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# The Spherical Functions Related to the Root System $B_2$

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*Abstract.* In this paper, we give an integral formula for the eigenfunctions of the ring of differential operators related to the root system $B_2$.

## 1 Introduction

In [13], Jiro Sekiguchi gave an Euler type formula for the spherical functions $\Phi_{\lambda,m}$ of the symmetric space $\mathrm{SO}_0(2, m+2)/\mathrm{SO}(2) \times \mathrm{SO}(m+2)$. He used that formula to propose a generalization of the functions $\Phi_{\lambda,m}$ for other values of the parameter $m$. He conjectured that these generalized spherical functions would satisfy a system of differential operators related to the root system $B_2$.

These generalized spherical functions are connected to the work of G. J. Heckman and E. M. Opdam in [2], [3], [7], [8] and to the work of R. J. Beerends in [1]. We did a similar investigation for the root system $A_{n-1}$ in [11].

In [12], we found a new expression for the spherical functions of the space $\mathrm{SO}_0(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$, $q \geq p$. Based on that expression, we will propose in Section 2 a generalization of these spherical functions for $p = 2$ and for $q$ with $\Re q > 3$. We will show that these generalized spherical functions satisfy a system of differential operators related to the root system $B_2$.

In Section 3, we will show that our definition and that of Sekiguchi are equivalent thus proving his conjecture. This gives an added relevance to the results of [13] whose interest relied on the validity of his conjecture.

## 2 Spherical Functions on $B_2$

We are concerned with the symmetric space $G/K = \mathrm{SO}_0(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ where $q \geq p$.

If $\lambda \in \mathfrak{a}^*$, the space of complex-valued functionals on the Cartan subalgebra $\mathfrak{a}$, the corresponding spherical function is $\phi_\lambda(e^H) = \int_K e^{(i\lambda-\rho)(\mathcal{H}(e^H k))}\,dk$ where $g = ke^{\mathcal{H}(g)}n \in KAN$ (the Iwasawa decomposition of $G$).

From [12], we know that the element $H \in \mathbf{a}$ is of the form

$$H = \begin{bmatrix} 0_{p\times p} & \mathcal{D}_{p\times p} & 0_{p\times(q-p)} \\ \mathcal{D}_{p\times p} & 0_{p\times p} & 0_{p\times(q-p)} \\ 0_{(q-p)\times p} & 0_{(q-p)\times p} & 0_{(q-p)\times(q-p)} \end{bmatrix}$$

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where the subscripts indicate the size of the matrix blocks and where $\mathcal{D} = \operatorname{diag}[H_1, \ldots, H_p]$.

Suppose $\lambda \in \mathfrak{a}^*$ is defined on $\mathfrak{a}$ by $\lambda(H) = \sum_{k=1}^{p} \lambda_k H_k$. In [12], we gave a relationship between the spherical functions on $\mathrm{SO}_0(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ and those on the symmetric cone $\mathrm{GL}_+(p,\mathbf{R})/\mathrm{SO}(p)$. If $\tilde{\phi}_\Lambda$ denotes a spherical function on $\mathrm{GL}_+(p,\mathbf{R})/\mathrm{SO}(p)$, then

$$\phi_\lambda(e^H) = \int_{\mathrm{SO}(q)} \tilde{\phi}_\Lambda\big(\cosh\mathcal{D} + (\sinh\mathcal{D})A(k)\big)\,dk \tag{1}$$

where $A(k)$ denotes the left-upper $p \times p$ submatrix of $k$ and $\Lambda(H) = \sum_{k=1}^{p}\big(\lambda_k + i(q-1)/2\big)H_k$. The relationship between $\lambda$ and $\Lambda$ will be assumed throughout the paper.

Except when otherwise stated, we will assume from now on that $p = 2$.

***Remark 1*** Let $H = \operatorname{diag}[H_1, H_2]$, $k_1, k_2 \in \mathrm{SO}(2)$ and $g = k_1 e^H k_2$. We recall from [10] that

$$\tilde{\phi}_\Lambda(g) = \frac{e^{i\Lambda_2(H_1+H_2)}}{\pi}\int_{H_2}^{H_1} \frac{e^{i(\Lambda_1-\Lambda_2)(\xi)}}{\sqrt{\sinh(H_1-\xi)\sinh(\xi-H_2)}}\,d\xi. \tag{2}$$

Let $E = \frac{\partial}{\partial H_1} + \frac{\partial}{\partial H_2}$ and $\Delta$ be the Laplace-Beltrami operator on $\mathrm{GL}_+(2,\mathbf{R})/\mathrm{SO}(2)$. We then have $E\tilde{\phi}_\Lambda = i(\Lambda_1+\Lambda_2)\tilde{\phi}_\Lambda$ and $\Delta\tilde{\phi}_\Lambda = -(\Lambda_1^2+\Lambda_2^2+1/4)\tilde{\phi}_\Lambda$. It will be useful later (in Definition 14) to note that every symmetric polynomial in $\Lambda_1$ and $\Lambda_2$ can be written in a unique way as a polynomial in $i(\Lambda_1+\Lambda_2)$ and $-(\Lambda_1^2+\Lambda_2^2+1/4)$.

It will be necessary to integrate with respect to $A = A(k)$ in (1). Lemma 2 and Proposition 4 will let us do that.

