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UNIVALENT HARMONIC RING MAPPINGS VANISHING ON THE INTERIOR BOUNDARY

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ABSTRACT. We give a characterization of univalent positively oriented harmonic mappings f defined on an exterior neighbourhood of the closed unit disk $\{z : |z| \leq 1\}$ such that $\lim_{\substack{|z|>1 \\ z\to e^{it}}} f(z) \equiv 0$.

1. **Introduction.** Let K be a compact continuum of the complex plane $\mathbb{C}$ such that $\bar{\mathbb{C}}\setminus K$ is simply connected. Denote by D a domain of $\mathbb{C}$ containing K. We shall call $D\setminus K = V_e(K)$ an exterior neighbourhood of K.

Suppose that $f = u+iv$ is a univalent (one-to-one) harmonic ($\Delta f \equiv 0$) mapping defined on $V_e(K)$. Then f is either orientation preserving or orientation reversing on $V_e(K)$. With no loss of generality we may assume that the first case holds, since if not, replace $f(z)$ by $f(\bar{z})$. This yields to the fact that the function

$$a(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} \in H\big(V_e(K)\big) \text{ and } |a(z)| < 1 \text{ on } V_e(K), \tag{1.0}$$

where $H(E)$ stands for the set of all analytic functions on an open neighbourhood of E. The fact that f is univalent and orientation preserving on $V_e(K)$ implies that $0 \notin f_z\big(V_e(K)\big)$ [1], and that the Jacobian determinant $J = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ on $V_e(K)$. Moreover, f_z and $\overline{f_{\bar{z}}}$ are analytic on $V_e(K)$ [3]. Hence, f is *locally* quasiconformal and pseudoanalytic of the second kind (in the sense of L. Bers [1]) on $V_e(K)$. But contrary to the case of quasiconformal mappings it is possible that $\lim_{\substack{z\to\partial K \\ z\in V_e(K)}} f(z) \equiv 0$. For instance, the harmonic function

$$f(z) = z - \frac{1}{\bar{z}} + 2A \ln|z|, \quad |A| \leq 1, \tag{1.1}$$

maps the complement $\mathbb{C}\setminus U$ of the unit disk U one-to-one onto $\mathbb{C}\setminus\{0\}$ (see [4]).

Let χ be the conformal univalent mapping from $\mathbb{C}\setminus K$ onto $\mathbb{C}\setminus U$ normalized by $\chi(z) = \alpha z + O(1)$, $\alpha > 0$ in a neighbourhood of infinity. Then $f\circ\chi$ is univalent, harmonic and orientation preserving on an exterior neighbourhood $V_e(\bar{U})$ of $\bar{U}$. Therefore, we may restrict our attention to the case $K = \bar{U}$ and $V_e(\bar{U}) = V_R = \{z : 1 < |z| < R\}$ for some $R > 1$.

The following notion will be used often.

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DEFINITION. We say that a harmonic mapping is *positively oriented* on an exterior neighbourhood V_R of ∂U if f is orientation-preserving and $\oint_\gamma d\arg f > 0$ on any simply closed curve γ in V_R winding in the positive sense around the origin.

REMARK. A positively oriented harmonic mapping on V_R is orientation-preserving. But the converse does not hold as the example

$$f(z) = \frac{1}{z}, \quad 1 < |z| < 2$$

shows.

As in the case of analytic functions, we say that f is harmonic on a set E if f is harmonic on open neighbourhood of E.

The purpose of this paper is to characterize univalent, positively oriented harmonic mappings f defined on an exterior neighbourhood V_R (R is not prescribed) of $\bar{U}$ such that

$$\lim_{\substack{|z|>1 \\ z\to e^{it}}} f(z) \equiv 0. \tag{1.2}$$

In connection with this problem, a modified version of the following result was shown in [4, Theorem 3.3].

THEOREM A. *Let f be a harmonic mapping defined on $\{z : |z| > 1\}$. Then f is positively oriented and univalent on $\{z : |z| > 1\}$ satisfying $f(|z| > 1) = \mathbb{C}\setminus\{0\}$ if and only if f is of the form*

$$f(z) = C\Big[z + B\bar{z} + 2A\ln|z| - \frac{1}{\bar{z}} - \frac{B}{z}\Big], \tag{1.3}$$

where $C \in \mathbb{C}$, $B = cd$, $A = c + d$, $|c| < 1$, $|d| \leq 1$.

Observe that we allow that $|d| = 1$ but that $|c|$ has to be strictly less than one. Put

$$h(z) = C\Big[z - \frac{B}{z}\Big], \quad g(z) = \bar{C}\Big[\bar{B}z - \frac{1}{z}\Big]$$

and

$$\psi(z) = zf_z(z) = zh'(z) + AC = C\Big[z + \frac{B}{z} + A\Big].$$

Then h and g belong to $H(\mathbb{C}\setminus\{0\})$ and we have the following properties:

$$h(z) = -\overline{g\big(\tfrac{1}{\bar{z}}\big)} \text{ in } \mathbb{C}\setminus\{0\}; \tag{1.4}$$

$$\psi'(z) = C\Big[1 - \frac{B}{z^2}\Big] \text{ does not vanish on the unit circle } \partial U; \tag{1.5}$$

$$p(z) = \frac{z\psi'(z)}{\psi(z)} = \frac{1 - \frac{B}{z^2}}{1 + \frac{B}{z^2} + \frac{A}{z}} = 1 + \sum_{j\in\mathbf{Z}\setminus\{0\}} p_j z^j \in H(|z| > 1) \tag{1.6}$$

satisfies $\operatorname{Re} p \geq k > 0$ on $|z| > 1$ for some $k > 0$.

