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VECTOR VALUED MEAN-PERIODIC FUNCTIONS ON GROUPS

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Abstract

Let G be a locally compact Hausdorff abelian group and X be a complex Banach space. Let $C(G, X)$ denote the space of all continuous functions $f : G \to X$, with the topology of uniform convergence on compact sets. Let X' denote the dual of X with the weak* topology. Let $M_c(G, X')$ denote the space of all X'-valued compactly supported regular measures of finite variation on G. For a function $f \in C(G, X)$ and $\mu \in M_c(G, X')$, we define the notion of convolution $f \star \mu$. A function $f \in C(G, X)$ is called mean-periodic if there exists a non-trivial measure $\mu \in M_c(G, X')$ such that $f \star \mu = 0$. For $\mu \in M_c(G, X')$, let $MP(\mu) = \{f \in C(G, X) : f \star \mu = 0\}$ and let $MP(G, X) = \bigcup_\mu MP(\mu)$. In this paper we analyse the following questions: Is $MP(G, X) \neq \emptyset$? Is $MP(G, X) \neq C(G, X)$? Is $MP(G, X)$ dense in $C(G, X)$? Is $MP(\mu)$ generated by 'exponential monomials' in it? We answer these questions for the groups $G = \mathbb{R}$, the real line, and $G = \mathbb{T}$, the circle group. Problems of spectral analysis and spectral synthesis for $C(\mathbb{R}, X)$ and $C(\mathbb{T}, X)$ are also analysed.

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1. Introduction

The notion of mean-periodic functions was introduced in 1935 by Delsarte [5]. It is well known that every solution of a constant coefficient homogeneous ordinary differential equation is a finite linear combination of solutions of the type $t^k e^{\iota\lambda t}$, where $\lambda \in \mathbb{C}$, and $k \in \mathbb{Z}_+$. Delsarte was interested in knowing whether this result is still true for convolution equation of the following type

$$(1) \qquad \int_{\mathbb{R}} f(s-t)k(t)\,dt = 0, \quad \forall\, s \in \mathbb{R},$$

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where k is a continuous function which is zero out side some interval. For $\tau > 0$, periodic continuous functions of period τ are solutions of the convolution equation

$$(2) \qquad \frac{1}{\tau}\int_{s-\tau/2}^{s+\tau/2} f(t)\,dt = 0, \quad \forall\, s \in \mathbb{R}.$$

For this reason Delsarte called the continuous functions which are solutions of equation (1) as *mean-periodic*. In [35], Schwartz observed that the mean-periodicity of a continuous function does not depend upon the function k, and he extended Delsarte's definition as follows:

DEFINITION 1.1. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be *mean-periodic* if there exists a non-trivial regular measure μ of compact support and finite variation such that $(f \star \mu)(s) = \int_{\mathbb{R}} f(s-t)\,d\mu(t) = 0, \forall\, s \in \mathbb{R}$.

Schwartz also gave an intrinsic characterization of mean-periodic functions. Let $C(\mathbb{R})$ denote the vector space of complex valued continuous functions on $\mathbb{R}$ with the topology of uniform convergence on compact sets (u.c.c.). Let $M_c(\mathbb{R})$ denote the space of all regular measures of compact support and finite variation on $\mathbb{R}$. For $f \in C(\mathbb{R})$, let $\tau(f)$ denote the closed translation invariant subspace of $C(\mathbb{R})$ generated by f. Schwartz in [35] showed that $f \in C(\mathbb{R})$ is mean-periodic if and only if $\tau(f) \neq C(\mathbb{R})$. Further, if $f \star \mu = 0$ for some non-zero $\mu \in M_c(\mathbb{R})$, then f is a limit of finite linear combination of exponential monomials $t^k e^{i\lambda t}$ which satisfy $t^k e^{i\lambda t} \star \mu = 0$. More generally, convolution equation of the type

$$(3) \qquad f \star \mu = g,$$

where $\mu \in M_c(\mathbb{R})$ and $g \in C(\mathbb{R})$ are given, can be analysed as in the case of ordinary differential equations. If p is a particular solution of the equation (3), then every other solution is of the form $h + p$, where h is a solution of the homogeneous equation $f \star \mu = 0$. In general, equation (3) need not have any solution in $C(\mathbb{R})$. For instance, let μ be such that $d\mu(t) = \phi(t)\,dt$, where $\phi \in C_c^\infty(\mathbb{R})$, space of all infinitely differentiable functions on $\mathbb{R}$, and g is a nowhere differentiable continuous function on $\mathbb{R}$. Some particular cases of (3) were analysed in [31, 32]. In general, no necessary and sufficient conditions for the existence of solutions of equation (3) are known. A variant of the above problem is the following: Consider the following convolution equation

$$(4) \qquad f_1 \star \mu_1 = -f_2 \star \mu_2,$$

where $\mu_1, \mu_2 \in M_c(\mathbb{R})$ are given. Equation (4) can be written as a convolution equation for vector valued functions: let $\underline{f} = (f_1, f_2) : \mathbb{R} \to \mathbb{C}^2$ and $\underline{\mu} = (\mu_1, \mu_2) : \mathscr{B}_{\mathbb{R}} \to \mathbb{C}^2$.

Then equation (4) is a homogeneous equation $\underline{f} \star \underline{\mu} = 0$. This leads to consideration of vector valued mean-periodic functions, the main content of this paper. We consider such equations in a more general setting and analyse their solutions.

Let G be a locally compact abelian group. Let X be a complex Banach space and X' denote the weak*-dual of X. We denote by $\mathscr{B}_G$ the σ-algebra of Borel subsets of G. We recall some results on integration of functions $f : G \to X$ with respect to X'-valued measures on $\mathscr{B}_G$, denoted by $M(G, X')$. For details one may refer Schmets [34]. Let $\mu \in M(G, X')$ and for every x, let μ_x denote the scalar measure on $\mathscr{B}_G$ defined by $\mu_x(E) := \langle x, \mu(E) \rangle$ for every $E \in \mathscr{B}_G$. The measure μ is said to be *regular* if μ_x is regular for every $x \in X$. For $E \in \mathscr{B}_G$, if $E = \bigcup_{i=1}^n E_i$ for some $E_1, E_2, \dots, E_n \in \mathscr{B}_G$ such that $E_i \bigcap E_j = \emptyset$ for $i \neq j$, we call $\{E_1, E_2, \dots, E_n\}$ a *measurable partition* of E. Let $\mathscr{P}(E)$ denote the set of all measurable partitions of E. Let

$$V_\mu(E) := \sup \left\{ \sum_{i=1}^{n} \|\mu(E_i)\| : \{E_1, E_2, \dots, E_n\} \in \mathscr{P}(E) \right\}.$$

The scalar measure V_μ is called the *variation* of μ. We say μ has finite variation if $V_\mu(E) < +\infty$ for every $E \in \mathscr{B}_G$. Let $M(G, X')$ denote the set of all regular Borel measures μ on G such that μ has finite variation. For $\mu \in M(G, X')$ the smallest closed set S with $\mu(E) = 0$ for every $E \in \mathscr{B}_G$ with $E \cap S = \emptyset$ is called the support of μ. We write $S = \operatorname{supp}(\mu)$ if S is the support of μ. Let $M_c(G, X')$ denote the set of all $\mu \in M(G, X')$ such that support of μ is compact. Let $C(G, X)$ denote the space of all X-valued continuous functions on G with the topology of uniform convergence on compact sets. Let $f \in C(G, X)$ and $\mu \in M_c(G, X')$ with $\operatorname{supp}(\mu) \subseteq K$, a compact set. Then there exists a sequence $\mathscr{P}_k(K) := \{B^k_{k_1}, B^k_{k_2}, \dots, B^k_{k_n}\}$ of measurable partitions of K with the following property : for arbitrary choice of $t_i \in B_{k_i}$, the sequence $\left\{\sum_{i=1}^n \langle f(t_i), \mu(B^k_{k_i})\rangle\right\}_{k \geq 1}$ is convergent and is independent of the choice of $t_i's$. This limit is called the *integral* of f with respect to μ and is denoted by $\int f \, d\mu$. For $f \in C(G, X)$ and $\mu \in M_c(G, X')$ the scalar valued function

$$(f \star \mu)(g) := \int_G f(g-h)\, d\mu(h), \quad \forall\, g \in G$$

is called the *convolution* of f with μ, that is, $(f \star \mu)(g) = \mu(f_g) = \langle \mu, f_g \rangle$, where $f_g(h) = f(g+h)$ and $\langle \mu, f \rangle = \mu(f) = \int_G f(-g)\, d\mu(g)$ is the duality pairing of $M_c(G, X')$ with $C(G, X)$.

DEFINITION 1.2. We say $f \in C(G, X)$ is *mean-periodic* if there exists a non-trivial $\mu \in M_c(G, X')$ such that $(f \star \mu)(g) = \int_G f(g-h)\, d\mu(h) = 0, \forall\, g \in G$.

The aim of this paper is to answer the following questions: let $MP(G, X)$ denote the space of all X-valued mean-periodic functions on G.

- Is $MP(G, X) \neq \emptyset$? That is, when does there exist non-zero mean-periodic functions?
- Is $MP(G, X) \neq C(G, X)$? That is, do there exist continuous functions which are not mean-periodic?
- Is $MP(G, X)$ dense in $C(G, X)$? That is, how large is $MP(G, X)$ as a subspace of $C(G, X)$?

We answer these questions for the particular cases $G = \mathbb{R}$, in Section 2 and $G = \mathbb{T}$, circle group, in Section 3. Analysis of such questions for more general groups remain open.

The problem of analysing mean-periodic functions is also related to the problem of 'spectral analysis' and 'spectral synthesis'. In order to carry-out the analysis, we define next vector valued exponential monomials and exponential polynomials.

An *additive function* on a locally compact abelian group is a complex valued continuous function a on G such that $a(g_1 + g_2) = a(g_1) + a(g_2)$ for all g_1 and g_2 in G. A *polynomial* on G is a function of the form $p(a_1, a_2, \dots, a_m)$, where p is a polynomial in m variables and $a_1, a_2, \dots, a_m$ are additive functions on G. A *monomial* on G is a function of the form $p(a_1, a_2, \dots, a_m)$, where p is a monomial in m variables and $a_1, a_2, \dots, a_m$ are additive functions on G. An *exponential* on G is a non-zero continuous complex valued function ω such that $\omega(g_1 + g_2) = \omega(g_1)\omega(g_2)$ for all g_1 and g_2 in G. An *exponential monomial* is a point-wise product of a monomial and an exponential. An *exponential polynomial* is a point-wise product of a polynomial and an exponential. The set of all exponentials is denoted by Ω. Note that $\Omega \subset C(G)$.