***Lemma 2*** *Let $d\nu$ stand for the rotation-invariant measure on the sphere with total mass 1. Suppose $J(\mathbf{x})$ is a matrix of size $q \times (q-1)$ which depends smoothly on $\mathbf{x} \in \mathbf{S}^{q-1}$ and is such that $[\mathbf{x} \quad J(\mathbf{x})] \in \mathrm{SO}(q)$. Suppose also that $J(\mathbf{x},\mathbf{y})$ is a matrix of size $q \times (q-2)$ which depends smoothly on $(\mathbf{x},\mathbf{y}) \in \mathbf{S}^{q-1}\times\mathbf{S}^{q-2}$ and is such that $[\mathbf{x} \quad J(\mathbf{x})\mathbf{y} \quad J(\mathbf{x},\mathbf{y})] \in \mathrm{SO}(q)$. Then*

$$\int_{\mathrm{SO}(q)} f(k)\,dk = \int_{\mathrm{SO}(q-2)}\int_{\mathbf{S}^{q-2}}\int_{\mathbf{S}^{q-1}} f\big([\mathbf{x} \quad J(\mathbf{x})\mathbf{y} \quad J(\mathbf{x},\mathbf{y})k_0]\big)\,d\nu(\mathbf{x})\,d\nu(\mathbf{y})\,dk_0. \tag{3}$$

**Proof** Every element of $k \in \mathrm{SO}(q)$ can be written as $k = [\mathbf{x} \quad J(\mathbf{x})\mathbf{y} \quad J(\mathbf{x},\mathbf{y})k_0]$. Indeed, $\mathbf{x}$ corresponds to the first column of $k$ and the columns of $J(\mathbf{x})$ span $\mathbf{x}^\perp$ so choosing $\mathbf{y}$ appropriately gives us the second column of $k$. The columns of $J(\mathbf{x},\mathbf{y})$ span $\{\mathbf{x}, J(\mathbf{x})\mathbf{y}\}^\perp$ so choosing $k_0$ appropriately gives us the rest of $k$. It now suffices to show that the right-hand side integral is invariant under the action of $\tilde{k} \in \mathrm{SO}(q)$. Now, $\tilde{k}J(\mathbf{x}) = J(\tilde{k}\mathbf{x})k_{\mathbf{x}}$ with $k_{\mathbf{x}} \in \mathrm{SO}(q-1)$ and $\tilde{k}J(\mathbf{x},\mathbf{y}) = J(\tilde{k}\mathbf{x}, k_{\mathbf{x}}\mathbf{y})k_{\mathbf{x},\mathbf{y}}$ with

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$k_{\mathbf{x},\mathbf{y}} \in \mathrm{SO}(q-2)$. The existence of $k_{\mathbf{x}}$ comes from the fact that both $\tilde{k}J(\mathbf{x})$ and $J(\tilde{k}\mathbf{x})$ span $(\tilde{k}\mathbf{x})^{\perp}$. A similar reasoning apply to the existence of $k_{\mathbf{x},\mathbf{y}}$.

The rest then follows from the invariance properties of the measures $d\nu(\mathbf{x})$, $d\nu(\mathbf{y})$ and $dk_0$. ■

The author is indebted to Ken Richardson of Texas Christian University for his suggestion to parameterize the first two columns of an element of $\mathrm{SO}(q)$ using elements of $\mathbf{S}^{q-1} \times \mathbf{S}^{q-2}$.

***Remark 3*** We give here a simple construction of the maps $J(\mathbf{x})$ and $J(\mathbf{x}, \mathbf{y})$ (we are excluding a set of measure 0 in $\mathbf{S}^{q-1} \times \mathbf{S}^{q-2}$). Given $\mathbf{x} \in \mathbf{S}^{q-1}$, we apply the Gram-Schmidt process to the columns of the matrix $[\mathbf{x} \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_q]$ to obtain $[\mathbf{x} \quad J(\mathbf{x})] \in \mathbf{O}(q)$. Given $\mathbf{y} \in \mathbf{S}^{q-2}$ and $J(\mathbf{x})$, we apply the Gram-Schmidt process to the columns of the matrix $[\mathbf{x} \quad J(\mathbf{x})\mathbf{y} \quad \mathbf{e}_3 \quad \cdots \quad \mathbf{e}_q]$ to obtain $[\mathbf{x} \quad J(\mathbf{x})\mathbf{y} \quad J(\mathbf{x}, \mathbf{y})] \in \mathbf{O}(q)$. In each case, it may be necessary to multiply the last column by $-1$ to obtain matrices in $\mathrm{SO}(q)$.

***Proposition 4*** *Let $f\colon \mathrm{SO}(q) \to \mathbf{R}$ be a function of the upper left corner $p \times p$ submatrix. Abusing notation, write $f(A) = f\big((a_{ij})_{1 \le i,j \le p}\big)$. Then, if $p = 1, 2$ and $q > 2p - 1$,*

$$\int_{\mathrm{SO}(q)} f\big(A(k)\big)\, dk = (C_q^p)^{-1} \int_{\mathcal{M}_p} f(A)\big(\det(I - AA^T)\big)^{(q-1)/2-p}\, dA \tag{4}$$

*where $\mathcal{M}_p$ is the set of matrices $A$ with $\|A\|_2 \le 1$ and $dA$ is the Lebesgue measure on $\mathcal{M}_p$. Recall that $\|A\|_2$ is the largest singular value of $A$ or, equivalently, the square root of the largest eigenvalue of $AA^T$. Note that $C_q^p = \int_{\mathcal{M}_p} \big(\det(I - AA^T)\big)^{(q-1)/2-p}\, dA$.*

**Proof** Since the case $p = 1$ is simpler, we will only discuss the case $p = 2$.

Let $\mathbf{x} = (x_i)$ with $x_i = (\prod_{k=1}^{i-1} \sin a_k) \cos a_i$, $1 \le i \le q-1$, and $x_q = \prod_{k=1}^{q-1} \sin a_k$ where $0 \le a_j \le \pi$ for $j < q-1$ and $0 \le a_{q-1} \le 2\pi$. Let $\mathbf{y} = (y_i)$ with $y_i = (\prod_{k=1}^{i-1} \sin b_k) \cos b_i$, $1 \le i \le q-2$ and $y_q = \prod_{k=1}^{q-2} \sin b_k$ where $0 \le b_j \le \pi$ for $j < q-2$ and $0 \le b_{q-2} \le 2\pi$.

