Q_0 \in \mathbb{N}$ be such that (8.6) holds for every $Q \in \mathbb{N}$ such that $Q_0 \mid Q$. Then for every $\varepsilon > 0$, there exists $Q_1 = Q_1(f, \tilde{\chi}, \varepsilon) \in \mathbb{N}$ such that $Q_0 \mid Q_1$ (hence, (8.5) holds for $Q = Q_1$) and*

$$\sum_{k_1,k_2,k_3,k_2' :\, f^{k_2}\cdot\tilde{\chi}^{k_2'}\neq 1} \Re(L_{k_1,k_2,k_3,k_2'}(Q_1)) \geq -\varepsilon. \quad (8.8)$$

We prove the claim. Let $\varepsilon > 0$. It suffices to show that

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K}\, L_{k_1,k_2,k_3,k_2'}(Q) = 0, \quad \text{as long as } f^{k_2} \cdot \tilde{\chi}^{k_2'} \neq 1. \quad (8.9)$$

Note that

$$L_{k_1,k_2,k_3,k_2'}(Q) := (f^{k_2} \cdot \tilde{\chi}^{k_2'})(2Q) \cdot L'_{k_1,k_2,k_3,k_2'}(Q),$$

where

$$L'_{k_1,k_2,k_3,k_2'}(Q) := \lim_{l\to\infty} \mathbb{E}_{m,n\in[N_l],m>n}\, f^{k_1}(Q(m-n)+1) \cdot f^{k_1}(Q(m+n)+1)$$
$$f^{k_2}(Qm+1) \cdot f^{k_3}((Qm+1)^2+(Qn)^2) \cdot f^{k_2}(n) \cdot \tilde{\chi}^{k_2}(n). \quad (8.10)$$

We prove (8.9). Let $\varepsilon' > 0$. Using (8.10), Corollary 2.17 and Lemma 3.1, we get that there exists $Q_2 = Q_2(f, \varepsilon')$ such that the following holds: If $Q \in \mathbb{N}$ satisfies $Q_2 \mid Q$, then, for all $k_1, k_2, k_3, k_2' \in \mathcal{K}$, we have

$$|L'_{k_1,k_2,k_3,k_2'}(Q) - c_{k_2}| \ll \varepsilon', \quad (8.11)$$

where

$$c_{k_2} := \lim_{l\to\infty} \mathbb{E}_{n\in[N_l]}\, f^{k_2}(n) \cdot \tilde{\chi}^{k_2}(n).$$ [15]

Hence, by (8.4), (8.10), and (8.11), we have

$$|L_{k_1,k_2,k_3,k_2'}(Q) - c_{k_2} \cdot (f^{k_2} \cdot \tilde{\chi}^{k_2'})(2Q)| \ll \varepsilon' \quad \text{for all } Q \text{ with } Q_2 \mid Q. \quad (8.12)$$

Since by assumption, $f^{k_2} \cdot \tilde{\chi}^{k_2'} \neq 1$, we have

$$\lim_{K\to\infty} \mathbb{E}_{Q\in\Phi_K}\, (f^{k_2} \cdot \tilde{\chi}^{k_2'})(Q) = 0.$$

Combining this with (8.12), we get that (8.9) holds. This proves Claim 2.

[15] The limit exists since $f^{k_2} \cdot \tilde{\chi}^{k_2}$ is finite-valued, but we do not have to use this.

Putting together the two claims, in particular the estimates (8.6) and (8.8), we deduce that for every $\varepsilon > 0$, there exists $Q_1 = Q_1(f, \tilde{\chi}, \varepsilon) \in \mathbb{N}$ with $q \mid Q_1$, such that $L(Q_1) \geq 1 - \varepsilon$; hence, (8.3) holds for $Q = Q_1$. This completes the proof.

8.3. *More general equations*

Our methods allow us to extend Theorem 1.5 and cover more general equations of the form

$$ax^2 + by^2 = cz^2, \tag{8.13}$$

where $a, b, c \in \mathbb{N}$ are squares satisfying Rado's condition (i.e., we have either $a = c$, or $b = c$, or $a + b = c$). We summarize the key differences in the argument.