We define exponential polynomials in $C(G, X)$ as follows:

DEFINITION 1.3. (i) We call $f \in C(G, X)$ an X-valued *exponential* if for every $g \in G$, $f(g) = \omega(g)x$ for some $\omega \in \Omega$ and $x \in X$.

(ii) We call $f \in C(G, X)$ an X-valued *exponential monomial* if for every $g \in G$, $f(g) = p(g)\omega(g)x$ for some $x \in X$, p a monomial in $C(G)$ and ω an exponential in $C(G)$.

(iii) We call $f \in C(G, X)$ an X-valued *exponential polynomial* if for every $g \in G$, $f(g) = p(g)\omega(g)x$ for some $x \in X$, p a polynomial in $C(G)$ and ω an exponential in $C(G)$.

EXAMPLE 1. (1) Let $f \in C(\mathbb{R}, X)$. Then f is an exponential if and only if for every $t \in \mathbb{R}$, $f(t) = e^{\iota\lambda t}x$ for some $\lambda \in \mathbb{C}$ and $x \in X$. f is an exponential monomial if and only if for every $t \in \mathbb{R}$, $f(t) = t^k e^{\iota\lambda t}x$ for some $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$ and $x \in X$. Finally, f is an exponential polynomial if and only if for every $t \in \mathbb{R}$, $f(t) = p(t)e^{\iota\lambda t}x$ for some $\lambda \in \mathbb{C}$, polynomial $p(t)$ and $x \in X$. Thus the exponentials, exponential monomials and exponential polynomials are the scalar multiples of the ones defined by Schwartz [35].

(2) A function $f \in C(\mathbb{T}, X)$ is an exponential if and only if for every $t \in \mathbb{R}$, $f(e^{it}) = e^{int}x$ for some non-negative integer n and $x \in X$.

REMARK. We shall use the following convention: When $X = \mathbb{C}$ we choose the $x \in X$ appearing in the exponential, exponential monomial and exponential polynomial to be the scalar constant 1. The generality is not lost due to this choice, since if a closed translation invariant subspace contains an exponential or exponential monomial or exponential polynomial if and only if it contains their scalar multiples.

DEFINITION 1.4. Let V be a closed translation invariant subspace of $C(G, X)$. We say

(i) *spectral analysis holds for* V if V contains an exponential;

(ii) *spectral synthesis holds for* V if the linear span of the set of all exponential monomials in V is dense in V;

(iii) if spectral analysis (synthesis) holds for every closed translation invariant subspace V of $C(G, X)$, then we say that *spectral analysis* (*synthesis*) *holds in* $C(G, X)$.

DEFINITION 1.5. Let V be a closed translation invariant subspace of $C(G, X)$ and $f \in C(G, X)$ be mean-periodic. Let $\tau(f)$ denote the closed translation invariant subspace of $C(G, X)$ generated by f.

(i) The *spectrum* of V is defined to be the set of all exponential monomials in V and is denoted by $\operatorname{spec}(V)$ or $\sigma(V)$.

(ii) The *spectrum* of f is defined to be $\operatorname{spec}(\tau(f))$ and is denoted by $\operatorname{spec}(f)$ or $\sigma(f)$.

Some of the known results for spectral analysis and spectral synthesis for $G = \mathbb{R}^n$ are as follows: Let $E(\mathbb{R}^n)$ be the space of all infinitely differentiable functions on $\mathbb{R}^n$ in the topology of compact convergence of functions and their derivatives. Then its dual $E(\mathbb{R}^n)'$ is the space of all compactly supported distributions on $\mathbb{R}^n$. Schwartz [35] proved the following theorem:

THEOREM 1.6 ([35]). *In $E(\mathbb{R})$, every closed translation invariant subspace is the closure of finite linear combinations of the exponential monomials in it.*

As a consequence of this theorem, the linear span of exponential monomials in every closed translation invariant subspace V of $C(\mathbb{R})$ is dense in V. That is, spectral analysis and spectral synthesis hold in $C(\mathbb{R})$. Using this Schwartz [35] described mean-periodic functions on $\mathbb{R}$.

Let V be the closed translation invariant subspace of $E(\mathbb{R}^n)$ generated by the solutions of the homogeneous constant coefficient partial differential equation $p(D)f = 0$. Malgrange [28] proved that spectral synthesis holds for V.

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In 1975 Gurevich [17] proved that Theorem 1.6 cannot be extended for $\mathbb{R}^n$, $n > 1$. Though Theorem 1.6 fails for $\mathbb{R}^n$, $n > 1$, spectral analysis and spectral synthesis hold in $C(G)$ for certain groups, for example, for $G = \mathbb{Z}^n$ (see [26]) and for discrete abelian groups (see [12, 13]). Consider the following example from [15].

EXAMPLE 2 ([10, 15]). Define $f_1, f_2 : \mathbb{R}^2 \to \mathbb{C}$ by

$$f_1(x_1, x_2) := 1 \quad \text{and} \quad f_2(x_1, x_2) := x_1 + x_2, \quad \forall\, (x_1, x_2) \in \mathbb{R}^2.$$

Let V be the closed translation invariant subspace of $C(\mathbb{R}^2)$ generated by f_1 and f_2. Then the spectrum of V is $\{f_1\}$. But the closed linear span of the spectrum of V is a proper subspace of V. Thus spectral synthesis fails in $C(\mathbb{R}^2)$ and spectral synthesis fails for V even if V is finite dimensional.

However, for certain closed translation invariant subspaces $V \subset C(\mathbb{R}^2)$ the linear span of all exponential polynomials in V is dense in V. These subspaces are described in the following three theorems.

THEOREM 1.7 ([4]). *Let V be a closed translation and rotation invariant subspace of $C(\mathbb{R}^2)$. Then the linear span of exponential polynomials in V is dense in V.*

THEOREM 1.8 ([16]). *Let $\mu \in M_c(\mathbb{R}^n)$. Then the linear span of exponential polynomials in $\tau_\mu := \{f \in C(\mathbb{R}^n) : f \star \mu = 0\}$ is dense in τ_μ.*

THEOREM 1.9 ([14]). *Let V be a finite dimensional translation invariant subspace of $C(\mathbb{R}^n)$. Then every element of V is a finite linear combination of exponential polynomials.*

The following question is raised in [15] and the answer is not known: Let V be closed translation invariant subspace of $C(\mathbb{R}^2)$.

- Does there exist an exponential in V?

In Section 4, we answer this question affirmatively when V is either finite dimensional or rotation invariant or $V = \tau_\mu := \{f \in C(\mathbb{R}^2) : f \star \mu = 0\}$ for some $\mu \in M_c(\mathbb{R}^2)$.

Let V be a closed translation invariant subspace of $C(G, X)$. Then the *problems of spectral analysis and synthesis* are the following:

- Is every exponential monomial in $C(G, X)$ mean-periodic?
- Are exponential monomials dense in $C(G, X)$?
- When does there exist an exponential monomial in V ?
- When is the linear span of exponential monomials in V dense V?
- Does there exist an exponential monomial solution for the convolution equation $f \star \mu = 0$ for a given $\mu \in M_c(G, X')$?

We analyse these problems for $G = \mathbb{R}$ in Section 2 and $G = \mathbb{T}$ in Section 3.

2. Mean-periodic functions on $G = \mathbb{R}$

For $G = \mathbb{R}$ and $X = \mathbb{C}$, it is known (see Schwartz [35]) that $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic if and only if $\tau(f)$, the closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$ is proper. We first extend this result to X, arbitrary Banach space.

THEOREM 2.1. *The following are equivalent*:

(i) *f is mean-periodic*;
(ii) $\tau(f) \neq C(\mathbb{R}, X)$.

PROOF. We use the fact that $C(\mathbb{R}, X)$ is a locally convex space and its dual is $M_c(\mathbb{R}, X')$. To show that (i) implies (ii): let $\mu \in M_c(\mathbb{R}, X')$ be non-trivial such that $f \star \mu = 0$. Then $\mu(g) = 0$ for every $g \in \tau(f)$. Hence $\tau(f) \neq C(\mathbb{R}, X)$, for otherwise $\mu(g) = 0$ for every $g \in C(\mathbb{R}, X)$, which is not possible, since μ is non-trivial. The implication (ii) implies (i) follows from the Hahn-Banach theorem for locally convex spaces and the fact that $\tau(f)$ is a proper closed translation invariant subspace of $C(\mathbb{R}, X)$. □

We show next that there exist nontrivial X-valued mean-periodic functions on $\mathbb{R}$.

PROPOSITION 2.2. $MP(\mathbb{R}, X) \neq \emptyset$.

PROOF. Let $0 \neq x \in X$ and $0 \neq x' \in X'$. Choose $g \in MP(\mathbb{R})$, scalar valued function mean-periodic with respect to some $\mu \in M_c(\mathbb{R})$. Define $\nu : \mathscr{B}_{\mathbb{R}} \to X'$ by $\nu(E) := \mu(E)x'$ and define $f : \mathbb{R} \to \mathbb{C}$ by $f(t) := g(t)x$. Then μ is a X'-valued measure and f is a continuous X-valued function with $f \star \nu = (g \star \mu)\langle x, x'\rangle = 0$. Thus f is mean-periodic with respect to ν. □

We prove next that existence of functions which are not mean-periodic is related to the X being separable.

THEOREM 2.3. *$MP(\mathbb{R}, X)$ is a proper subset of $C(\mathbb{R}, X)$ if and only if X is separable.*

PROOF. Suppose that X is a non-separable complex Banach space and $f \in C(\mathbb{R}, X)$. Since f continuous, $f(\mathbb{R})$ is separable and hence $\overline{[f(\mathbb{R})]}$ is separable. Since, for every $g \in \tau(f)$, $g(\mathbb{R}) \subseteq \overline{[f(\mathbb{R})]}$, $\tau(f) \neq C(\mathbb{R}, X)$. Hence f is mean-periodic.