We now describe the matrix $J(\mathbf{x}) = (J_{ij})$ of size $q \times (q-1)$:

$$J_{ij} = \begin{cases} 0 & i < j \\ -\sin a_j & i = j \\ \cos a_j (\prod_{k=j+1}^{i-1} \sin a_k) \cos a_i & j < i < q \\ \cos a_j \prod_{k=j+1}^{q-1} \sin a_k & i = q. \end{cases}$$

We only need the upper left corner $2 \times 2$ submatrix $A$ which we compute using (3). Written in the $(a_i, b_j)$ coordinates,

$$A = \begin{pmatrix} \cos a_1 & -\sin a_1 \cos b_1 \\ \sin a_1 \cos a_2 & \cos a_1 \cos a_2 \cos b_1 - \sin a_2 \sin b_1 \cos b_2 \end{pmatrix}. \tag{5}$$

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Using Lemma 2, the Haar measure on $\mathrm{SO}(q)$ is given by

$$C\Big(\prod_{i=1}^{q-1}\sin^{q-1-i}a_i\Big)\Big(\prod_{i=1}^{q-2}\sin^{q-2-i}b_i\Big)\,da_1\cdots da_{q-1}db_1\cdots db_{q-2}dk_0$$

if $dk_0$ represents the Haar measure on $\mathrm{SO}(q-2)$ (the measure $d\nu$ is given in [4, p. 223]). Integrating out the variables that do not intervene in $A$, we have

$$\begin{aligned}\int_{\mathrm{SO}(q)} & f\big(A(k)\big)\,dk\\ &= C\int_{[0,\pi]^4} f(A)\sin^{q-2}a_1\sin^{q-3}a_2\sin^{q-3}b_1\sin^{q-4}b_2\,da_1\,da_2\,db_1\,db_2\end{aligned}$$

where $A$ is as in (5). Let $a_{11}=\cos a_1$, $a_{12}=-\sin a_1\cos b_1$, $a_{21}=\sin a_1\cos a_2$ and $a_{22}=\cos a_1\cos a_2\cos b_1-\sin a_2\sin b_1\cos b_2$. It is a simple calculus exercise to show that

$$\int_{\mathrm{SO}(q)} f\big(A(k)\big)\,dk = C'\int_{\mathcal{M}_2} f(A)\big(\det(I-AA^T)\big)^{(q-5)/2}\,da_{ij}. \qquad \blacksquare$$

***Corollary 5*** *Using the same notation as in the lemma, we have for* $p=2$

(6)

$$\begin{aligned}\int_{\mathrm{SO}(q)} f\big(A(k)\big)\,dk &= (C_q^p)^{-1}\int_{\mathrm{SO}(2)}\int_{\mathrm{SO}(2)}\int_{-1}^{1}\int_{0}^{|x_1|} f\left(k_1\begin{bmatrix}x_1 & 0\\ 0 & x_2\end{bmatrix}k_2\right)\\ &\quad\cdot(1-x_1^2)^{(q-5)/2}(1-x_2^2)^{(q-5)/2}(x_1^2-x_2^2)\,dx_2\,dx_1\,dk_2\,dk_1.\end{aligned}$$

**Proof** We write what is essentially the singular value decomposition of $A$:

(7)

$$\begin{aligned}A &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1 & 0\\ 0 & x_2\end{bmatrix}\begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}^T\\ &= \begin{bmatrix}x_1\cos\theta\cos\psi + x_2\sin\theta\sin\psi & x_1\cos\theta\sin\psi - x_2\sin\theta\cos\psi\\ x_1\sin\theta\cos\psi - x_2\cos\theta\sin\psi & x_1\sin\theta\sin\psi + x_2\cos\theta\cos\psi\end{bmatrix},\end{aligned}$$

and compute $da_{11}\,da_{12}\,da_{21}\,da_{22} = (x_1^2-x_2^2)\,dx_1\,dx_2\,d\theta\,d\psi$, with $-1\le x_1\le 1$, $0\le x_2\le |x_1|$ and $\theta,\psi\in[0,2\pi]$. $\blacksquare$

***Corollary 6*** *If* $p=1$ *or* $2$ *and* $q>2p-1$ *then*

$$\phi_\lambda(e^H) = (C_q^p)^{-1}\int_{\mathcal{M}_p}\tilde{\phi}_\Lambda\big(\cosh\mathcal{D}+(\sinh\mathcal{D})A\big)\big(\det(I-AA^T)\big)^{(q-1)/2-p}\,dA$$

*where* $\Lambda_r=\lambda_r+i(q-1)/2$.