Indeed, we have for $|z| > 1$:

$$\operatorname{Re}\frac{z\psi'(z)}{\psi(z)} = \operatorname{Re}\frac{1-\frac{dc}{z^2}}{(1+\frac{c}{z})(1+\frac{d}{z})}$$
$$= \frac{1}{2}\operatorname{Re}\Big(\frac{1-\frac{c}{z}}{1+\frac{c}{z}}\Big) + \frac{1}{2}\operatorname{Re}\Big(\frac{1-\frac{d}{z}}{1+\frac{d}{z}}\Big) \geq \frac{1}{2}\frac{1-|\frac{c}{z}|}{1+|\frac{c}{z}|} \geq \frac{1}{2}\frac{1-|c|}{1+|c|} = k > 0.$$

Let f be a harmonic mapping defined on an exterior neighbourhood V_R of $\bar{U}$. Then (Lemma 2.1) $\lim_{\substack{|z|>1 \\ z\to e^{it}}} f(z) \equiv 0$ if and only if f is of the form

(1.7) $$f(z) = h(z) - h(\frac{1}{\bar{z}}) + 2A\ln|z|, \quad A \in \mathbb{C},$$

where $h \in H\big(\frac{1}{R} < |z| < R\big)$.

A second version of our main result, Theorem 3.2, states the following: The mapping (1.7) is univalent and positively oriented in V_R for some $R > 1$ if and only if (1.4), (1.5) and (1.6) hold in $V_{\hat{R}}$ for some $\hat{R} > 1$, where $\psi(z) = zf_z(z) = zh'(z) + A$.

The property (1.5) can be replaced by the following condition (Remark 3.5):

(1.8) ψ has at most one zero on ∂U, which is of order one.

Furthermore, (Lemma 2.5), the statement (1.6) can be replaced by the condition

(1.9) $$\oint_{|z|=\rho} d\arg\psi = 1, \quad \rho \in (1,\hat{R}) \text{ and } \operatorname{Re} p(z) \geq k > 0 \text{ on } V_{\hat{R}}$$

or by

(1.10) ψ is univalent on $V_{\hat{R}}$ satisfying $\operatorname{Re} p(z) \geq k > 0$

where $p(z) = \frac{z\psi'(z)}{\psi(z)}$.

The functions which appear in our considerations are closely related to the Carathéodory class

$$P_q = \Big\{p \in H(q < |z| < 1) : p(z) = 1 + \sum_{j\in \mathbf{Z}\setminus\{0\}} p_j z^j$$
$$\text{and } \operatorname{Re} p(z) > 0 \text{ on } \{z : q < |z| < 1\}\Big\}, \quad 0 < q < 1,$$

which has been studied by several authors (e.g.) [5], [6], [7]. The class S_q^* of starlike functions defined by the relation:

$$F \in S_q^* \Leftrightarrow \frac{zF'(z)}{F(z)} \in P_q,$$

was considered in [2] and in [7].

In order to simplify a rather lengthy proof (Section 4) of Theorem 3.1 we state several lemmas in Section 2. Finally, in Section 5 we discuss the region of values of A.

2. **Some auxiliary lemmas.** We start this section with the following lemma.

LEMMA 2.1. *Let f be any harmonic function on V_R (for some $R > 1$) satisfying (1.2). Then f has a harmonic continuation across $\partial U = \{z : |z| = 1\}$ which is of the form*

$$f(z) = h(z) - \overline{h(\frac{1}{\bar{z}})} + 2A\ln|z|, \quad A \in \mathbb{C}, \tag{2.1}$$

where $h(z) = \sum_{j\in\mathbf{Z}} a_j z^j \in H(\frac{1}{R} < |z| < R)$. Observe that $f(\frac{1}{\bar{z}}) = -f(z)$ on $\{z : \frac{1}{R} < |z| < R\}$.

Conversely, each harmonic mapping on ∂U satisfying (2.1) has the property (1.2).

PROOF. Since f is harmonic on V_R, f admits the representation

$$f(z) = h(z) + \overline{g(z)} + 2A\ln|z|, \tag{2.2}$$

where $A \in \mathbb{C}$, $h(z) = \sum_{j\in\mathbf{Z}} a_j z^j \in H(V_R)$, $g(z) = \sum_{j\in\mathbf{Z}} b_j z^j \in H(V_R)$. This implies that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2\,dt = \sum_{j\in\mathbf{Z}} |a_j + \bar{b}_{-j}|^2 = 0,$$

and therefore h and g admit an analytic continuation onto $\{z : \frac{1}{R} < |z| < R\}$ satisfying

$$g(\frac{1}{\bar{z}}) = -\overline{h(z)} \quad\text{and}\quad h(\frac{1}{\bar{z}}) = -\overline{g(z)}. \tag{2.3}$$ ∎

In the next Lemma we consider again harmonic mappings having the properties stated in Lemma 2.1. We give a necessary and sufficient condition in order that f is orientation-preserving on an exterior neighbourhood of $\bar{U}$.

LEMMA 2.2. *Let*

$$f(z) = h(z) - \overline{h(\frac{1}{\bar{z}})} + 2A\ln|z|, \quad A \in \mathbb{C}$$

be harmonic on $\{z : |z| = 1\}$, $f \not\equiv const$. There is an exterior neighbourhood V_R of $\bar{U}$ such that f is orientation-preserving on V_R if and only if there exists a constant $k > 0$ such that $\psi(z) = zh'(z) + A$ satisfies

$$+\infty > \operatorname{Re}\frac{e^{it}\psi'(e^{it})}{\psi(e^{it})} \geq k > 0 \tag{2.4}$$

whenever $\psi(e^{it}) \neq 0$.

PROOF. By Lemma 2.1. $h \in H(\frac{1}{R_1} < |z| < R_1)$ for some $R_1 > 1$. Therefore, the dilatation function

$$a(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} = \frac{\overline{\psi(\frac{1}{\bar{z}})}}{\psi(z)} \tag{2.5}$$

is meromorphic on $\{z : \frac{1}{R_1} < |z| < R_1\}$. We may choose R_1 so close to one such that the only possible zeros of ψ in $\{z : \frac{1}{R_1} < |z| < R_1\}$ are on ∂U. Furthermore, we have

$$|a(e^{it})| = \left|\frac{\overline{\psi(e^{it})}}{\psi(e^{it})}\right| = 1$$

whenever $\psi(e^{it}) \neq 0$. Therefore, the zeros of ψ on ∂U are removable singularities of the function $a(z)$ and we conclude that

$$a \in H(\frac{1}{R_1} < |z| < R_1) \text{ and } |a(e^{it})| \equiv 1 \text{ for all } t \in [0, 2\pi]. \tag{2.6}$$