Conversely, suppose that X is separable. We show that $MP(\mathbb{R}, X) \neq C(\mathbb{R}, X)$. For every $n \in \mathbb{N}$, let

$$f_n(t) := \sum_{j=1}^{\infty} a_{nj} e^{i\lambda_{nj} t}, \quad t \in \mathbb{R}, \tag{5}$$

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where λ_{nj} and a_{nj} satisfy the following conditions:

(i) $0 \neq a_{nj} \in \mathbb{C}$.

(ii) $\lambda_{nj} \in [\alpha, \beta]$ for some $\alpha < \beta$.

(iii) $\{\lambda_{nj} : j \in \mathbb{N}\} \cap \{\lambda_{mj} : j \in \mathbb{N}\} = \emptyset$ for $m \neq n$ and for every n, $\{\lambda_{nj}\}_{j=1}^{\infty}$ has a limit $\lambda_n \in \mathbb{R}$.

(iv) The convergence in (5) is uniform on compact sets with each f_n bounded by 1.

(v) $\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |a_{nj}| < \infty$.

Let $\{x_1, x_2, \dots\}$ be a dense subset of X. Define $f : \mathbb{R} \to X$ by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} f_n(t) x_n, \quad t \in \mathbb{R}. \tag{6}$$

We show that f is not mean-periodic. Since $\{e^{i\lambda_{nj} t}\}_{n,j=1}^{\infty}$ is an equicontinuous family, $\{f_n\}_{n=1}^{\infty}$ is an equicontinuous family. Therefore, for $\mu \in M_c(\mathbb{R}, X')$,

$$f \star \mu = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} (f_n x_n) \star \mu = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} f_n \star \mu_{x_n}.$$

Thus $f \star \mu = 0$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} (f_n \star \mu_{x_n})(t) = 0, \quad \forall\, t \in \mathbb{R},$$

that is, for every $t \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \frac{1}{2^n(1 + \|x_n\|)} \sum_{j=1}^{\infty} a_{nj}\, \hat{\mu}_{x_n}(\lambda_{nj}) e^{i\lambda_{nj} t} = 0. \tag{7}$$

Let $S_{pq}(t) = \sum_{n=1}^{p} \sum_{j=1}^{q} e^{i\lambda_{nj} t} a_{nj}\, \hat{\mu}_{x_n}(\lambda_{nj}) / 2^n(1 + \|x_n\|)$. Notice that S_{pq} is almost periodic and its Fourier coefficients $a(S_{pq}; \lambda)$ satisfy the following:

$$a(S_{pq}; \lambda) = \begin{cases} \dfrac{a_{nj}\, \hat{\mu}_{x_n}(\lambda_{nj})}{2^n(1 + \|x_n\|)} & \text{if } \lambda = \lambda_{nj},\ 1 \leq n \leq p,\ 1 \leq j \leq q; \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Since the convergence in (6) is uniform, the convergence in (7) also is uniform. Therefore S_{pq} converges to 0 uniformly as $p, q \to \infty$. Further, the Fourier coefficients $a(S_{pq}; \lambda)$ converges to 0 as $p, q \to \infty$ ([27]). In view of (8), $a(S_{pq}; \lambda) = 0$ for every λ. Moreover, $\hat{\mu}_{x_n}(\lambda_{nj}) = 0$ for every n and j. Since $\{\lambda_{nj}\}_{j=1}^{\infty}$ has limit point, this implies $\mu_{x_n} = 0$ for all n. Therefore, $\mu = 0$. Hence f is not mean-periodic. □

Let $f \in C(\mathbb{R}, X)$ and let $x' \in X'$. Then $x' \circ f \in C(\mathbb{R})$. It is natural to ask the following question: Is $x' \circ f$ mean-periodic for every $x' \neq 0$ if f is mean-periodic? We analyse this in the following theorem.

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THEOREM 2.4. *For* $f \in C(\mathbb{R}, X)$ *and* $x', y' \in X'$ *with* $x' \neq y'$ *the following hold*:

(i) *If* $x' \circ f$ *is mean-periodic, then* f *is mean periodic.*
(ii) *If* $x' \circ f = y' \circ f$, *then* f *is mean-periodic.*
(iii) *If* $X = \mathbb{C}^n$, $n > 1$, *then* f *is a finite sum of mean-periodic functions.*
(iv) *There exists* $f \in MP(\mathbb{R}, \mathbb{C}^n)$ *such that* $x' \circ f$ *is not mean-periodic for any* $x' \in X', x' \neq 0$.

PROOF. (i) By Theorem 2.1, it suffices to show that $\tau(f) \neq C(\mathbb{R})$. For this, let $g \in C(\mathbb{R})$, $g \neq 0$ be such that $g \notin \tau(x' \circ f)$. Choose $v \in X$ such that $\langle x', v \rangle \neq 0$ and define $h : \mathbb{R} \to X$ by $h(t) = g(t)v/\langle x', v \rangle$. Then h is continuous and $(x' \circ h)(t) = g(t)$. We show that h is not in $\tau(f)$. If possible let, $h \in \tau(f)$. Then there exists $\sum c_i f_{t_i} \to h$, which implies $x'(\sum c_i f_{t_i}) \to x' \circ h = g$, a contradiction.

(ii) Choose $g \in C(\mathbb{R}, X)$ such that $x'(g) \neq y'(g)$. We show that $g \notin \tau(f)$. If possible, let $g \in \tau(f)$. Since $\sum c_i f_{t_i} \to g \Rightarrow x'(\sum c_i f_{t_i}) \to x'(g)$ and $y'(\sum c_i f_{t_i}) \to y'(g)$, and also since $x'(f) = y'(f)$, $x'(\sum c_i f_{t_i}) = y'(\sum c_i f_{t_i})$. This implies $x'(g) = y'(g)$, a contradiction.

(iii) Let $f = (f_1, f_2, \dots, f_n)$. Obviously $(0, \dots, 0, f_i, 0, \dots, 0)$ is mean-periodic for every i with respect to $\mu = (\mu_1, \dots, \mu_n)$ where $0 \neq \mu_j \in M_c(\mathbb{R})$ are arbitrary and for $j = i$, $\mu_j = 0$. Hence f is a finite sum of mean-periodic functions.

(iv) Choose a non zero, compactly supported complex valued continuous function g. Let $f = (g, g, \dots, g)$. Then f is a $\mathbb{C}^n$-valued continuous function on $\mathbb{R}$. Clearly f is mean-periodic with respect to $\mu = (\nu_1, -\nu_1, 0, \dots, 0)$, where $0 \neq \nu_1 \in M_c(\mathbb{R})$ is arbitrary but $x' \circ f$ is not mean periodic for any $0 \neq x' \in X'$. □

REMARK. When $X = \mathbb{C}$, $MP(\mathbb{R}, X)$ is a subspace of $C(\mathbb{R}, X)$. It follows from Theorem 2.4 (iii) that sum of mean-periodic functions in $C(\mathbb{R}, X)$ need not be mean-periodic and hence $MP(\mathbb{R}, X)$ in general need not be a vector subspace of $C(\mathbb{R}, X)$. Moreover, the same argument works for separable complex Hilbert spaces.

THEOREM 2.5. $MP(\mathbb{R}, X)$ *is dense in* $C(\mathbb{R}, X)$.

PROOF. Case (i): $X = \mathbb{C}$. It suffices to show that the annihilator of $MP(\mathbb{R})$ is $\{0\}$. Let $\mu \in M_c(\mathbb{R})$ be such that $\mu(MP(\mathbb{R})) = \{0\}$. In particular $\mu(e^{i\lambda t}) = \hat{\mu}(\lambda) = 0$ for every $\lambda \in \mathbb{C}$. Hence $\mu = 0$.

Case (ii): Let X be finite dimensional, $X = \mathbb{C}^n$. Consider $C(\mathbb{R}) \times C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$. This is a finite product of locally convex spaces. Hence it is a locally convex space in the product topology. It is easy to see that $C(\mathbb{R}, X)$ is isomorphic to $C(\mathbb{R}) \times \cdots \times C(\mathbb{R})$ as locally convex spaces. Also $MP(\mathbb{R}) \times MP(\mathbb{R}) \times \cdots \times MP(\mathbb{R}) \subseteq MP(\mathbb{R}, X)$ and $MP(\mathbb{R})$ is dense in $C(\mathbb{R})$. Thus it follows that $MP(\mathbb{R}, X)$ is dense in $C(\mathbb{R}, X)$.

Case (iii): X is not finite dimensional. Consider the set $\mathrm{Exp}(\mathbb{R}, X) = \{e^{\iota\lambda t}x : \lambda \in \mathbb{C}, x \in X\}$. We show that the linear span of $\mathrm{Exp}(\mathbb{R}, X)$ is contained in $MP(\mathbb{R}, X)$ and it is dense in $C(\mathbb{R}, X)$. Let $f(t) = e^{\iota\lambda_1 t}x_1$, $g(t) = e^{\iota\lambda_2 t}x_2 \in \mathrm{Exp}(\mathbb{R}, X)$ and $\alpha, \beta \in \mathbb{C}$. Choose $0 \neq x' \in X'$ such that $x'(x_1) = x'(x_2) = 0$ and $\mu_1, \mu_2 \in M_c(\mathbb{R})$ such that $e^{\iota\lambda_1 t} \star \mu_1 = 0 = e^{\iota\lambda_2 t} \star \mu_2$. Define $\mu(E) = (\mu_1 \star \mu_2)(E)x'$, for every $E \in \mathscr{B}_{\mathbb{R}}$. Then $(\alpha f + \beta g) \star \mu = 0$. To prove the denseness, let $\mu \in M_c(\mathbb{R}, X')$ be such that μ annihilates the linear span of $\mathrm{Exp}(\mathbb{R}, X)$. Then $\hat{\mu}_x(\lambda) = 0, \forall \lambda \in \mathbb{C}, \forall x \in X$. It follows that $\mu = 0$. This completes the proof. □

We analyse next the problem of spectral analysis and spectral synthesis in $C(\mathbb{R}, X)$. Let V be a closed translation invariant subspace of $C(\mathbb{R}, X)$. For $X = \mathbb{C}$, Schwartz [35] proved that V contains exponential monomials and the linear span of exponential monomials in V is dense in V. It is well known [17] that spectral synthesis fails for $\mathbb{R}^n, n > 1$. Further, it holds for certain locally compact abelian groups, namely for $\mathbb{Z}^n$ due to Lefranc [26] and discrete groups due to Gilbert [16, 15] and Elliott [12, 13]. However, nothing is known for vector valued functions. In this section, we extend Schwartz's result for finite dimensional closed translation invariant subspace of $C(\mathbb{R}, X)$, X an arbitrary Banach space. For this we need the following lemmas.