**Proof** A direct consequence of (1) and of Proposition 4. $\blacksquare$

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***Remark 7*** If $p = 1$ then

$$\phi_\lambda(e^H) = \frac{\int_{-1}^{1} \big(\cosh H + (\sinh H)x\big)^{i\lambda} (1-x^2)^{(q-3)/2}\,dx}{\int_{-1}^{1} (1-x^2)^{(q-3)/2}\,dx}.$$

We are now ready to propose a generalization of the spherical functions related to the root system $B_p$.

***Definition 8*** For $\lambda \in \mathfrak{a}^*$ and $\Re q > 2p - 1$, let

(8)
$$\phi_\lambda^{(q)}(e^H) = (C_q^p)^{-1} \int_{\mathcal{M}_p} \tilde{\phi}_\Lambda\big(\cosh \mathcal{D} + (\sinh \mathcal{D})A\big) \big(\det(I - AA^T)\big)^{(q-1)/2-p}\,dA.$$

Although Definition 8 is given for all integers $p \geq 1$, we continue to focus on the case $p = 2$. We still need to explain in which way the $\phi_\lambda^{(q)}$'s generalize the ordinary spherical functions.

***Definition 9*** Let $D(G/K)$ denote the commutative algebra of left-invariant differential operators on $G/K$.

***Definition 10*** Adapting [13, p. 99] to our notation, let

$$\begin{aligned}
\Delta_2^{(q)} &= \frac{\partial^2}{\partial H_1^2} + \frac{\partial^2}{\partial H_2^2} + [(q-2)\coth H_1 + \coth(H_1+H_2) + \coth(H_1-H_2)]\frac{\partial}{\partial H_1} \\
&\qquad + [(q-2)\coth H_2 + \coth(H_1+H_2) - \coth(H_1-H_2)]\frac{\partial}{\partial H_2}, \\
L_1^{(q)} &= \frac{\partial^2}{\partial H_1^2} - \frac{\partial^2}{\partial H_2^2} + (q-2)\coth(H_1)\frac{\partial}{\partial H_1} - (q-2)\coth(H_2)\frac{\partial}{\partial H_2}, \\
L_2^{(q)} &= \frac{\partial^2}{\partial H_1^2} - \frac{\partial^2}{\partial H_2^2} \\
&\qquad + \big(2\coth(H_1-H_2) + 2\coth(H_1+H_2) + (q-2)\coth(H_1)\big)\frac{\partial}{\partial H_1} \\
&\qquad + \big(2\coth(H_1-H_2) - 2\coth(H_1+H_2) - (q-2)\coth(H_2)\big)\frac{\partial}{\partial H_2} \\
&\qquad + (q-2)\Big(\coth(H_1)\big(\coth(H_1-H_2) + \coth(H_1+H_2)\big) \\
&\qquad\qquad + \coth(H_2)\big(\coth(H_1-H_2) - \coth(H_1+H_2)\big)\Big) \\
&\qquad + 4\coth(H_1-H_2)\coth(H_1+H_2), \\
\Delta_4^{(q)} &= L_2^{(q)} L_1^{(q)}.
\end{aligned}$$

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***Remark 11*** In Remark 1, we gave the two generators of $D\big(\mathrm{GL}_+(2,\mathbf{R})/\,\mathrm{SO}(2)\big)$, namely $E$ and $\Delta$. The two generators of $D\big(\mathrm{SO}_0(2,q)/\,\mathrm{SO}(2)\times\mathrm{SO}(q)\big)$ are $\Delta_2^{(q)}$ and $\Delta_4^{(q)}$. The generators of $D(G/K)$ are given in [6] in all rank 2 cases except for $\mathbf{G}_2/\,\mathrm{SU}(2)\times\mathrm{SU}(2)$.

The following result is mentioned in [13, Lemma 9] without proof.

***Lemma 12*** *We have*

1. $[\Delta_2^{(q)}, L_1^{(q)}] = 2(\frac{1}{\sinh^2(H_1-H_2)} + \frac{1}{\sinh^2(H_1-H_2)})L_1^{(q)}$.
2. $[\Delta_2^{(q)}, L_2^{(q)}] = -2L_2^{(q)} \circ (\frac{1}{\sinh^2(H_1-H_2)} + \frac{1}{\sinh^2(H_1-H_2)})$.
3. $[\Delta_2^{(q)}, \Delta_4^{(q)}] = 0$.

**Proof** The proof of 1. and 2. is rather tedious but can be done using a system such as Maple or Mathematica (we did both). To alleviate some of the computations, one applies the differential operators to the function $e^{a_1H_1+a_2H_2}$ and then we factor out that function. Proving those equalities then becomes a matter of checking the equality of polynomials in the $a_i$'s of degree at most 3.

The proof of 3. follows directly from 1., 2. and $[\Delta_2^{(q)}, L_2^{(q)}L_1^{(q)}] = L_2^{(q)}[\Delta_2^{(q)}, L_1^{(q)}] + [\Delta_2^{(q)}, L_2^{(q)}]L_1^{(q)}$. ∎

***Remark 13*** There are some mistakes in the statement of [13, Lemma 9]. Referring to the notation in [13]: the terms $\coth(2h_1)$ and $\coth(2h_2)$ that appear in the 0 order term of $L_2^{(\nu,\mu)}$ should be interchanged and the sign preceding $4\nu$ in (i) should be +.

***Definition 14*** Let $D^{(q)}$ be the algebra generated by $\Delta_2^{(q)}$ and $\Delta_4^{(q)}$. Let $\chi(\Delta_2^{(q)})(\lambda) = -\Big(\lambda_1^2+\lambda_2^2+\big((q-1)^2+1\big)/2\Big)$ and $\chi(\Delta_4^{(q)})(\lambda) = \big((\lambda_1-\lambda_2)^2+1\big)\big((\lambda_1+\lambda_2)^2+1\big)$ and extend $\chi$ on $D^{(q)}$ as an algebra homomorphism (refer to [13, p 99] with $\lambda$ replaced by $i\lambda$).