Moreover, by the reflection principle, we have

$$a(z) = \frac{1}{\overline{a(\frac{1}{\bar{z}})}}. \tag{2.7}$$

(a) Suppose now that f is orientation-preserving on V_R. Then, by (2.7), $|a(z)| < 1$ on V_R and $|a(z)| > 1$ on $\{z : \frac{1}{R} < |z| < 1\}$, from which follows that $a'(e^{it}) \neq 0$ for all $t \in [0, 2\pi]$ and therefore $\frac{za'(z)}{a(z)}$ is a nonvanishing analytic function on ∂U. We have, for all $z \in \partial U$,

$$\frac{za'(z)}{a(z)} = \text{Re}\,\frac{za'(z)}{a(z)} = \frac{1}{2}\frac{\partial |a|^2}{\partial |z|} < 0, \tag{2.8}$$

which implies that there is a constant $c > 0$ such that

$$\frac{za'(z)}{a(z)} = \text{Re}\,\frac{za'(z)}{a(z)} = \frac{1}{2}\frac{\partial |a|^2}{\partial |z|} \leq -c < 0 \text{ for all } z \in \partial U. \tag{2.9}$$

Finally, we have from (2.5), whenever $\psi(z) \neq 0$,

$$\frac{za'(z)}{a(z)} = -\left[\frac{z\psi'(z)}{\psi(z)} + \frac{1}{z}\frac{\overline{\psi'(\frac{1}{\bar{z}})}}{\overline{\psi(\frac{1}{\bar{z}})}}\right], \tag{2.10}$$

which implies that

$$\infty > \text{Re}\,\frac{e^{it}\psi'(e^{it})}{\psi(e^{it})} = -\frac{1}{2}\frac{e^{it}a'(e^{it})}{a(e^{it})} \geq \frac{c}{2} = k > 0, \tag{2.11}$$

for all e^{it} for which $\psi(e^{it}) \neq 0$ and (2.4) holds.

(b) Now, suppose that the function ψ satisfies (2.4). Then, by (2.11) and (2.9), we have in the case of $\psi(e^{it}) \neq 0$

$$\frac{e^{it}a'(e^{it})}{a(e^{it})} = \frac{1}{2}\frac{\partial |a|^2}{\partial |z|}(e^{it}) \leq -2k < 0. \tag{2.12}$$

Since, by (2.6), $a \in H(\frac{1}{R_1} < |z| < R_1)$ and $|a(e^{it})| \equiv 1$, we conclude that (2.12) holds for all $t \in [0, 2\pi]$. Put $R \in (1, R_1)$ such that $\text{Re}\,\frac{za'(z)}{a(z)} \leq -k$ on $\{z : \frac{1}{R} < |z| < R\}$. Then, the relation

$$\frac{1}{2}\frac{\partial |a|^2}{\partial |z|} = \frac{|a|^2}{|z|}\text{Re}\,\frac{za'(z)}{a(z)} \tag{2.13}$$

shows that $|a(z)| < 1$ on V_R, and Lemma 2.2 is proved. ∎

For the completeness we give a short proof of the following lemma.

LEMMA 2.3. *Let Φ be in $H(\frac{1}{R} < |z| < R)$ for some $R > 1$, such that Φ is real on ∂U and $\frac{1}{2\pi}\oint_{|z|=\rho} d\arg z\Phi' = 1$ for all $\rho \in (1,R)$. Then Φ' has exactly two zeros on ∂U which are of order one.*

PROOF. By the reflection principle, we have

$$\overline{\Phi(\frac{1}{\bar{z}})} = \Phi(z) \text{ and } -\frac{1}{z}\overline{\Phi'(\frac{1}{\bar{z}})} = z\Phi'(z).$$

Since $\Phi(\partial U)$ is a bounded real interval, there exists an $e^{i\beta}$ and an $e^{i\gamma}$, $e^{i\beta} \neq e^{i\gamma}$, such that $\Phi'(e^{i\beta}) = \Phi'(e^{i\gamma}) = 0$. Applying the argument principle, we get for $\rho \in (1,R)$

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg z\Phi' - \frac{1}{2\pi}\oint_{|z|=\frac{1}{\rho}} d\arg z\Phi' = 2\cdot\frac{1}{2\pi}\oint_{|z|=\rho} d\arg z\Phi' = 2$$

and the result follows. ∎

The next lemma gives the important relation of some auxiliary analytic function Φ to a given harmonic mapping f.

LEMMA 2.4. *Let*

$$f(z) = h(z) - \overline{h(\frac{1}{\bar{z}})} + 2A\ln|z|, \quad A \in \mathbb{C}$$

be harmonic on $\{z : \frac{1}{R} < |z| < R\}$ and suppose that f is orientation-preserving on V_R.

Put

$$e^{i\alpha} = \begin{cases} \bar{A}/|A| & \text{if } A \neq 0 \\ 1 & \text{if } A = 0, \end{cases}$$

and define

$$\Phi(z) = e^{i\alpha}h(z) + e^{-i\alpha}\overline{h(\frac{1}{\bar{z}})}.$$

Then f is univalent on V_{R_1} for some $R_1 \in (1,R)$ if and only if Φ is univalent on V_{R_2} for some $R_2 \in (1,R)$.

PROOF. Observe that $\Phi \in H(\frac{1}{R} < |z| < R)$ and that Φ is real on ∂U. Furthermore we have

$$\begin{aligned} z\Phi'(z) &= ze^{i\alpha}\left[h'(z) - e^{-2i\alpha}\overline{h'(\frac{1}{\bar{z}})}\frac{1}{z^2}\right] + |A| - |A| \\ &= e^{i\alpha}[zh'(z) + A] - e^{-i\alpha}\left[\overline{\frac{1}{\bar z}h'(\frac{1}{\bar{z}})} + \bar{A}\right] \\ &= e^{i\alpha}\psi(z) - e^{-i\alpha}\overline{\psi(\frac{1}{\bar{z}})} \\ &= e^{i\alpha}\psi(z)[1 - a(z)e^{-2i\alpha}], \end{aligned} \tag{2.14}$$

where $\psi(z) = zf_z(z) = zh'(z) + A$ and $a(z) = \overline{\psi(\frac{1}{\bar{z}})}/\psi(z)$. Since f is orientation-preserving on V_R, $|a(z)| < 1$ on V_R. Therefore Φ' and f_z vanishes simultaneously. In other words, Φ is locally univalent on V_R if and only if f is locally univalent on V_R.