LEMMA 2.6. *Let $v^1, v^2, \ldots, v^n \in X^n$, $v^i = (v_1^i, v_2^i, \ldots, v_n^i)$, be linearly independent. Then there exist $x_1', x_2', \ldots, x_n' \in X'$ which satisfy*

$$\begin{aligned} x_1'(v_1^1) + x_2'(v_2^1) + \cdots + x_n'(v_n^1) &= 1, \\ x_1'(v_1^2) + x_2'(v_2^2) + \cdots + x_n'(v_n^2) &= 0, \\ &\cdots\cdots \\ x_1'(v_1^n) + x_2'(v_2^n) + \cdots + x_n'(v_n^n) &= 0. \end{aligned}$$

PROOF. Let Y be the linear span of $\{v^2, v^3, \ldots, v^n\}$. Then Y being a finite dimensional subspace of X^n is closed. Since $v^1, v^2, \ldots, v^n$ are linearly independent, $v^1 \notin Y$. Thus by Hahn-Banach theorem, there exists $\Lambda \in (X^n)'$ such that $\Lambda(Y) = \{0\}$ and $\Lambda(v^1) = 1$. Clearly Λ can be written as $\Lambda = (x_1', x_2', \ldots, x_n')$, where $x_i' \in X'$ satisfy $\Lambda(x_1, x_2, \ldots, x_n) = x_1'(x_1) + x_2'(x_2) + \cdots + x_n'(x_n)$. Therefore,

$$\begin{aligned} x_1'(v_1^1) + x_2'(v_2^1) + \cdots + x_n'(v_n^1) &= \Lambda(v_1^1, 0, \ldots, 0) + \cdots + \Lambda(0, \ldots, 0, v_n^1) \\ &= \Lambda((v_1^1, v_2^1, \ldots, v_n^1)) = 1. \end{aligned}$$

For every i, $2 \leq i \leq n$,

$$\begin{aligned} x_1'(v_1^i) + x_2'(v_2^i) + \cdots + x_n'(v_n^i) &= \Lambda(v_1^i, 0, \ldots, 0) + \cdots + \Lambda(0, \ldots, 0, v_n^i) \\ &= \Lambda((v_1^i, v_2^i, \ldots, v_n^i)) = 0. \end{aligned}$$

This completes the proof of the lemma. □

For sets A and B, let $\mathscr{F}(A, B)$ denote the set of all functions from A to B. For a set $E \subseteq V$, a vector space, let $LS(E)$ denote the linear span of E.

LEMMA 2.7. *Let S be any set containing at-least n points and V be a vector space over $\mathbb{C}$. Let $\{f_1, f_2, \dots, f_n\} \subset \mathscr{F}(S, V)$. Then $\{f_1, f_2, \dots, f_n\}$ is linearly independent in $\mathscr{F}(S, V)$ if and only if there exists n distinct points $t_1, t_2, \dots, t_n \in S$ such that $\{f_1, f_2, \dots, f_n\}$ is linearly independent in $\mathscr{F}(\{t_1, t_2, \dots, t_n\}, V)$.*

PROOF. We prove the straight implication by induction. Suppose that $\{f_1, f_2, \dots, f_n\}$ is a linearly independent set in $\mathscr{F}(S, V)$. As $\{f_1\}$ is linearly independent, there exists $t_1 \in S$ such that $f_1(t_1) \neq 0$. Then $\{f_1\}$ is linearly independent on $\{t_1\}$. Thus the lemma is true when $n = 1$. If $f_1(t_1) = \alpha f_2(t_1)$, for some nonzero $\alpha \in \mathbb{C}$, choose $t_2 \in S$ such that $f_1(t_2) \neq \alpha f_2(t_2)$, which is possible, since $f_1, f_2, \dots, f_n$ are linearly independent on S. Then it is easy to check that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. If $f_1(t_1) \neq \alpha f_2(t_1)$ for any non zero scalar and $f_2(t_1) \neq 0$, then choose any $t_2 \neq t_1$. It is easy to see that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. If $f_2(t_1) = 0$, then choose t_2 such that $f_2(t_2) \neq 0$. In this case also one can easily verify that $\{f_1, f_2\}$ is linearly independent on $\{t_1, t_2\}$. Assume that $\{f_1, f_2, \dots, f_{n-1}\}$ is linearly independent on $\{t_1, t_2, \dots, t_{n-1}\}$. If $\{f_1, f_2, \dots, f_{n-1}, f_n\}$ is linearly independent on $\{t_1, t_2, \dots, t_{n-1}\}$ then choose any t_n which is different from $t_1, t_2, \dots, t_{n-1}$. If $\{f_1, f_2, \dots, f_{n-1}, f_n\}$ is linearly dependent, then there exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ such that $\alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_{n-1} f_{n-1} = f_n$ on $\{t_1, t_2, \dots, t_{n-1}\}$. Since $\{f_1, f_2, \dots, f_n\}$ is linearly independent on S, there exists $t_n \in S$ such that $\alpha_1 f_1(t_n) + \alpha_2 f_2(t_n) + \dots + \alpha_{n-1} f_{n-1}(t_n) \neq f_n(t_n)$. It follows from this that $\{f_1, f_2, \dots, f_n\}$ is linearly independent on $\{t_1, t_2, \dots, t_n\}$. This proves the required claim.

The converse is trivial. □

Using these lemmas we prove that every finite dimensional translation invariant subspace V of $C(\mathbb{R}, X)$ includes an exponential and every element in V is a finite sum of exponential monomials.

THEOREM 2.8. *Let V be an n-dimensional translation invariant subspace of $C(\mathbb{R}, X)$. Then the following hold:*

(i) *There exist $\lambda_1, \lambda_2, \dots, \lambda_q \in \mathbb{C}$ and $m_1, m_2, \dots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \dots + m_q = n$, and $w_1, w_2, \dots, w_q \in X$, not all zero, such that $e^{i\lambda_j t} w_j \in V$, for $1 \leq j \leq q$.*

(ii) *There exist $\lambda_1, \lambda_2, \dots, \lambda_q \in \mathbb{C}$, $m_1, m_2, \dots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \dots + m_q = n$ and $x_1, x_2, \dots, x_n \in X$ such that every $f \in V$ is of the form $f = \sum_{l=1}^{n} g_l x_l$, where each $g_l \in LS\{t^k e^{i\lambda_j t} : 0 \leq k \leq m_j - 1,\ 1 \leq j \leq q\}$.*

(iii) *There exist $\lambda_1, \lambda_2, \dots, \lambda_q \in \mathbb{C}$ and $m_1, m_2, \dots, m_q \in \mathbb{N}$ with $m_1 + m_2 + \dots + m_q = n$ such that every $f \in V$ is of the form $f = \sum_{j=1}^{q} \sum_{k=0}^{m_j - 1} \alpha_{kj} t^k e^{i\lambda_j t} y_{kj}$, where $\alpha_{kj} \in \mathbb{C}$ and $y_{kj} \in X$ for $0 \leq k \leq m_j - 1,\ 1 \leq j \leq q$.*

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PROOF. Fix a basis $\{f_1, f_2, \dots, f_n\}$ of V. Since V is translation invariant, $(f_i)_s \in V$ for every $s \in \mathbb{R}$. Therefore there exist unique scalars $\alpha_{ij} \in \mathbb{C}$ such that $(f_i)_s = \sum_{j=1}^n \alpha_{ij}(s) f_j$. Let f denote the $n \times 1$ matrix $f = [f_1, f_2, \dots, f_n]'$ and $A(s)$ denote the $n \times n$ matrix $(\alpha_{ij}(s))$. Then

$$f_s = A(s)[f_1, f_2, \dots, f_n]' = A(s)f. \tag{9}$$

Now

$$(f_s - f)/s = ((A(s) - A(0))/s)f. \tag{10}$$

CLAIM. $s \mapsto A(s)$ is continuous. We give two proofs of this claim.

PROOF 1. By the Lemma 2.7 there exist n distinct points $\{t_1, t_2, \dots, t_n\} \subset \mathbb{R}$ such that $\{f_1, f_2, \dots, f_n\}$ is linearly independent on $\{t_1, t_2, \dots, t_n\}$. In view of (9), $f(s+t_j) = A(s)f(t_j)$ for $j = 1, 2, \dots, n$. That is $(f_i(s+t_j))_{i,j=1}^n = A(s)(f_i(t_j))_{i,j=1}^n$. Let $v^i = (f_i(t_1), f_i(t_2), \dots, f_i(t_n))$, $1 \le i \le n$. Then $\{v^1, v^2, \dots, v^n\}$ is a linearly independent subset of X^n. By the Lemma 2.6 there exists $x'_{ij} \in X'$ such that

$$\sum_{k=1}^n \langle f_i(t_k), x'_{kj} \rangle = \delta_{ij}, \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus we have

$$(f_i(s+t_j))_{i,j=1}^n (x'_{ij})_{i,j=1}^n = A(s)(f_i(t_j))_{i,j=1}^n (x'_{ij})_{i,j=1}^n = A(s)(\delta_{ij})_{i,j=1}^n = A(s).$$

The entries of the matrix obtained by multiplying the matrices on the left side of the above equation are continuous. This shows that $s \mapsto A(s)$ is continuous from $\mathbb{R}$ to $BL(\mathbb{C}^n)$.

PROOF 2. For every $t \in \mathbb{R}$, define an operator $T_t : V \to V$ by

$$(T_t f)(s) := f(t+s), \quad \forall f \in V,\ s \in \mathbb{R}.$$

Then $T_t \in BL(V)$ and satisfies the following properties: For every $s, t \in \mathbb{R}$

(i) $T_s \circ T_t = T_{s+t}$;
(ii) $T_0 = I$;
(iii) $T_s \circ T_t = T_t \circ T_s$.