We define the map $T\colon D\big(\mathrm{SO}_0(2,q)/\,\mathrm{SO}(2)\times\mathrm{SO}(q)\big) \to D\big(\mathrm{GL}_+(2,\mathbf{R})/\,\mathrm{SO}(2)\big)$ the following way. Let $D \in D\big(\mathrm{SO}_0(2,q)/\,\mathrm{SO}(2)\times\mathrm{SO}(q)\big)$. We have $D\phi_\lambda = \chi(D)(\lambda)\phi_\lambda$. We define $T(D)$ to be the unique differential operator in $D\big(\mathrm{GL}_+(2,\mathbf{R})/\,\mathrm{SO}(2)\big)$ such that $T(D)\tilde{\phi}_\Lambda = \chi(D)(\lambda)\tilde{\phi}_\Lambda$ (recall the relationship between $\lambda$ and $\Lambda$). This is possible by the end of Remark 1.

***Remark 15*** The map $\chi$ is well defined since $\Delta_2^{(q)}$ and $\Delta_4^{(q)}$ commute and are algebraically independent. Note that for $q \geq 2$ an integer, $D^{(q)}$ corresponds to $D\big(SO_0(2,q)/\,\mathrm{SO}(2)\times\mathrm{SO}(q)\big)$ and $D\phi_\lambda = \chi(D)(\lambda)\phi_\lambda$ whenever $D \in D^{(q)}$. Note also that in the notation of [13], $D^{(q)} = A_{1,q-2}$.

We justify our statement that $\phi_\lambda^{(q)}$ is a generalized spherical function of the root system $B_2$ by showing that $D\phi_\lambda^{(q)} = \chi(D)(\lambda)\phi_\lambda^{(q)}$ for every $D \in D^{(q)}$. This is done in Theorem 20 given toward the end of this section. Our strategy to prove that result is simple. First, we show that for an important range of $\lambda$'s, $\phi_\lambda^{(q)}$ is a rational function of $q$ which is known when $q \geq 4$ is an integer. The next step is to show that the other $\phi_\lambda^{(q)}$'s can be approximated by the smaller class.

To this end, we need to know more about integration on $\mathcal{M}_2$ as given in (4).

**Lemma 16** *If $p(A) = p(a_{11}, a_{1,2}, a_{21}, a_{22})$ is a polynomial then $\int_{\mathcal{M}_2} p(A)\big(\det(I - AA^T)\big)^{(q-5)/2}\, dA$ is a rational function of $q$.*

**Proof** It is enough to show this for terms of the form $a_{11}^{n_{11}} a_{12}^{n_{12}} a_{21}^{n_{21}} a_{22}^{n_{22}}$. If $\sum n_{ij}$ is odd, then we can show that the integral is 0 using the invariance properties of $\big(\det(I - AA^T)\big)^{(q-5)/2} dA$. If $\sum n_{ij}$ is even then $a_{11}^{n_{11}} a_{12}^{n_{12}} a_{21}^{n_{21}} a_{22}^{n_{22}}$ will be a sum of terms $x_1^r x_2^s F(\theta, \psi)$ with $r + s = \sum n_{ij}$. If $r$ and $s$ are both odd then the integral is 0. We can therefore assume that $r$ and $s$ are both even. Note also that $\tilde{p}(x_1, x_2) = \int_0^{2\pi}\int_0^{2\pi} p(a_{11}, a_{12}, a_{21}, a_{22})\, d\theta\, d\psi$ is symmetric in $x_1^2$ and $x_2^2$. To see this, it suffices to replace $\theta$ by $\theta + \pi/2$ and $\psi$ by $\psi + \pi/2$ in (7). We also note that every symmetric polynomial in $x_1^2$ and $x_2^2$ can be written as a linear combination of terms of the form $(x_1^{2k} + x_2^{2k})(1 - x_1^2)^i(1 - x_2^2)^i$ (if $i \leq j$, then $x_1^{2i} x_2^{2j} + x_1^{2j} x_2^{2i} = (x_1^{2(j-i)} + x_2^{2(j-i)}) \cdot (1 - x_1^2)^i (1 - x_2^2)^i$+terms of lower degree). It is therefore enough to show that

$$R_k(\nu) = \int_0^1 \int_0^{x_1} (x_1^{2k} + x_2^{2k})(x_1^2 - x_2^2)(1 - x_1^2)^{\nu-1}(1 - x_2^2)^{\nu-1}\, dx_2\, dx_1$$

is a rational function of $\nu$ for every fixed integer $k \geq 0$.