(a) Suppose first that Φ is univalent in V_{R_2} for some $R_2 \in (1, R)$. Put $w = u+iv = f(z)$ and $\zeta = \xi + i\eta = \Phi(z)$. Define

$$(2.15)\quad \begin{aligned} w = F(\zeta) &= e^{i\alpha} f \circ \Phi^{-1}(\zeta) \\ &= \zeta - 2\,\mathrm{Re}\Big\{e^{i\alpha} h\Big(\frac{1}{\bar{z}}\Big)\Big\} + 2|A| \ln|z| \\ &= \zeta - 2\,\mathrm{Re}\Big\{e^{i\alpha} h\Big(\frac{1}{\overline{\Phi^{-1}(\zeta)}}\Big)\Big\} + 2|A| \ln \Phi^{-1}(\zeta)| \\ &= \zeta - q(\zeta), \end{aligned}$$

where q is real on $\Phi(V_{R_2})$. Put $J = \Phi(\partial U)$ and let γ be a convex closed Jordan around $J, \gamma \in \Phi(V_{R_2})$.

Denote by G the doubly connected domain bounded by γ and J. Then, F is locally univalent on G satisfying

$$(2.16)\quad v(\zeta) = \mathrm{Im}\, F(\zeta) = \mathrm{Im}\, \zeta = \eta,$$

and

$$(2.17)\quad \frac{\partial u}{\partial \xi} = \mathrm{Re}\Big\{\frac{1 + a\big(\Phi^{-1}(\zeta)\big)}{1 - a(\Phi^{-1}(\zeta))}\Big\} > 0 \text{ on } G.$$

It follows that F is univalent on $G \setminus \{\zeta : \mathrm{Im}\, \zeta = 0\}$. Let ξ_1 and ξ_2 belong to $G \cap \{\zeta : \mathrm{Im}\, \zeta = 0\}$ such that, $\xi_1 < \xi_2$. Note that $F(\xi_1)$ and $F(\xi_2)$ are real. We will show that $F(\xi_1) < F(\xi_2)$, from which it follows that F is univalent in G. We can find closed intervals $\Gamma_n = [\xi_1 + i\eta_n, \xi_2 + i\eta_n]$ in G with $\eta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Since, by (2.17), $\mathrm{Re}\, F$ is strictly increasing on Γ_n, we conclude that $F(\xi_1) \leq F(\xi_2)$. On the other hand, $\frac{\partial\, \mathrm{Re}\, F}{\partial \xi}(\xi_i) > 0$, $i = 1, 2$, implies that $F(\xi_1) < F(\xi_2)$.

Choose now $R_1 \in (1, R)$ such that $V_{R_1} \subset \Phi^{-1}(G)$. Then f is univalent on V_{R_1} and in one direction Lemma 2.4 is proved.

(b) Suppose now that f is univalent on V_{R_1}. Then, by the same reasoning, we get that Φ is univalent on an exterior neighbourhood V_{R_2} of $\bar{U}$.

Indeed,

$$\zeta = F^{-1}(w) = \Phi \circ f^{-1}(e^{-i\alpha} w) = w + q_1(w),$$

where $q_1 = 2\,\mathrm{Re}\Big\{e^{i\alpha} h\Big(1/\overline{f^{-1}(e^{-i\alpha} w)}\Big)\Big\} - 2|A| \ln|f^{-1}(e^{-i\alpha} w)|$ is real on $f(V_{R_1})$. Define $G_1 = \{w : 0 < |w| < \gamma\} \subset f(V_{R_2})$. Then, from the equality

$$\eta = \mathrm{Im}\, F^{-1}(w) \equiv \mathrm{Im}\, w = v$$

and the condition $\frac{\partial u}{\partial \xi} > 0$, it follows that $\frac{\partial \xi}{\partial u} > 0$. The rest of the proof goes as in (a) by reversing the role of $w = u + iv$ and $\zeta = \xi + i\eta$ and by replacing F by F^{-1}. ∎

The next lemma is known [7]. For completeness we give a short proof of it.

LEMMA 2.5. *Let F be analytic on $A = \{z : r_1 < |z| < r_2\}$, $0 < r_1 < r_2$, such that*

$$0 < \operatorname{Re} \frac{zF'(z)}{F(z)} < \infty \text{ on } A. \tag{2.18}$$

Then the following statements are equivalent:

(i) *F is univalent on A;*

(ii) *$\frac{1}{2\pi} \oint_{|z|=\rho} d \arg F = 1$, for some $\rho \in (r_1, r_2)$;*

(iii) *$p(z) = \frac{zF'(z)}{F(z)} = 1 + \sum_{j \in \mathbb{Z} \setminus \{0\}} p_j z^j \in H(A)$.*

PROOF. (a) The fact (ii)$\Leftrightarrow$(iii) follows from the relation

$$\frac{1}{2\pi} \oint_{|z|=\rho} d \arg F = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{zF'(z)}{F(z)} \cdot \frac{dz}{z} = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{p(z)}{z} dz.$$

(b) The implication (i)$\Rightarrow$(ii) is trivial since (2.18) excludes the case $\frac{1}{2\pi} \oint_{|z|=\rho} d \arg F = -1$.

(c) Going to prove (ii)$\Rightarrow$(i), observe first that F and F' do not vanish on A. Indeed, if F has a zero of order m at $z_0 \in A$, then, by (2.18), F' also has to have a zero of order m at z_0 which is impossible. We conclude therefore that F is locally univalent on A and that

$$\frac{1}{2\pi} \oint_{|z|=\rho} d \arg zF' = \frac{1}{2\pi} \oint_{|z|=\rho} d \arg F = 1$$

for all $\rho \in (r_1, r_2)$.