Let $\{t_1, t_2, \dots, t_n\}$ be as given by Lemma 2.7. Let $\{K_n\}_{n \ge 1}$ be compact subsets of $\mathbb{R}$ such that $\bigcup_{m=1}^\infty K_m = \mathbb{R}$ with $\{t_1, t_2, \dots, t_n\} \subseteq K_1 \subseteq K_2 \subseteq \cdots$. To show the required claim we have to show that $t \mapsto T_t$ is continuous in $BL(V)$. We shall show first that $t \mapsto T_t$ is continuous point-wise. Let $s_n \to s$ as $n \to \infty$. Now $T_{s_n}(f) = f_{s_n}$ and $T_s(f) = f_s$, for every $f \in V$. Since f is uniformly continuous on compact sets, $f_{s_n} \to f_s$ in $C(\mathbb{R}, X)$. Therefore $T_{s_n} \to T_s$ point-wise. To show that $T_{s_n} \to T_s$ in $BL(V)$, it is sufficient to show that for every m, $\| T_{s_n} - T_s \|_{K_m} \to 0$ as $n \to \infty$, where

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$\|T_{s_n} - T_s\|_{K_m} = \sup_{\|f\|_{K_m} \le 1} \|T_{s_n}(f) - T_s(f)\|_{K_m}$. Let $\epsilon > 0$. Since $\{f_1, f_2, \dots, f_n\}$ is a basis of V, for every $f \in V$, there exist unique scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ such that $f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$. Also since $\{f_1, f_2, \dots, f_n\}$ is linearly independent on K_m, $\{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n : \|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\|_{K_m} \le 1\}$ is bounded in $\mathbb{C}^n$, that is, there exists $M > 0$ such that $\|\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n\|_{K_m} \le 1$ implies that $\|(\alpha_1, \alpha_2, \dots, \alpha_n)\| \le M$. Since $\{f_1, f_2, \dots, f_n\}$ is equicontinuous, there exists a $\delta > 0$, with $\delta < 1$, such that whenever $t_1, t_2 \in s + K_m + [0, 1]$ with $|t_1 - t_2| < \delta$, $\|f_j(t_1) - f_j(t_2)\| < \epsilon/M$, for every $j = 1, 2, \dots, n$. Choose $N \in \mathbb{N}$ such that $|s_n - s| < \delta$, whenever $n \ge N$. Then for every $f \in V$ with $\|f\|_{K_m} \le 1$, for every $t \in K_m$, and $n \ge N$, we have

$$\begin{aligned}
\|f_{s_n}(t) - f_s(t)\| &= \|f(s_n + t) - f(s + t)\| \\
&= \|(\alpha_1 f_1 + \cdots + \alpha_n f_n)(s_n + t) - (\alpha_1 f_1 + \cdots + \alpha_n f_n)(s + t)\| \\
&\le |\alpha_1| \|f_1(s_n + t) - f_1(s + t)\| + \cdots + |\alpha_n| \|f_n(s_n + t) - f_n(s + t)\| \\
&\le \epsilon.
\end{aligned}$$

Thus $\|T_{s_n} - T_s\|_{K_m} \to 0$ as $n \to \infty$ for every m and hence $T_{s_n} \to T_s$ in $BL(V)$ as $n \to \infty$. This completes the second proof of the claim.

Thus $A(s)$ satisfies the following properties:

(i) $s \mapsto A(s)$ is continuous.

(ii) $A(0) = I$.

(iii) $A(s + t) = A(s)A(t) = A(t)A(s)$.

Therefore, $s \mapsto A(s)$ is differentiable (refer [18]) and

$$A(s) = e^{sA'(0)}. \tag{11}$$

By virtue of equations (10) and (11),

$$f' = A'(0)f. \tag{12}$$

This equation can be solved ([21]) and the solution is given by

$$f(t) = e^{tA'(0)}[x_1, x_2, \dots, x_n]^t.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_q \in \mathbb{C}$ be the eigen values of $A'(0)$ with multiplicities $m_1, m_2, \dots, m_q$, respectively. Let the Jordan canonical form of $A'(0)$ be given by

$$BA'(0)B^{-1} = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_q} \end{bmatrix},$$

where $J_1, \dots, J_q$ are the Jordan blocks of $A'(0)$, B is an invertible matrix. This gives

$$e^{tA'(0)} = B^{-1} \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_q} \end{bmatrix} B,$$

where each B_k is an $m_k \times m_k$ matrix given by

$$B_k = \begin{bmatrix} e^{\iota\lambda_k t} & te^{\iota\lambda_k t} & \cdots & e^{\iota\lambda_k t}t^{m_k-1}/(m_k-1)! \\ 0 & e^{\iota\lambda_k t} & \cdots & e^{\iota\lambda_k t}t^{m_k-2}/(m_k-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\iota\lambda_k t} \end{bmatrix}.$$

Thus $f(t) = C[x_1, x_2, \dots, x_n]'$, where $C = (c_{ij})$ and each $c_{ij} \in LS\{t^k e^{\iota\lambda_j t} : 0 \le k \le m_j - 1, 1 \le j \le q\}$, that is, for every i, $f_i(t) = \sum_{j=1}^n g_{ij}(t)x_j$, where $g_{ij} \in LS\{t^k e^{\iota\lambda_j t} : 0 \le k \le m_j - 1, 1 \le j \le q\}$. Hence every element h of V is of the form $h(t) = \sum_{j=1}^n g_j(t)x_j$, where each $g_j \in LS\{t^k e^{\iota\lambda_j t} : 0 \le k \le m_j - 1, 1 \le j \le q\}$. This proves (ii).

(iii) By the discussion above, each f_i can be expressed as follows:

$$f_i = \sum_{j=1}^{q} \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} \beta_{kj}^i x_{kj}^i.$$

Every $h \in V$ is of the form

$$\begin{aligned} h = \sum_{i=1}^n \alpha_i f_i &= \sum_{i=1}^n \sum_{j=1}^q \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} \alpha_i \beta_{kj}^i x_{kj}^i \\ &= \sum_{j=1}^q \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} \left(\sum_{i=1}^n \alpha_i \beta_{kj}^i x_{kj}^i \right) = \sum_{j=1}^q \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} y_{kj}. \end{aligned}$$

This proves (iii). For (i), $f_i = \sum_{j=1}^q \sum_{k=0}^{m_j-1} t^k e^{\iota\lambda_j t} y_{kj}^i$. For every j choose largest k such that $y_{kj}^i \ne 0$, let it be k_j. We will show that $e^{\iota\lambda_j t} y_{k_j j}^i \in V$. To prove this, let $\mu \in M_c(\mathbb{R}, X')$ be such that $\mu(V) = \{0\}$. Then $f \star \mu = 0$ for every $f \in V$, since V is translation invariant. Hence $f_i \star \mu = 0$, for every i. As $f_i \star \mu$ is a finite sum of complex valued exponential monomials and $\hat{\mu}_{y_{k_j j}^i}(\lambda_j)$ is the coefficient of $e^{\iota\lambda_j t}$, $\hat{\mu}_{y_{k_j j}^i}(\lambda_j) = 0$. This implies that $e^{\iota\lambda t} y_{k_j j}^i \in V$. □

COROLLARY 2.9. *Let $f \in C(\mathbb{R}, X)$. Then $\tau(f)$ is finite dimensional if and only if f is a finite linear combination of exponential monomials in $C(\mathbb{R}, X)$.*

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PROOF. Suppose that $\tau(f)$ is finite dimensional. Then it follows from the above theorem that f is a finite linear combination of exponential monomials. Conversely, suppose f is a finite linear combination of exponential monomials. Let $f = \sum_{j=1}^{q}\sum_{k=0}^{m_j-1}\alpha_{jk}t^k e^{i\lambda_j t}x_{jk}$. Then $\tau(f) \subseteq LS\{t^{k-l}e^{i\lambda_j t}x_{jk} : 0 \le l \le k, 0 \le k \le m_j - 1, 1 \le j \le q\}$. Therefore $\tau(f)$ is finite dimensional. □

REMARK. (i) Some authors (see [14, 25]) define exponential polynomials to be functions of the form $\sum_{j=1}^{m} f_j$, where f_j are exponential polynomials defined as in Definition 1.3. With this definition, our result states that every finite dimensional translation invariant subspace V of $C(\mathbb{R}, X)$ is generated by exponential polynomials in V.

(ii) Anselone and Korevaar [1] have proved that when $X = \mathbb{C}$, $V \subset C(\mathbb{R})$ is finite dimensional if and only if V is the solution space of a homogeneous constant coefficient ordinary differential equation. This result is not true for arbitrary X which can be seen by the following examples.

EXAMPLE 3. Let X be a separable infinite dimensional complex Hilbert space. Let $\{e_n\}$ be a complete orthonormal basis. Consider the homogeneous ordinary differential equation with constant coefficient.

$$a_0 f + a_1 f' + \cdots + a_n f^{(n)} = 0. \tag{13}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_q$ with multiplicities $m_1, m_2, \ldots, m_q$ be the roots of the characteristic polynomial $p(t)$. Then for every $n \in \mathbb{N}$, $0 \le k \le m_j$, $1 \le j \le q$, $t^k e^{i\lambda_j t}e_n$ is a solution of the differential equation (13). Thus the solution space is not finite dimensional.

EXAMPLE 4. Let X be a complex Banach space. Fix $A \in BL(X)$. Consider the following differential equation $du/dt = Au$. Then the solution space $\{u \in C(\mathbb{R}, X) : du/dt = Au\} = \{e^{tA}x : x \in X\}$ is a closed translation invariant subspace of $C(\mathbb{R}, X)$. Further, it is finite dimensional if and only if X is finite dimensional.

Let $\mu \in M_c(\mathbb{R}, X')$. In the case when $X = \mathbb{C}$ it is known [35] that for a given μ the linear span of exponential monomial solutions of the convolution equation $f \star \mu = 0$ is dense in the space of all solutions. We extend this for $X = \mathbb{C}^n$ as follows:

THEOREM 2.10. *Let $f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}, \mathbb{C}^n)$ satisfies the following*:

(i) *f_j is mean-periodic, for every $1 \le j \le n$*;

(ii) *$\sigma(f_j) \cap \sigma(f_k) = \emptyset$ for $j \ne k$.*

Then $\tau(f)$ contains exponential monomials and the linear span of exponential monomials in $\tau(f)$ is dense in $\tau(f)$.