Let $\gamma_1(t) = (t, t)$, $-\gamma_2(t) = (t, 1)$ and $-\gamma_3(t) = (0, t)$ where $0 \leq t \leq 1$. Let $P = -x_1^{2k+1}(1 - x_1^2)^{\nu}(1 - x_2^2)^{\nu-1}$ and $Q = x_2^{2k+1}(1 - x_1^2)^{\nu-1}(1 - x_2^2)^{\nu}$ and note that $\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} = \big[(2\nu + 2k + 1)\big(x_1^{2k} + x_2^{2k} + S_{k-1}(x_1^2, x_2^2)\big)\big](x_1^2 - x_2^2)(1 - x_1^2)^{\nu-1}(1 - x_2^2)^{\nu-1}$ where $S_{k-1}$ is a symmetric polynomial of degree at most $k - 1$ (in particular, if $k = 0$ then $S_{-1} = 0$). Using Green's theorem, we have

$$\int_0^1 \int_0^{x_1} \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}\right) dx_2\, dx_1 = \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} (Q\, dx_1 - P\, dx_2).$$

The latter expression is rational in $\nu$. The result follows then using induction in $k$. ■

**Corollary 17** *We have $C_q^2 = \frac{1}{(q-2)(q-3)}$.*

**Proof** This requires a simple computation using $P$ and $Q$ with $k = 0$ and $\nu = (q - 3)/2$. ■

We state a Weierstrass theorem for functions of several variables. The explicit construction of the polynomials allows us to say something about the derivatives of order at most 2.

**Lemma 18** *Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be such all that its derivatives up to order 2 are continuous. Let $K$ be a compact set. Then there exists a sequence of polynomials $(f_n)$ such that $f_n$, $\frac{\partial f_n}{\partial x_i}$ and $\frac{\partial^2 f_n}{\partial x_i x_j}$ converge uniformly to $f$, $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i x_j}$ respectively for $i, j = 1, 2$.*

**Proof** By scaling and by multiplying by an appropriate cutoff function, we may assume that $K \subset \{x : \|x\| < 1/2\}$ and that $f$ vanishes when $\|x\| \geq 1/2$.

Let $p_n(x) = c_n(1 - \|x\|^2)^n$ where $n$ is chosen so that $\int_{\|x\|<1} p_n(y)\,dy = 1$. Let

$$f_n(x) = \int_{\mathbf{R}^2} f(y)p_n(x-y)\,dy = \int_{\mathbf{R}^2} f(x-y)p_n(y)\,dy.$$

The first equality ensures that $f_n$ is a polynomial while the second equality shows that the relationship between the derivatives of $f_n$ and those of $f$ is the same as that between $f_n$ and $f$. It is therefore enough to show that the sequence $(f_n)$ converges uniformly to $f$ on the set $\|x\| < 1/2$. The rest of the proof is very close to [14, p. 117]. ∎

***Remark 19*** If $f = f(x_1, x_2)$ is symmetric in $x_1$ and $x_2$ then we can choose symmetric polynomials. Indeed, it suffices to replace $f_n$ from the proof by $\big(f_n(x_1, x_2) + f_n(x_2, x_1)\big)/2$.

***Theorem 20*** *Let $p = 2$ and suppose $\Re q > 3$. Then for every $D \in D^{(q)}$, we have*

$$D\phi_\lambda^{(q)} = \chi(D)(\lambda)\phi_\lambda^{(q)}. \tag{9}$$

**Proof** We will proceed in two steps. It is useful to refer to (2).

1. Let $n_1 \geq n_2 \geq 0$ be integers. Let $\Lambda(H) = -i(4n_1+1)H_1/2 - i(4n_2-1)H_2/2$. This ensures that $\tilde\phi_\Lambda(g)$ is a symmetric polynomial in $e^{2H_1}$ and $e^{2H_2}$, *i.e.*, a polynomial in $r_1 = \operatorname{tr} gg^T = e^{2H_1} + e^{2H_2}$ and in $r_2 = (\det g)^2 = e^{2H_1+2H_2}$. These polynomials are known as "Jack polynomials" (to know more about them, refer to [5]). If $g = \cosh \mathcal{D} + \sinh \mathcal{D}A$ then $r_1$ and $r_2$ are polynomials in $\sinh H_i$, $\cosh H_i$ and in the $a_{ij}$.
   When we apply the operator $D$ to $\phi_\lambda$, equation (8), Lemma 16 and the above imply that both sides are rational functions of $q$. We know that they are equal when $q \geq 4$ is an integer since then $\phi_\lambda^{(q)}$ is a spherical function of a symmetric space. This means that they have to be equal for every $q$. Therefore (9) holds for the corresponding class of $\lambda$'s.
2. Let $T$ be as in Definition 14. We claim that

$$\begin{aligned} D\int_{\mathcal{M}_2} f(\cosh D + \sinh DA)\big(\det(I - AA^T)\big)^{(q-5)/2}\,dA \\ = \int_{\mathcal{M}_2} \big(T(D)f\big)(\cosh D + \sinh DA)\big(\det(I - AA^T)\big)^{(q-5)/2}\,dA \end{aligned}$$

   where $f\colon \mathbf{R}^2 \to \mathbf{R}$ is such all that its derivatives up to order 2 are continuous. This clearly implies the desired result (one just has to take $f = \tilde\phi_\Lambda$).
   We know that the claim is true for every $\tilde\phi_\Lambda$ with $\Lambda$ chosen as in part 1, *i.e.*, for every Jack polynomial and, by linearity, for every symmetric polynomials. We then use Proposition 18 to do the rest. ∎

***Remark 21*** We could have done much of the same thing with $p = 1$. We already know that for $p = 1$, $\phi_\lambda^{(q)}$ can be described by an ordinary hypergeometric function (see for instance [4, Problem 8, p. 484]).

The following lemma gives a crude estimate which will be useful next section.