Furthermore, using (2.18) and (ii), we conclude that for each $\rho \in (r_1, r_2)$ F is univalent on $\{z : |z| = \rho\}$ and that $\Gamma_\rho = \{F(\rho^{it}) : 0 \le t \le 2\pi\}$ is a simple closed analytic and strictly starlike curve which winds once around the origin. Moreover, for each $r_1 < \rho_1 < \rho_2 < r_2$ we have $\Gamma_{\rho_1} \cap \Gamma_{\rho_2} = \emptyset$. In fact, F is analytic and hence bounded on $\{z : \rho_1 \le |z| \le \rho_2\}$ and the image of it has to be a domain. Therefore, F is univalent on $\{z : \rho_1 \le |z| \le \rho_2\}$. This holds for all $r_1 < \rho_1 < \rho_2 < r_2$ and hence F is univalent on A. ∎

We will close this section with the following lemma.

LEMMA 2.6. *Let F be in $H(|z| = 1)$ such that*

(i) *$|F(e^{it})| \equiv 1$ and $F'(e^{it}) \neq 0$ for all $t \in [0, 2\pi]$;*

(ii) *$\frac{1}{2\pi} \oint_{|z|=1} d \arg F = 1$.*

Then there exists an $R_1 > 1$ such that F is univalent on

$$\{z : \frac{1}{R_1} < |z| < R_1\}.$$

PROOF. The fact that $\frac{zF'(z)}{F(z)} \in \mathbb{R} \setminus \{0\}$ on ∂U and that

$$\frac{1}{2\pi} \oint_{|z|=1} d \arg zF' = \frac{1}{2\pi} \oint_{|z|=1} d \arg F = 1,$$

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implies that there is a $k > 0$ such that $\frac{zF'(z)}{F(z)} = \operatorname{Re}\frac{zF'(z)}{F(z)} \geq k > 0$ and is finite on ∂U. Hence, there is an $R_1 > 1$ such that F is analytic and $0 < \operatorname{Re}\frac{zF'(z)}{F(z)} < \infty$ on $\{z : \frac{1}{R_1} < |z| < R_1\}$. Applying Lemma 2.5, we conclude that F is univalent on $\{z : \frac{1}{R_1} < |z| < R_1\}$. ∎

3. A characterization theorem.

THEOREM 3.1. *Let*

$$f(z) = h(z) + \overline{g(z)} + 2A\ln|z|, \quad A \in \mathbb{C},$$

be a harmonic mapping defined on the unit circle $\partial U = \{z : |z| = 1\}$. *Put*

$$\psi(z) = zh'(z) + A = zf_z(z).$$

Then there exists an exterior neighbourhood V_R *of* $\bar{U}$ *such that* f *is univalent and positively oriented on* V_R *and* $\lim_{\substack{|z|>1 \\ z\to e^{it}}} f(z) \equiv 0$, *if and only if the following conditions are satisfied:*

(a) h and g admit an analytic continuation across ∂U *such that* $h(\frac{1}{\bar{z}}) = \overline{-g(z)}$ *for all* z, $\frac{1}{R} < |z| < R$;

(b) ψ *has at most one zero on* ∂U *which is of order one;*

(c) there exists an exterior neighbourhood $V_{\hat{R}}$ *of* $\bar{U}$ *and a constant* $k > 0$ *such that*

$$\infty > \operatorname{Re}\frac{z\psi'(z)}{\psi(z)} \geq k > 0 \text{ on } V_{\hat{R}}, \tag{3.1}$$

and

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi = 1 \text{ for all } \rho \in (1, \hat{R}). \tag{3.2}$$

Let us give some remarks about this Theorem.

REMARK 3.1. The statement (c) says that $\psi(z) = zf_z(z)$ maps $\{z : |z| = r\}$, $1 < r < \hat{R}$, univalently onto an analytic strictly starlike Jordan curve with respect to the origin.

For $r = 1$, $\psi(\partial U)$ is still an analytic curve, but it may pass through the origin as the following example shows: the mapping

$$f(z) = z - \frac{1}{\bar{z}} + 2\ln|z|$$

is univalent, harmonic and orientation-preserving on $\mathbb{C}\setminus U$ and $\psi(z) = 1 + z$ vanishes at $z = -1$. Observe that this zero is of order one.

REMARK 3.2. The next example shows that (3.2) is essential. Let $\psi(z) = z + z^2$. Then ψ satisfies (b) and (3.1) with $k = \frac{3}{2}$ but not (3.2). The function $h(z) = z + z^2/2$ is analytic in $\mathbb{C}$, but

$$f(z) = z + \frac{z^2}{2} - \frac{1}{\bar{z}} - \frac{1}{2\bar{z}^2}$$

is not univalent on any circle $\{z : |z| = r\}$, $r > 1$. Indeed, we have

$$f(re^{it}) = \frac{r^2 - 1}{r}[e^{it} + \frac{r^2 + 1}{2r}e^{2it}].$$

Putting $d = \frac{r^2+1}{r} > 1$ we see that the equation $\eta_1 + d\eta_1^2 = \eta_2 + d\eta_2^2$ has a solution for $|\eta_1| = |\eta_2| = 1$, $\eta_1 \neq \eta_2$.

REMARK 3.3. The condition (3.1) cannot be replaced by

$$\operatorname{Re} \frac{e^{it}\psi'(e^{it})}{\psi(e^{it})} \geq 0.$$

Indeed, consider the function $\psi(z) = z + \frac{1}{2}z^2$. Then we have

$$a(z) = \frac{1}{z^3} \cdot \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z},$$

and, for ε positive close to zero,

$$a\big(-(1 + \varepsilon)\big) = 1 + 2\varepsilon^3 + 0(\varepsilon^4) > 1.$$

Therefore f is not orientation-preserving on V_R.

REMARK 3.4. Put $p(z) = \frac{z\psi'(z)}{\psi(z)}$. By Lemma 2.5 the statement (c) is equivalent to:

(c$'$) There exists an exterior neighbourhood $V_{\hat{R}}$ of $\bar{U}$ and a constant $k > 0$ such that

$$p \in H(V_{\hat{R}}),\ p(z) = 1 + \sum_{j \in \mathbf{Z} \setminus \{0\}} p_j z^j \text{ and } \operatorname{Re} p(z) \geq k \text{ on } V_{\hat{R}}.$$

REMARK 3.5. The statements (b) can be replaced by

(b$'$) $\psi'(e^{it}) \neq 0$ for all $t \in [0, 2\pi]$.