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PROOF. Clearly $\tau(f) \subseteq \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. We show that these two sets are equal. Let $g \in \tau(f_1) \cap \tau(f_2)$. Then $\tau(g) \subseteq \tau(f_1) \cap \tau(f_2)$ and hence by Schwartz's theorem, $\tau(f_1) \cap \tau(f_2) = (0)$. Thus $\tau(f_i) \cap \tau(f_j) = (0)$ for $i \neq j$. Let $\mu \in M_c(\mathbb{R}, X')$ be such that $\mu(\tau(f)) = \{0\}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. Since $\tau(f)$ is translation invariant, $f \star \mu = \sum_{j=1}^n f_j \star \mu_j = 0$. Let $e^{\iota\lambda t}, te^{\iota\lambda t}, \ldots, t^{m_1-1}e^{\iota\lambda t} \in \tau(f_1)$ and $t^{m_1}e^{\iota\lambda t} \notin \tau(f_1)$. By Hahn-Banach theorem there exists a measure $\nu_1 \in M_c(\mathbb{R})$ such that $\nu_1(\tau(f_l)) = \{0\}$ for every $l \neq 1$ and $\nu_1(e^{\iota\lambda t}) \neq 0$. Therefore $f_l \star \nu_1 = 0$, for $l \neq 1$. Now $f_1 \star \mu_1 \star \nu_1 = (f \star \mu) \star \mu_1 = 0$. Therefore $(\hat{\mu}_1\hat{\nu}_1)(\lambda) = (\hat{\mu}_1\hat{\nu}_1)'(\lambda) = \cdots = (\hat{\mu}_1\hat{\nu}_1)^{(m_1-1)}(\lambda) = 0$. As $\hat{\mu}_1(\lambda)\hat{\nu}_1(\lambda) = 0$ and $\hat{\nu}_1(\lambda) \neq 0$, $\hat{\mu}_1(\lambda) = 0$. Also $(\hat{\mu}_1\hat{\nu}_1)'(\lambda) = 0$ implies $\hat{\mu}_1'(\lambda)(\hat{\nu}_1)(\lambda) + \hat{\mu}_1(\lambda)\hat{\nu}_1'(\lambda) = 0$. This implies $\hat{\mu}_1'(\lambda) = 0$. Similarly we can show that $\hat{\mu}_1''(\lambda) = \cdots = \hat{\mu}_1^{(m_1-1)}(\lambda) = 0$. Thus λ is a zero of $\hat{\mu}_1$ with multiplicity at-least m_1. This shows that $f_1 \star \mu_1 = 0$. Similarly, $f_j \star \mu_j = 0$ for every j. Thus $\mu(\tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)) = 0$. It follows that $\tau(f) = \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. This completes the proof. □

COROLLARY 2.11. *Let $X = \mathbb{C}^n$. Let $f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}, X)$ and $\mu \in M_c(\mathbb{R}, X')$. Suppose that each f_j is mean-periodic and $\sigma(f_j) \cap \sigma(f_k) = \emptyset$ for $j \neq k$. If $f \star \mu = 0$, then f is a finite linear combination of exponential monomials solutions.*

PROOF. Since spectral synthesis holds for $\mathbb{R}$, $LS(\sigma(f_j))$ is dense in $\tau(f_j)$, for every j. It is easy to see that $\sigma(f_1) \times \sigma(f_2) \times \cdots \times \sigma(f_n) \subset LS(E)$, where $E = \{t^k e^{\iota\lambda t}x : x \neq 0, t^k e^{\iota\lambda t}x \star \mu = 0\}$. Thus $\overline{LS(E)} = \tau(f_1) \times \tau(f_2) \times \cdots \times \tau(f_n)$. The required result follows from the Theorem 2.10. □

EXAMPLE 5. (1) When $G = \mathbb{R}$ and $X = \mathbb{C}$, the notion of mean-periodic functions was introduced by Delsarte in 1935 [5]. In [35] Schwartz gave an intrinsic characterization of mean-periodic functions: $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic if and only if $\tau(f)$, the closed translation invariant subspace of $C(\mathbb{R}, \mathbb{C})$ is proper. Clearly, for every $\lambda \in \mathbb{C}$, $f_\lambda(t) = e^{\iota\lambda t}$, $t \in \mathbb{R}$, is mean-periodic, $f \star \mu = 0$ for $\mu = \delta_0 - e^{\iota\lambda}\delta_1$, where δ_x denote the Dirac measure on $\mathbb{R}$ at $x \in \mathbb{R}$. Schwartz [35] showed that if $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic with $f \star \mu = 0$, then f is a limit of finite linear combinations of functions of the type $f_\lambda(t) = t^k e^{\iota\lambda t}$, such that $f_\lambda \star \mu = 0$. In Laird [22] it is shown that if $f \in C(\mathbb{R}, \mathbb{C})$ is mean-periodic and g is an exponential polynomial, that is, $g(t) = p(t)e^{\iota\lambda t}$, where $p(t)$ is a polynomial, then fg is mean-periodic.

(2) Let G be a compact abelian group. Then every character of G is mean-periodic, as observed in Rana [33].

(3) For $X = \mathbb{C}$, mean-periodic functions on various locally compact groups have been analysed by various authors (see [2, 3, 5, 7, 10, 11, 17, 19, 20, 23, 24, 22, 29, 30, 36, 38, 37, 39]).

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In general setting, even when $G = \mathbb{R}$ and X is an arbitrary Banach space, nothing seem to be known.

NOTE. The following questions still remain unanswered:

(1) Let V be a closed translation invariant subspace of $C(\mathbb{R}, X)$. Does V always include a monomial exponential? Is V the closed linear span of the monomial exponentials in it?

(2) The problem of finding solutions for $f \star \mu = g$, for a given μ and g, seems to be much more difficult even for the case $G = \mathbb{R}$ and $X = \mathbb{C}$: Some particular situations are analysed in [31] and [32]. Another particular case is given in the next theorem.

THEOREM 2.12. *For a given $\mu \in M_c(\mathbb{R})$ and g a finite sum of exponential polynomials in $C(\mathbb{R})$, there exists $f \in C(\mathbb{R})$ such that $f \star \mu = g$.*

PROOF. First suppose that g is an exponential polynomial. Let $g(t) = e^{\iota\lambda t}\sum_{k=0}^{n} a_k t^k$. Let $Z(\hat{\mu}) = \{\lambda \in \mathbb{C} : \hat{\mu}(\lambda) = 0\}$. We say

(i) $\lambda \in Z(\hat{\mu})$ is of multiplicity 0 if $\hat{\mu}(\lambda) \neq 0$.

(ii) $\lambda \in Z(\hat{\mu})$ of multiplicity $m \in \mathbb{N}$, if $\hat{\mu}(\lambda) = 0, \hat{\mu}'(\lambda) = 0, \dots, \hat{\mu}^{(m-1)}(\lambda) = 0$ and $\hat{\mu}^{(m)}(\lambda) \neq 0$.

Let m be the multiplicity of $\lambda \in Z(\hat{\mu})$. Define $f(t) := \sum_{k=0}^{n} b_k t^{m+k} e^{\iota\lambda t}$, where

$$b_n = \frac{(\iota)^m}{\binom{n+m}{m}\hat{\mu}^{(m)}(\lambda)} a_n, \quad b_{n-1} = \left[a_{n-1} - b_n \frac{\binom{m+n}{m+1}\hat{\mu}^{(m+1)}(\lambda)}{(\iota)^{m+1}}\right] \frac{(\iota)^m}{\binom{m+n-1}{m}\hat{\mu}^{(m)}(\lambda)}, \dots,$$

$$b_0 = \left[a_0 - b_1 \frac{\binom{m+1}{m+1}\hat{\mu}^{(m+1)}(\lambda)}{(\iota)^{m+1}} - b_2 \frac{\binom{m+2}{m+2}\hat{\mu}^{(m+2)}(\lambda)}{(\iota)^{m+2}} - \dots \right.$$
$$\left. - b_n \frac{\binom{m+n}{m+n}\hat{\mu}^{(m+n)}(\lambda)}{(\iota)^{m+n}}\right] \frac{(\iota)^m}{\binom{m}{m}\hat{\mu}^{(m)}(\lambda)}.$$

A simple computation of $f \star \mu$ gives $f \star \mu = g$. In the general case, suppose that $g = \sum_{j=1}^{p} g_j$, where $g_j(t) = p_j(t)e^{\iota\lambda_j t}$, for every j and $\lambda_k \neq \lambda_j$ for $k \neq j$. Let f_j be the exponential polynomial function corresponding to g_j obtained as in the first case, that is, $f_j \star \mu = g_j$. Then $f = \sum_{j=1}^{p} f_j$ is a solution of the given convolution equation. □

3. Mean-periodic functions on $G = \mathbb{T}$

We shall consider integrals of X-valued functions with respect to scalar measures in the sense of Bochner integral, and the integrals of scalar valued continuous functions with respect to X'-valued measures in the sense similar to that of Bochner discussed in the last section.

DEFINITION 3.1. Let $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$. For every $n \in \mathbb{Z}$,

$$\hat{f}(n) := \int_{\mathbb{T}} z^{-n} f(z)\, dz \quad \text{and} \quad \hat{\mu}(n) := \int_{\mathbb{T}} z^{-n} d\mu(z)$$

are called the *nth-Fourier coefficient* of f and μ, respectively.

For $f \in C(\mathbb{T}, X)$, let $\tau(f)$ denote the closed translation invariant subspace generated by f.

PROPOSITION 3.2. *$f \in C(\mathbb{T}, X)$ is mean-periodic if and only if $\tau(f) \neq C(\mathbb{T}, X)$.*

PROOF. Follows from the fact that the dual of $C(\mathbb{T}, X)$ is $M(\mathbb{T}, X')$. □

LEMMA 3.3. *For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, the following hold:*
(i) *$f \star \mu$ is a uniformly continuous function on $\mathbb{T}$;*
(ii) $(f \star \mu)\hat{} = \langle \hat{f}(n), \hat{\mu}(n) \rangle$.

PROOF. (i) Follows from the facts that f is uniformly continuous, μ has finite variation and that $|(f \star \mu)(z) - (f \star \mu)(w)| \leq \int_{\mathbb{T}} \|f(z\bar{s}) - f(w\bar{s})\|\, dV_\mu(s)$.