***Lemma 22*** *Suppose $p = 2$. Suppose $K_1$ and $K_2$ be compact subsets of $\mathfrak{a}$ and $\mathfrak{a}^*$ respectively and suppose $\delta > 0$. Then there exist a constant $C = C(K_1, K_2, \delta)$ such that for $\Re q > 3 + \delta$, we have $|\phi_\lambda^{(q)}(e^H)| \le C|q|^2$.*

—Proof Note first that the set $\{\cosh \mathcal{D} + (\sinh \mathcal{D})A : \mathcal{D} \in K_1, A \in \mathcal{M}_2\}$ is compact. If we refer to (8) and to Corollary 17 then

$$\begin{aligned}
|\phi_\lambda^{(q)}(e^H)| &\le \frac{1}{|C_q^2|}\left|\int_{\mathcal{M}_2} \tilde{\phi}_\Lambda\big(\cosh \mathcal{D} + (\sinh \mathcal{D})A\big)\big(\det(I - AA^T)\big)^{(q-5)/2}\, dA\right| \\
&\le \frac{1}{|C_q^2|}\int_{\mathcal{M}_2} \left|\tilde{\phi}_\Lambda\big(\cosh \mathcal{D} + (\sinh \mathcal{D})A\big)\right|\big(\det(I - AA^T)\big)^{\Re(q-5)/2}\, dA \\
&\le \frac{\tilde{C}(K_1, K_2)}{|C_q^2|}\int_{\mathcal{M}_2} \big(\det(I - AA^T)\big)^{\Re(q-5)/2}\, dA \\
&\le \frac{\tilde{C}(K_1, K_2) C_{\Re q}^2}{|C_q^2|} = \frac{\tilde{C}(K_1, K_2)|(q-2)(q-3)|}{(\Re q - 2)(\Re q - 3)}.
\end{aligned}$$

■

## 3 On a Conjecture by Jiro Sekiguchi

As explained in the introduction, the work of Sekiguchi in [13] was the inspiration for this paper. In Section 2, we accomplished the generalization of the spherical functions he wanted but with another "candidate". We will now show that our generalizations are equivalent thus proving his conjecture.

We first recall some of the notation and results of [13]. Let

$$\begin{aligned}
L(b_1, b_2, t) = &\left\{1 + b_1^2 t_1 + (1 + b_1 b_2 t_2)\left(1 + \frac{b_1 t_2}{b_2}\right) t_3\right\} \\
&\cdot \left\{1 + \frac{1}{b_1^2} t_1 + \left(1 + \frac{t_2}{b_1 b_2}\right)\left(1 + \frac{b_2 t_2}{b_1}\right) t_3\right\} \\
&+ \Bigg(2(1 + t_1) + \bigg\{(1 + b_1 b_2 t_2)\left(1 + \frac{t_2}{b_1 b_2}\right) \\
&+ \left(1 + \frac{b_2 t_2}{b_1}\right)\left(1 + \frac{b_1 t_2}{b_2}\right)\bigg\}\Bigg) t_4 + t_4^2.
\end{aligned}$$

If $m \ge 1$ is an integer, let $\Phi_{\lambda,m}(a_1, a_2) = \phi_{-i\lambda}^{(m+2)}(e^H) = \int_K e^{(\lambda-\rho)(\mathcal{H}(e^H k))}\, dk$ where $a_i = e^{H_i}$. Suppose $0 < \Re(\lambda_1 - \lambda_2) < 1$ and $-m/2 < \Re\lambda_2 < m/2$. In [13,

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Theorem 5], Sekiguchi showed that

(10)

$$\Phi_{\lambda,m}(a_1,a_2) = C(\lambda,m)\int_0^\infty t_1^{-(\lambda_1-\lambda_2+1)/2}\,dt_1\int_0^\infty t_2^{-\lambda_2+m/2-1}\,dt_2\int_0^\infty t_3^{m/2-1}\,dt_3 \cdot\int_0^\infty t_4^{m/2-1}\,dt_4\int_{-\infty}^\infty (1+y^2)^{(m-1)/2}(L_1+L_2y^2)^{-(m+1)/2}\,dy$$

where $C(\lambda,m) = 2^{2m-2}m\pi^{-3}\Gamma(\frac{m+1}{2})^2\frac{\cos\frac{\pi(\lambda_1-\lambda_2)}{2}}{\Gamma(\lambda_2+m/2)\Gamma(-\lambda_2+m/2)}$, $L_1 = L(a_1,a_2,t)$ and $L_2 = L(a_2,a_1,t)$.

***Definition 23* (Sekiguchi)** Let $\Phi_{\lambda,m}$ be as in (10) in the domain $\Re m > 0$, $0 < \Re(\lambda_1-\lambda_2) < 1$ and $-\Re m/2 < \Re\lambda_2 < \Re m/2$.

Sekiguchi conjectured in [13] that the $\Phi_{\lambda,m}$'s would be eigenfunctions of the operators given in Definition 10 (modulo a change of variables).

To prove the conjecture Sekiguchi made, it will suffice to show that $\Phi_{\lambda,m}(a_1,a_2) = \phi^{(m+2)}_{-i\lambda}(a_1,a_2)$ whenever $\Re m > 1$. We already know that this equality is valid when $m$ is an integer greater than 1. This will suffice once we show that the two functions satisfy similar bounds.