In fact we have:

(α) (a), (b) and (c) imply (a), (b$'$) and (c).

If e^{it_0} is a zero of order 1 of ψ then evidently $\psi'(e^{it_0}) \neq 0$. Let $\psi(e^{it}) \neq 0$. By (a), ψ is analytic and hence bounded on ∂U and by (c) we have $0 < k \leq \operatorname{Re} \frac{e^{it}\psi'(e^{it})}{\psi(e^{it})} < \infty$, which implies that $\psi'(e^{it}) \neq 0$ for all $t \in [0, 2\pi]$.

(β) (a), (b$'$), (c) imply (a), (b), (c).

Note that $\psi \in H(\partial U)$ and that all zeros of ψ on ∂U are of order one. On the other hand, if $\psi(e^{it}) \neq 0$, then we conclude by (c) that

$$0 < k \leq \operatorname{Re} \frac{e^{it}\psi'(e^{it})}{\psi(e^{it})} < \infty. \tag{3.3}$$

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Furthermore, for $\rho \in (1, \hat{R})$, we have

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg z\psi' = \frac{1}{2\pi}\oint_{|z|=\rho} d\arg \psi = 1,$$

and, from (b′), we conclude that $\frac{1}{2\pi}\oint_{|z|=1} d\arg z\psi' = 1$. Suppose that $\eta_k = e^{it_k}$, $1 \le k \le N$, are the zeros of ψ on ∂U. Then, by (3.3),

$$\frac{1}{2\pi}\int_{(t_k, t_{k+1})} d\arg(z\psi')(e^{it}) = \frac{1}{2\pi}\int_{(\eta_k, \eta_{k+1})} d\arg z\psi' > \frac{1}{2}, \quad \eta_{N+1} = \eta_1,$$

and therefore

$$\begin{aligned} 1 &= \frac{1}{2\pi}\oint_{|z|=1} d\arg z\psi' \\ &= \frac{1}{2\pi}\sum_{k=1}^{N}\oint_{(\eta_k, \eta_{k+1})} d\arg z\psi' + \frac{1}{2\pi}\sum_{k=1}^{N}\Delta[\arg(z\psi')(\eta_k)]. \end{aligned}$$

Since the zeros of ψ are of order one, we get $\Delta\arg(z\psi')(\eta_k) = 0$. Therefore $N = 0$ or 1.

Together with Remarks 4 and 5, Theorem 3.1 can be restated as follows.

THEOREM 3.2. *Let*

$$f(z) = h(z) + \overline{g(z)} + 2A\ln|z|, \quad A \in \mathbb{C},$$

be a harmonic mapping defined on the unit circle $\partial U = \{z : |z| = 1\}$. *Put* $\psi(z) = zf_z(z) = zh'(z) + A$ *and* $p(z) = \frac{z\psi'(z)}{\psi(z)}$. *Then there exists an exterior neighbourhood* V_R *of* $\bar{U}$ *such that* f *is univalent, positively oriented on* V_R *and* $\lim_{\substack{|z|>1 \\ z\to e^{it}}} f(z) \equiv 0$, *if and only if the following conditions are satisfied:*

(a) *h and g admit an analytic continuation across* ∂U *such that* $h(\frac{1}{\bar{z}}) = \overline{-g(z)}$, $z \in \{z : \frac{1}{R} < |z| < R\}$;

(b′) $\psi'(e^{it}) \neq 0$ *for all* $t \in [0, 2\pi]$;

(c′) *there is an exterior neighbourhood* $V_{\hat{R}}$ *and a constant* $k > 0$ *such that* $p \in H(V_{\hat{R}})$, *is of the form* $p(z) = 1 + \sum_{j\in\mathbb{Z}\setminus\{0\}} p_j z^j$ *and satisfies the condition* $\operatorname{Re} p(z) \ge k > 0$ *on* $V_{\hat{R}}$.

4. **Proof of Theorem 3.1.** Necessity: suppose that f is a univalent, positively oriented and harmonic mapping defined on an exterior neighbourhood V_R of $\bar{U}$ such that $\lim_{\substack{|z|>1 \\ z\to e^{it}}} f(z) \equiv 0$.

The statement (a) has been already proved in Lemma 2.1.

Let us prove (3.2) of the statement (c). Since f is univalent and positively oriented we have for $\rho \in (1, R)$:

$$1 = \frac{1}{2\pi}\oint_{|z|=\rho} \arg[izf_z - i\bar{z}f_{\bar{z}}].$$

Recall that $f_z \neq 0$ on V_R and, since f is orientation-preserving, we have

$$a(z) = \frac{\overline{f_{\bar{z}}(z)}}{f_z(z)} \in H(V_R) \text{ and } |a(z)| < 1, \quad z \in V_R.$$

Therefore, we get for $\rho \in (1, R)$

$$(4.1)\qquad \begin{aligned} 1 &= \frac{1}{2\pi}\oint_{|z|=\rho} d\arg[izf_z - i\bar{z}f_{\bar{z}}] = \frac{1}{2\pi}\oint_{|z|=\rho} d\arg zf_z[1 - \bar{a}\frac{\bar{z}\overline{f_z}}{zf_z}] \\ &= \frac{1}{2\pi}\oint_{|z|=\rho} d\arg zf_z = \frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi, \end{aligned}$$

and (3.2) is shown.