(ii) Since $\mathbb{T}$ is compact, f is uniformly continuous on $\mathbb{T}$. Let $\epsilon_k > 0$ be such that $\epsilon_k \to 0$ as $k \to \infty$. Since the metric on $\mathbb{T}$ is invariant under rotation, there exist finite Borel partitions P_k of $\mathbb{T} = \sqcup B_{ki}$ such that if $z_{ki}, w_{ki} \in B_{ki}$, then $\|f(z_{ki}\overline{w}) - f(w_{ki}\overline{w})\| < \epsilon_k$ whenever $|w| = 1$. Now

$$(14) \qquad (f \star \mu)\hat{}(n) = \int_{\mathbb{T}} (f \star \mu)(z) z^{-n}\, dz = \int_{\mathbb{T}} \int_{\mathbb{T}} f(z\overline{w})\, d\mu(w) z^{-n}\, dz$$
$$= \int_{\mathbb{T}} \lim_{k\to\infty} \left(\sum_j \langle f(z\overline{w_{kj}}),\ \mu(B_{kj}) \rangle \right) z^{-n}\, dz.$$

Since f is continuous on $\mathbb{T}$, $f(\mathbb{T}) \subset B(0, r) = rB(0, 1)$ for some $r > 0$. We have

$$\left| \sum_j \langle f(z\overline{w_{kj}}),\ \mu(B_{kj}) \rangle \right| \leq \sum_j |\langle f(z\overline{w_{kj}}),\ \mu(B_{kj}) \rangle|$$
$$\leq \sum_j r V_\mu(B_{kj}) \leq r V_\mu(\mathbb{T}) \leq rC.$$

Applying dominated convergence theorem in (14) for the functions

$$z \mapsto \sum_j \langle f(z\overline{w_{kj}}), \mu(B_{kj}) \rangle z^{-n}$$

we obtain

$$
\begin{aligned}
(f \star \mu)\hat{}(n) &= \lim_{k\to\infty} \int_{\mathbb{T}} \sum_j \langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle z^{-n}\, dz \\
&= \lim_{k\to\infty} \sum_j \int_{\mathbb{T}} \langle f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle z^{-n}\, dz \\
&= \lim_{k\to\infty} \sum_j \int_{\mathbb{T}} \langle z^{-n} f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle\, dz.
\end{aligned}
$$

Now apply change of variable formula for the function $z \mapsto \langle z^{-n} f(z\overline{w_{kj}}),\ \mu(B_{kj})\rangle$, to get

$$
\begin{aligned}
(f \star \mu)\hat{}(n) &= \lim_{k\to\infty} \sum_j \int_{\mathbb{T}} \left\langle \left(\frac{z}{\overline{w_{kj}}}\right)^{-n} f(z),\ \mu(B_{kj})\right\rangle dz \\
&= \lim_{k\to\infty} \sum_j \left\langle \int_{\mathbb{T}} \left(\frac{z}{\overline{w_{kj}}}\right)^{-n} f(z)dz,\ \mu(B_{kj})\right\rangle \\
&= \lim_{k\to\infty} \sum_j (\overline{w_{kj}})^{-n} \langle \hat{f}(n),\ \mu(B_{kj})\rangle \\
&= \left\langle \hat{f}(n),\ \lim_{k\to\infty} \sum_j (\overline{w_{kj}})^{-n} \mu(B_{kj})\right\rangle = \langle \hat{f}(n),\ \hat{\mu}(n)\rangle. \qquad \square
\end{aligned}
$$

COROLLARY 3.4. *For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, $f \star \mu = 0$ if and only if $\langle \hat{f}(n),\ \hat{\mu}(n)\rangle = 0$ for all $n \in \mathbb{Z}$.*

PROOF. Follows from Lemma 3.3 and the uniqueness of Fourier-Stieltjes coefficients of scalar valued functions on $\mathbb{T}$. □

PROPOSITION 3.5. *Let $f \in C(\mathbb{T}, X)$. Then $\sigma(f) = \{\alpha z^n \hat{f}(n) : \hat{f}(n) \neq 0$ and $0 \neq \alpha \in \mathbb{C}\}$.*

PROOF. First we show that $\{\alpha z^n \hat{f}(n) : \hat{f}(n) \neq 0\} \subseteq \sigma(f)$. Let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(\tau(f)) = 0$. Then $f \star \mu = 0$, since $\tau(f)$ is translation invariant. Hence by Corollary 3.4, $\langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$ for every n. Thus $\mu(\alpha z^n \hat{f}(n)) = \alpha \langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$, and by Corollary 3.4, $\alpha z^n \hat{f}(n) \in \tau(f)$. Hence $\alpha z^n \hat{f}(n) \in \sigma(f)$.

On the other hand, let $z^m x \in \sigma(f)$. To show that $x = \alpha \hat{f}(m)$ for some scalar α. Let $x' \in X'$ be such that $x'(\hat{f}(m)) = 0$. Let $d\nu(z) = z^m x' dz$. Then

$$
\hat{\mu}(n) = \begin{cases} x' & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}
$$

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Thus by Corollary 3.4, $f \star \nu = 0$. Therefore $z^m x \star \nu = 0$ and hence $\langle x, \hat{\nu}(m)\rangle = 0$, that is, $\langle x, x'\rangle = 0$. Thus for $x' \in X'$, $\langle \hat{f}(m), x'\rangle = 0$ implies $\langle x, x'\rangle = 0$. Therefore $x = \alpha \hat{f}(m)$ for some $\alpha \in \mathbb{C}$. This completes the proof. □

PROPOSITION 3.6. *Let $f \in C(\mathbb{T}, X)$. Then $\sigma(f) = \emptyset$ if and only if $f = 0$.*

PROOF. By Proposition 3.5, it suffices to show that $\hat{f}(n) = 0$, for every $n \in \mathbb{Z}$ if and only if $f = 0$. Using the uniqueness of Fourier coefficients for scalar valued functions we obtain, for every $n \in \mathbb{Z}$ and $x' \in X'$,

$$\hat{f}(n) = 0 \Leftrightarrow \langle x', \hat{f}(n)\rangle = 0 \Leftrightarrow \left\langle x', \int_{\mathbb{T}} f(z) z^{-n}\, dz \right\rangle = 0 \Leftrightarrow \int_{\mathbb{T}} \langle x', f(z)\rangle z^{-n}\, dz = 0$$
$$\Leftrightarrow (x' \circ f)\hat{}(n) = 0 \Leftrightarrow x' \circ f = 0 \Leftrightarrow f = 0.$$ □

THEOREM 3.7. *For a complex Banach space $X \neq \mathbb{C}$ the following hold:*

(i) $MP(\mathbb{T}, X) = C(\mathbb{T}, X)$.

(ii) *For every* $0 \neq \mu \in M(\mathbb{T}, X')$, $\{0\} \neq MP(\mu) \neq C(\mathbb{T}, X)$.

PROOF. (i) Let $f : \mathbb{T} \to X$ be a non zero continuous function. Then $\hat{f}(n_0) \neq 0$ for some n_0. Chose $x' \in X'$ such that $x' \neq 0$ and $\langle x', \hat{f}(n_0)\rangle = 0$. Define $\mu(E) := \left(\int_E z^{n_0} dz\right) x'$, for every $E \in \mathscr{B}_{\mathbb{T}}$. Then $\mu \in M(\mathbb{T}, X')$ and

$$\hat{\mu}(n) = \begin{cases} x' & \text{if } n = n_0; \\ 0 & \text{if } n \neq n_0. \end{cases}$$

Thus $(f \star \mu)\hat{}(n) = \langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$, for every $n \in \mathbb{Z}$. Hence it follows from Corollary 3.4, $f \star \mu = 0$.

(ii) Let $0 \neq \mu \in M(\mathbb{T}, X')$. Then $\hat{\mu}(n_0) \neq 0$ for some n_0. Let $0 \neq x \in X$ be such that $\langle \hat{\mu}(n_0), x\rangle = 0$, and $y \in X$ be such that $\langle \hat{\mu}(n_0), y\rangle \neq 0$. Define $f, g : \mathbb{T} \to X$, by $f(z) = z^{n_0} x$ and $g(z) = z^{n_0} y$. Then

$$\hat{f}(n) = \begin{cases} x & \text{if } n = n_0; \\ 0 & \text{if } n \neq n_0 \end{cases} \quad \text{and} \quad \hat{g}(n) = \begin{cases} y & \text{if } n = n_0; \\ 0 & \text{if } n \neq n_0. \end{cases}$$

Therefore, $\langle \hat{f}(n), \hat{\mu}(n)\rangle = 0$ for all $n \in \mathbb{Z}$ and $(g \star \mu)\hat{}(n_0) = \langle \hat{g}(n_0), \hat{\mu}(n_0)\rangle \neq 0$. Thus f is mean-periodic with respect to μ and g is not mean-periodic with respect μ. □

REMARK. (1) Theorem 3.7 (i) is not true when $X = \mathbb{C}$. For instance, the function $f : \mathbb{T} \to \mathbb{C}$ defined by $f(z) := \sum_{-\infty}^{\infty} a_n z^n$, $z \in \mathbb{T}$, where $a_n \in \mathbb{C}$, $a_n \neq 0$ for every n and $\sum_{-\infty}^{\infty} |a_n| < \infty$ is not mean-periodic.

(2) Let G be a locally compact abelian group and X a complex Banach space. A function $f \in C(G, X)$ is said to be *almost periodic* if the set of all translates of f is relatively compact in $C(G, X)$. Every $f \in C(\mathbb{T}, X)$ is almost periodic and if $X \neq \mathbb{C}$, then every $f \in C(\mathbb{T}, X)$ is mean-periodic. When $X = \mathbb{C}$, there are complex valued continuous functions on the circle group $\mathbb{T}$ which are not mean-periodic.

We have the following result for spectral analysis and spectral synthesis for $\mathbb{T}$.