Suppose first that $a = c$ (the case $b = c$ is similar). Then, as in Section 1.4, we get parametrizations of (8.13) of the form

$$x = \ell_1\,(m^2 - n^2), \quad y = \ell_2\, mn, \quad z = \ell_3\,(m^2 + n^2),$$

for some $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$, and our hypothesis $a = c$ implies that we can take $\ell_1 = \ell_3$. This fact is then used to handle Claim 1 in the proof of Proposition 8.1, and the rest of the argument remains unchanged. To see how Claim 1 is handled, note that in our setting, the expressions $L_{k_1,k_2,k_3,k_2'}(Q)$ in (8.4) take the form

$$\begin{aligned} L_{k_1,k_2,k_3,k_2'}(Q) := c_{k_1,k_2,k_3} \cdot \lim_{l\to\infty} \mathbb{E}_{m,n\in[N_l],m>n}\, f^{k_1}((Qm+1)^2 - (Qn)^2) \cdot f^{k_2}(2(Qm+1)(Qn))\cdot \\ f^{k_3}((Qm+1)^2 + (Qn)^2) \cdot \tilde{\chi}^{k_2'}(2(Qn)), \end{aligned} \tag{8.14}$$

where

$$c_{k_1,k_2,k_3} := (f^{k_1} \cdot \tilde{\chi}^{k_1'})(\ell_1) \cdot (f^{k_2} \cdot \tilde{\chi}^{k_2'})(\ell_2) \cdot (f^{k_3} \cdot \tilde{\chi}^{k_3'})(\ell_3).$$

Using additionally that $\ell_1 = \ell_3$ and that $f^{k_2} \cdot \tilde{\chi}^{k_2'} = 1$, $f^{k_1+k_2+k_3} \cdot \tilde{\chi}^{k_1'+k_2'+k_3'} = 1$, which are standing assumptions in Claim 1, we deduce that $c_{k_1,k_2,k_3} = 1$. With this information at hand, the proof of Claim 1 in our setting is exactly the same as in the case of Pythagorean triples.

Now suppose that $a + b = c$, in which case, the argument is a bit different and somewhat simpler. As shown in Step 2 of [21, Appendix C], we can obtain parametrizations of (8.13) of the form

$$x = k\,(m + \ell_1 n)\cdot(m + \ell_2 n), \quad y = k\,(m + \ell_3 n)\cdot(m + \ell_4 n), \quad z = k\,(m^2 + (\ell_5 n)^2),$$

for suitable $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 \in \mathbb{N}$ that satisfy $\ell_1 \neq \ell_2$, $\ell_3 \neq \ell_4$ and $\{\ell_1, \ell_2\} \neq \{\ell_3, \ell_4\}$. Note that our assumption $a + b = c$ was used to ensure that the coefficient of m is 1 in all linear forms. We average on the grid $\{(Qm + 1, Qn) \colon m, n \in \mathbb{N}\}$. We will demonstrate how Claims 1 and 2 in the proof of Proposition 8.1 can be established within our framework. The remainder of the argument remains unaltered. In our context, the expressions $L_{k_1,k_2,k_3,k_2'}(Q)$ in (8.4) take the form

$$\begin{aligned} L_{k_1,k_2,k_3,k_2'}(Q) := \lim_{l\to\infty} \mathbb{E}_{m,n\in[N_l],m>n}\, f^{k_1}\big((Q(m+\ell_1 n)+1)(Q(m+\ell_2 n)+1)\big)\cdot \\ f^{k_2}\big((Q(m+\ell_3 n)+1)(Q(m+\ell_4 n)+1)\big) \cdot f^{k_3}((Qm+1)^2 + (Q\ell_5 n)^2). \end{aligned} \tag{8.15}$$

Using the concentration estimates of Corollary 2.17, we can see that Claim 1 holds without assuming that $f^{k_2} \cdot \tilde{\chi}^{k_2'} = 1$. Therefore, in our setting, Claim 2 in the proof of Proposition 8.1 is already addressed by this case and requires no further explanation.

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