Next we show (b). We apply Lemma 2.2. By (2.6), we have $|a(e^{it})| \equiv 1$ for all $t \in [0, 2\pi]$, and (2.9) implies that

$$(4.2)\qquad \frac{1}{2\pi}\oint_{|z|=1} d\arg a = \frac{1}{2\pi i}\oint_{|z|=1} \frac{za'(z)}{a(z)} \cdot \frac{dz}{z} < 0.$$

Now, by Lemma 2.4, we know that Φ is univalent on V_{R_1} for some $R_1 \in (1, R)$ and, from the formula

$$(2.14)\qquad z\Phi'(z) = e^{i\alpha}\psi(z)\big(1 - a(z)e^{-2i\alpha}\big) = e^{i\alpha}\psi(z) - e^{-i\alpha}\overline{\psi(\frac{1}{\bar{z}})},$$

we obtain

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg z\Phi' = \frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi = 1 \text{ for all } \rho \in (1, R_1).$$

Lemma 2.3 implies that the function Φ' has exactly two zeros on ∂U which are of order one. This information together with (4.2) leads to the conclusion that either

$$\frac{1}{2\pi}\oint_{|z|=1} d\arg a = -2 \text{ and } 0 \notin \psi(\partial U)$$

or

$$\frac{1}{2\pi}\oint_{|z|=1} d\arg a = -1$$

and then ψ has exactly one zero of order one on ∂U. Therefore (b) has been established.

It remains to show the statement (3.1). If ψ does not vanish on ∂U then, by Lemma 2.2, there is a $k_1 > 0$ such that

$$\operatorname{Re}\frac{z\psi'(z)}{\psi(z)} \geq k_1 \text{ on } \partial U.$$

Choose $\hat{R}$ such that $\operatorname{Re}\frac{z\psi'(z)}{\psi(z)} \geq k = k_1/2$ on $V_{\hat{R}}$ and (3.1) is shown.

Suppose now that $\psi(e^{i\beta}) = 0$, $\psi'(e^{i\beta}) \neq 0$ and $\psi(e^{it}) \neq 0$ on $\partial U \setminus \{e^{i\beta}\}$.

By Lemma 2.2 we have

$$0 < k \leq \operatorname{Re}\frac{z\psi'(z)}{\psi(z)} < \infty \text{ on } \partial U \setminus \{e^{i\beta}\}$$

and therefore $\psi'(e^{it}) \neq 0$ for all $t \in [0, 2\pi]$ (by the same reasoning as in the first part of Remark 3.5).

Consider now the function

$$m(z) = \frac{\psi(z)}{z\psi'(z)}.$$

Then $m \in H(\partial U)$ and satisfies the condition

$$\Big|m(e^{it}) - \frac{1}{2k}\Big| \leq \frac{1}{2k}.$$

Let $K = \{z : |z - e^{i\beta}| \leq \kappa\}$, where $\kappa > 0$ is so small that $m(z)$ is analytic and univalent on K and such that either $m(K \cap \{z : |z| < 1\})$ or $m(K \cap \{z : |z| > 1\})$ is in the disk $\{w : |w - \frac{1}{2k}| < \frac{1}{2k}\}$ (this is possible since $m'(e^{i\beta}) \neq 0$). The condition $e^{i\beta}m'(e^{i\beta}) = 1$ implies that the second case holds.

Therefore, we have $\infty > \operatorname{Re}\frac{z\psi'(z)}{\psi(z)} \geq k$ on $[\partial U \setminus \{e^{i\beta}\}] \cup [K \cap \{z : |z| > 1\}]$. Hence there exists an exterior neighbourhood of $\bar{U} \cup K$ and hence an exterior neighbourhood V_{R_2}, $R_2 \in (1, R)$, such that $\operatorname{Re}\frac{z\psi'(z)}{\psi(z)} \geq k/2$.

Therefore (3.1) holds, and the proof of the necessity of the statements (a), (b), (c) is finished.

Sufficiency: we now show the sufficiency of the statements (a), (b) and (c). Let $h \in H(\partial U)$ and suppose that $\psi(z) = zh'(z) + A$, $A \in \mathbb{C}$, satisfies the statements (b) and (c) of Theorem 3.1, i.e. there is an $\hat{R} > 1$ such that $h \in H(\frac{1}{\hat{R}} < |z| < \hat{R})$, $\infty > \operatorname{Re}\frac{z\psi'(z)}{\psi(z)} \geq k > 0$ on $V_{\hat{R}}$, $\frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi = 1$ for $\rho \in (1, \hat{R})$ and ψ has at most one zero on ∂U which is of order 1.

Put

$$e^{i\alpha} = \begin{cases} \bar{A}/|A| & \text{if } A \neq 0 \\ 1 & \text{if } A = 0 \end{cases}$$

and define:

$$\Phi(z) = e^{i\alpha}h(z) + e^{-i\alpha}\overline{h(\frac{1}{\bar{z}})}, \quad z \in V_{\hat{R}},$$

$$f(z) = h(z) - h(\frac{1}{\bar{z}}) + 2A\ln|z|, \quad z \in V_{\hat{R}}.$$

Evidently, f is harmonic on $\{z : \frac{1}{\hat{R}} < |z| < \hat{R}\}$ and $f(z) \equiv 0$ on ∂U.

First, we observe that f is orientation preserving on an exterior neighbourhood of $\bar{U}$. This follows from the fact that $\operatorname{Re}\frac{e^{it}\psi'(e^{it})}{\psi(e^{it})} \geq k > 0$ for all t for which $\psi(e^{it}) \neq 0$ and from Lemma 2.2. Call this exterior neighbourhood V_{R_1}, $R_1 \in (1, \hat{R})$.

Next, we will show that f is univalent on V_R for some $R \in (1, R_1)$. Indeed, since by (2.14) $z\Phi'(z) = e^{i\alpha}\psi(z)\big(1 - a(z)e^{-2i\alpha}\big)$, the condition (3.2) implies that for all $\rho \in (1, R_1)$

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg z\Phi' = \frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi = 1.$$

By Lemma 2.3, we conclude that Φ' has exactly two zeros of order one on ∂U; call them $e^{i\beta}$, $e^{i\gamma}$, $e^{i\beta} \neq e^{i\gamma}$.