***Lemma 24*** *Suppose $\Re m > 0$, $0 < \Re(\lambda_1-\lambda_2) < 1$ and $-\Re m/2 < \Re\lambda_2 < \Re m/2$. Then $\Phi_{\lambda,m}(1,1) = 1$.*

**Proof** Take $a_1 = a_2 = 1$ in (10). We then have $L_1 = L_2 = \big(1+t_1+t_4+t_3(1+t_2)^2\big)^2$. Using (10), we have

$$\Phi_{\lambda,m}(1,1) = C(\lambda,m)\pi\int_0^\infty t_1^{-(\lambda_1-\lambda_2+1)/2}\,dt_1\int_0^\infty t_2^{-\lambda_2+m/2-1}\,dt_2\int_0^\infty t_3^{m/2-1}\,dt_3 \cdot\int_0^\infty t_4^{m/2-1}\big(1+t_1+t_4+t_3(1+t_2)^2\big)^{-(m+1)}\,dt_4.$$

Now, if we make the change of variables $s_3 = t_3(1+t_2)^2$, $s_4 = t_4/(1+t_1+s_3)$, $s_1 = t_1/(1+s_3)$ and $s_2 = t_2$, then the computations become straightforward. ■

***Corollary 25*** *Suppose $\Re m > 0$, $0 < \Re(\lambda_1-\lambda_2) < 1$ and $-\Re m/2 < \Re\lambda_2 < \Re m/2$. Then*

$$|\Phi_{\lambda,m}(a_1,a_2)| \le \frac{|C(\lambda,m)|}{C(\Re\lambda,\Re m)}.$$

**Proof** It is not difficult to see that the minimum of $L_1$ and $L_2$ occur when $a_1 = a_2 = 1$. Noting that for $a > 0$, $|a^z| = a^{\Re z}$, the result follows from the proof of the lemma. ■

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***Corollary 26*** *Let $M > 0$ and $0 < \delta < 1/2$ be fixed. Suppose $\delta < \Re(\lambda_1 - \lambda_2) < 1 - \delta$, $-M/2 < \Re\lambda_2 < M/2$, $|\Im\lambda_i| < M/2$, $i = 1, 2$. Then there exists a constant $C = C(M, \delta)$ such that on the domain $\Re m > 2M + 1$, we have*

$$|\Phi_{\lambda,m}(a_1, a_2)| \leq C|m^2|.$$

**Proof** We know that $\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}$ when $z \to \infty$ provided $|\arg| < \pi - \delta$ for some $\delta > 0$. This means that in our domain, there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1 |z|^{\Re z - 1/2} e^{-\Re z} \leq |\Gamma(z)| \leq C_1 |z|^{\Re z - 1/2} e^{-\Re z}$. This implies that $\frac{|C(\lambda,m)|}{C(\Re\lambda,\Re m)} \leq C \frac{|m|^2}{(\Re m)^2}$. ∎

***Lemma 27*** *Let $a \in \mathbf{R}$ and suppose $|f(z)| \leq |P(z)|$ where $P$ is a polynomial and $f$ is an analytic function on $\Omega = \{z : \Re z > a\}$ which is 0 at every integer in $\Omega$. Then $f(z) = 0$ for all $z \in \Omega$.*

**Proof** By taking $a$ larger if necessary, we may assume that $\Omega$ does not contain any zero of $P$ (if $f$ is zero on the smaller domain, it has to be zero on the larger one). Let $U$ denotes the unit disk and let $T \colon \Omega \to U$ be defined by $T(z) = (z-a-1)/(z-a+1)$ and note that $1 - T(n) = 2/(n - a + 1)$. Now, $(f/P) \circ T^{-1}$ is a bounded analytic function on $U$ with zeros at $T(n)$. Since $\sum \big(1 - T(n)\big)$ is divergent, we conclude from [9, Theorem 15.23] that $(f/P) \circ T^{-1}$ is identically zero. ∎

We now prove the conjecture made by Sekiguchi in [13].

***Theorem 28*** *Suppose $\Re m > 1$, $0 < \Re(\lambda_1 - \lambda_2) < 1$ and $-\Re m/2 < \Re\lambda_2 < \Re m/2$. Then*

$$\Phi_{\lambda,m}(e^H) = \phi^{(m+2)}_{-i\lambda}(e^H). \tag{11}$$

**Proof** It suffices to show (11) with the following stricter restriction: $\Re m > 2(1+\delta)$ with $\delta > 0$, $\delta < \Re(\lambda_1 - \lambda_2) < 1 - \delta$, $-(1+\delta)/2 < \Re\lambda_2 < (1+\delta)/2$, $|\Im\lambda_i| < (1+\delta)/2$, $i = 1, 2$ and $H$ in an arbitrary compact subset of $\mathfrak{a}$ since the result will then follow by analyticity. If we consider $f(m) = \Phi_{\lambda,m}(e^H) - \phi^{m+2}_{-i\lambda}(e^H)$ only as a function of $m$ then it is analytic and it is bounded by a multiple of $|m^2|$ according to Lemma 22 and Corollary 25. Since $f(m) = 0$ when $m \geq 2$ is an integer, we can use Lemma 27 to conclude. ∎

## 4 Conclusion

A natural question is whether this can be generalized to all $p$.

Let us outline what would need to be done. One would have to show that Proposition 4 (or something similar) is valid for all $p$. On the other hand, a general version of Corollary 5 would follow from the proposition without difficulty.

Defining the algebra $D^{(q)}$ for an arbitrary $p$ might be difficult to do explicitly but all we really needed was the fact that the operators in $D^{(q)}$ were rational (actually polynomial) functions of $q$. Note that a version of Lemma 16 for $p = 1$ would

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require the factor $(C_q^1)^{-1}$ in order to be valid. We presume that in general, it would depend on whether $p$ is even or odd. Lemma 18 is easily generalized to all $p$. If all these "details" were taken care of, the proof of Theorem 20 would hold for all integers $p \geq 1$ and all $q$ with $\Re q > 2p - 1$.

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