Denote by J the bounded real interval

$$\Phi(\partial U) = [\Phi(e^{i\beta}), \Phi(e^{i\gamma})].$$

Next, define the function

$$\zeta(s) = \frac{\Phi(e^{i\gamma}) - \Phi(e^{i\beta})}{4}(s + \frac{1}{s}) + \frac{\Phi(e^{i\gamma}) + \Phi(e^{i\beta})}{2},$$

and let $s = q(\zeta)$ be the univalent inverse function of $\zeta(s)$ which maps the exterior of $(\mathbb{C} \setminus J)$ onto the exterior of the unit disk. Put $Q = q \circ \Phi$. Then Q satisfies the conditions of Lemma 2.6 and we conclude that there is an $R_2 \in (1, R_1)$, such that Q is univalent. Hence Φ is univalent on V_{R_2}. Finally, Lemma 2.4 shows that f is univalent on some $V_R, R \in (1, R_2)$.

It remains to show that f is positively oriented. We already have seen that f is oreintation-preserving. By (3.2), we have for $\rho \in (1, R)$:

$$\begin{aligned}\frac{1}{2\pi}\oint_{|z|=\rho} d\arg(izf_z - i\bar{z}f_{\bar{z}}) &= \frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi\left(1 - a(z)\frac{\bar{z}\overline{f_z}}{zf_z}\right) \\ &= \frac{1}{2\pi}\oint_{|z|=\rho} d\arg\psi = 1.\end{aligned}$$

Since f is univalent, $f(|z| = \rho)$, $1 < \rho < R$, are disjoint positively oriented Jordan curves. It remains to show, that they wind around the origin.

Since f is harmonic on ∂U and $f(\partial U) \equiv 0$, $f_\rho(t) = f(\rho^{it})$ converges uniformly to $f_1 \equiv 0$ on $[0, 2\pi]$ as $\rho \downarrow 1$. Using the fact that f is univalent on V_R we conclude that

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg f = 1 \text{ for all } \rho \in (1, R)$$

and Theorem 3.1 is proved. ∎

5. **On the region of values of A.** Let $h \in H(\partial U)$ and consider the harmonic mapping

$$f(z) = h(z) - h(\frac{1}{\bar{z}}) + 2A\ln|z|, \quad A \in \mathbb{C}.$$

Denote by E_h the set of all $A \in \mathbb{C}$ for which f is univalent and positively oriented on V_R for some $R > 1$ (which may depend on A). Evidently E_h can be empty as the example $h(z) = z + \frac{1}{2}z^2$ or $h(z) = z^2$ shows. We have the following result.

THEOREM 5.1. *Let $h \in H(\partial U)$. If E_h is nonempty, then we have the following properties:*

(a) E_h is a convex set.

(b) If, in addition, $zh'(U)$ is a convex set, then the bounded component G of $\mathbb{C} \setminus [zh'(\partial U)]$ belongs to $-E_h$.

PROOF. (a) Let A_1 and A_2 be in E_h. Since E_h is non-empty, we conclude from Theorem 3.1 and Remark 3.5 that (a), (b′) and (c) are satisfied. Observe that (a) and (b′) are independent of A. Therefore we have only to consider the relation (c). There is a $k > 0$ and an $R > 1$ such that for $i = 1, 2$

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg(zh' + A_i) = 1, \quad \rho \in (1, R)$$

and

$$\operatorname{Re} p_i(z) = \operatorname{Re}\frac{\big(zh'(z)\big)'}{zh'(z) + A_i} \geq k > 0 \text{ on } V_R.$$

Put $A = \lambda A_1 + (1-\lambda)A_2$ for some $\lambda \in (0,1)$. Then we have for all $z \in V_R$.

$$\frac{1}{p_i(z)} = \frac{zh'(z) + A_i}{\big(zh'(z)\big)'} \in D = \Big\{w : \Big|w - \frac{1}{2k}\Big| \leq \frac{1}{2k}\Big\}.$$

Since D is convex

$$\frac{1}{p(z)} = \frac{zh'(z) + A}{(zh'(z))'} \in D,$$

and therefore, $\operatorname{Re} p(z) \geq k$. Furthermore, since $\Gamma_\rho = zh'(|z| = \rho)$, $\rho \in (1, R)$, is a positively oriented Jordan curve which is strictly starlike with respect to $-A_1$ and $-A_2$, it is also strictly starlike with respect to $-A$. Hence,

$$\frac{1}{2\pi}\oint_{|z|=\rho} d\arg(zh' + A) = 1,$$

which shows that E_h is a convex set.

(b) Put $\psi_0(z) = zh'(z)$. Since $E_h \neq \emptyset$, $\psi_0(\partial U)$ is an analytic closed Jordan curve. (This follows from (b), (c), (b′) and (c′) applied to $\psi_0 + A$ for an $A \in E_h$). Suppose, in addition, that $\psi_0(\partial U)$ is also a convex curve, i.e. that the bounded component G of $\mathbb{C} \setminus \psi_0(\partial U)$ is a convex domain. Let $-A$ be in G and put $\psi = \psi_0 + A$. Since $0 \notin \psi(\partial U)$, $0 \notin \psi'(\partial U)$ and $\operatorname{Re}\frac{z\psi'(z)}{\psi(z)} > 0$ on ∂U, we conclude that (c) holds on V_{R_1}, for some $R_1 > 1$. The statements (a) and (b) hold already, since $E_h \neq \emptyset$. Therefore $-G \subset E_h$. ∎

We finish this section with the following remarks.

REMARKS 5.2. (i) Define $\psi_0(z) = zh'(z)$ and suppose that $E_h \neq \emptyset$. Let, as before, G be the bounded component of $\mathbb{C} \setminus [\psi_0(\partial U)]$. Then $-E_h \subset \bar{G}$. Therefore, E_h is a bounded set.

(ii) We cannot conclude that E_h is closed (and hence compact) as the following example shows: Consider $h(z) = \frac{1}{1-z/2}$. Then $\psi_0(z) = zh'(z) = \frac{1}{2}\cdot\frac{z}{(1-z/2)^2}$ and $\frac{z\psi_0'(z)}{\psi_0(z)} = \frac{1+z/2}{1-z/2}$. It follows then that $0 \in E_h$. There is even a disk $\{A : |A| < r\}$, $r > 0$, which belongs to E_h. On the other hand $A = \frac{4}{9} \notin E_h$. Let $\hat{A} = \limsup\{A : A > 0, A \in E_h\}$. Then, $\hat{A}$ violates (3.1) and is therefore not in E_h.

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