THEOREM 3.8. *The following hold*:

(i) *Let $x \in X$, $x \neq 0$, and $n_0 \in \mathbb{Z}$. Then $\tau(z^{n_0}x)$, the closed translation invariant subspace generated by $z^{n_0}x$, does not contain any non-zero proper closed translation invariant subspace of $C(\mathbb{T}, X)$.*

(ii) *Every non-zero closed translation invariant subspace V of $C(\mathbb{T}, X)$ contains an exponential, that is, spectral analysis holds in $C(\mathbb{T}, X)$.*

(iii) *The linear span of the exponentials in every closed translation invariant subspace V of $C(\mathbb{T}, X)$ is dense in V, that is, spectral synthesis holds in $C(\mathbb{T}, X)$.*

PROOF. (i) Let V_1 be a non-zero closed translation invariant subspace of $C(\mathbb{T}, X)$ such that $V_1 \subseteq \tau(z^{n_0}x)$. Then for $f \in \tau(z^{n_0}x)$, $\hat{f}(n_0) = cx$ for some $0 \neq c \in \mathbb{C}$ and $\hat{f}(n) = 0$ if $n \neq n_0$. To show $V_1 = \tau(z^{n_0}x)$, let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(V_1) = \{0\}$. Then $\langle\hat{\mu}(n), x\rangle$ for every n. In particular $\langle\hat{\mu}(n_0), x\rangle$ and hence $\mu(V) = \{0\}$. Hence $V_1 = \tau(z^{n_0}x)$.

(ii) Choose $n_0 \in \mathbb{Z}$ and $f \in V$ such that $\hat{f}(n_0) \neq 0$. We will show that $z^{n_0}\hat{f}(n_0) \in V$. For, let $\mu \in M(\mathbb{T}, X')$ be such that $\mu(V) = \{0\}$. Since V is translation invariant and $\mu(V) = \{0\}$, $f \star \mu = 0$. This implies $\langle\hat{f}(n_0), \hat{\mu}(n_0)\rangle = 0$. Thus $z^{n_0}\hat{f}(n_0) \star \mu = 0$. Hence $z^{n_0}\hat{f}(n_0) \in V$.

(iii) Let V be closed translation invariant subspace of $C(\mathbb{T}, X)$. Let V_0 be the closed linear span of $z^n\hat{f}(n)$, $f \in V$. Then by (ii), $V_0 \subseteq V$. Let $f \in V$. Let $\mu \in M(\mathbb{T}, X')$ such that $\mu(V_0) = 0$. Then $\langle\hat{f}(n), \hat{\mu}(n)\rangle = 0$, for every $n \in \mathbb{Z}$. Thus $f \star \mu = 0$. Therefore, $\mu(f) = 0$. □

COROLLARY 3.9. *For $f \in C(\mathbb{T}, X)$ and $\mu \in M(\mathbb{T}, X')$, the following are equivalent*:

(i) $f \star \mu = 0$.

(ii) *f is a limit of finite linear combinations of functions $z^n x$ which satisfy the equation $z^n x \star \mu = 0$.*

PROOF. First observe that for a given μ, $MP(\mu) = \{f \in C(\mathbb{T}, X) : f \star \mu = 0\}$ is a closed translation invariant subspace of $C(\mathbb{T}, X)$. The result follows from Theorem 3.8 (iii). □

4. Some results for general groups

As mentioned earlier, problem of analysing mean-periodic functions, the problem of spectral analysis and spectral synthesis seems difficult to answer for general groups. However, it is not difficult to show that if G is compact and $X = \mathbb{C}$ then every nontrivial closed translation invariant subspace V of $C(K, \mathbb{C})$ includes exponentials and the linear span of exponentials in V is dense in it. Hence every mean-periodic (scalar valued) function on a compact group is a limit of finite linear combination of exponentials.

For G arbitrary locally compact abelian, and $X = \mathbb{C}$ we have the following: recall, $\Omega = \{\omega : G \to \mathbb{C}^* : \omega \in C(G) \text{ and } \omega(g_1 + g_2) = \omega(g_1)\omega(g_2)\}$.

THEOREM 4.1. (i) *Every $\omega \in \Omega$ is mean-periodic.*

(ii) *Let G be an infinite locally compact T_1 abelian group. Then every exponential polynomial on G is mean-periodic.*

(iii) *Let $MP(G)$ be the set of all mean-periodic functions on G. Then $MP(G)$ is dense in $C(G)$ if and only if G is not finite.*

PROOF. (i) Clearly, every translate ω_g of ω is a constant multiple of ω, and hence every finite linear combination of translates of ω is a constant multiple of ω. Therefore the closed translation invariant subspace $\tau(\omega)$ is a one dimensional subspace of $C(G)$. Thus $\tau(\omega) \neq C(G)$, if G is non-trivial.

(ii) Let f be an exponential polynomial on G,

$$f(g) := \left(\sum_{\alpha} c_\alpha a_1(g)^{\alpha_1} a_2(g)^{\alpha_2} \cdots a_m(g)^{\alpha_m}\right) \omega(g),$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, c_α are complex constants and $a_1, \dots, a_m$ are additive functions. Let $V = LS\{a_1(g)^{\beta_1} a_2(g)^{\beta_2} \cdots a_m(g)^{\beta_m}\omega(g) : \beta_j \in \mathbb{Z}_+, \beta_j \leq \alpha_j \text{ for } 1 \leq j \leq m\}$. It is easy to see that $f \in V$ and V is a finite dimensional translation invariant subspace of $C(G)$. Since V is finite dimensional, it is closed and it follows that $\tau(f) \subseteq V$. But $C(G)$ is infinite dimensional as G is not finite. Hence $\tau(f) \neq C(G)$.

(iii) Suppose that G is finite, $G = \{g_1, g_2, \dots, g_n\}$. Let $f \in C(G)$ and $\mu \in M_c(G)$. Let $\mu(g_i) = c_i$. Then $f \star \mu = 0$ for a non-trivial μ if and only if

$$\begin{vmatrix} f(g_1 - g_1) & f(g_1 - g_2) & \cdots & f(g_1 - g_n) \\ f(g_2 - g_1) & f(g_2 - g_2) & \cdots & f(g_2 - g_n) \\ \vdots & \vdots & & \vdots \\ f(g_n - g_1) & f(g_n - g_2) & \cdots & f(g_n - g_n) \end{vmatrix} = 0.$$

The columns of the above matrix are permutations of $[f(g_1), f(g_2), \dots, f(g_n)]$. Thus f is mean-periodic if and only if $(f(g_1), f(g_2), \dots, f(g_n))$ is a root of some fixed polynomial P in the variables $z_1, z_2, \ldots, z_n$. The roots of this polynomial P form a closed set $Z(P)$ in $\mathbb{C}^n$ of $2n$-dimensional Lebesgue measure zero. Therefore $Z(P)$ is not dense in $\mathbb{C}^n$. But $MP(G) = Z(P)$. Hence $MP(G)$ is not dense in $C(G)$.

Conversely, suppose that G is not finite. Let $EP(G)$ be the set of all exponential polynomials in $C(G)$. By (ii), $EP(G) \subseteq MP(G)$, that is, $\Gamma \subseteq \Omega \subseteq EP(G) \subseteq MP(G)$. Moreover Ω separates points of G. Since the pointwise product of finite number of exponentials is again an exponential, it is easy to see that product of two exponential polynomials f and g is a finite sum of exponential polynomials and hence $\tau(fg)$ is finite dimensional. Therefore the algebra $A(EP(G))$, generated by $EP(G)$, is contained in $MP(G)$, that is, $A(EP(G)) \subseteq MP(G)$. Hence by Stone Weierstrass theorem ([9]) $A(EP(G))$ is dense in $C(G)$. Since $A(EP(G)) \subseteq MP(G)$, $MP(G)$ is dense in $C(G)$. □

COROLLARY 4.2. *If G is a finite T_1 topological abelian group, then* $\{0\} \neq MP(G) \neq C(G)$.

LEMMA 4.3. *Let G be a locally compact abelian group having no nontrivial compact subgroups. Let $\hat{G}$ be the dual group of G. Then for $\mu \in M_c(G)$, $\lambda_\Gamma(\{\gamma \in \Gamma : \hat{\mu}(\gamma) = 0\}) = 0$.*

PROOF. Refer [6]. □

THEOREM 4.4. *If G does not have compact elements, then* $\{0\} \neq MP(G) \neq C(G)$.

PROOF. Let $f \in C(G)$ be compactly supported. By Lemma 4.3, f is not mean-periodic. Thus $MP(G) \neq C(G)$. □

As we have pointed earlier, the problem of spectral synthesis does not hold for every closed translation invariant subspace V of $C(\mathbb{R}^2, \mathbb{C})$. However, with some conditions on V this is true. First we prove the following lemma.

LEMMA 4.5. *The following hold*:

(i) *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct complex numbers and $m_1, m_2, \ldots, m_n \in \mathbb{N}$. Then the set $\{e^{i\lambda_j t}, te^{i\lambda_j t}, \ldots, t^{m_j} e^{i\lambda_j t} : 1 \leq j \leq n\} \subseteq C(\mathbb{R})$ is linearly independent over $\mathbb{C}$.*

(ii) *Let $\lambda_1, \lambda_2, \ldots, \lambda_n; \eta_1, \eta_2, \ldots, \eta_n$ be complex numbers and for $1 \leq j, k, l \leq n$, α_{lj}, β_{kr} be non-negative integers. Then $\{t_1^{\alpha_{lj}} t_2^{\beta_{lj}} e^{i(\lambda_l t_1 + \eta_l t_2)} : 1 \leq l, j \leq n\}$ is a linearly independent subset of $C(\mathbb{R}^2)$ over $\mathbb{C}$ if $(\lambda_j, \eta_j) \neq (\lambda_k, \eta_k)$ or $(\alpha_{lj}, \beta_{lj}) \neq (\alpha_{lk}, \beta_{lk})$.*

PROOF. (i) Without loss of generality, we may assume that

$$\operatorname{Im}(\lambda_n) = \max_{1 \leq j \leq n} \operatorname{Im}(\lambda_j),$$

where Im denotes the imaginary part of a complex number. Then $\mathrm{Im}(\lambda_n)-\mathrm{Im}(\lambda_j)>0$ for $1\le j\le n-1$. Now for $a_{ij}\in\mathbb{C}$,

$$\sum_{j=1}^{n}\left(a_{0j}e^{\iota\lambda_j t}+a_{1j}te^{\iota\lambda_j t}+\cdots+a_{m_j j}t^{m_j}e^{\iota\lambda_j t}\right)=0 \quad\Longrightarrow$$

$$\sum_{j=1}^{n-1}\left(a_{0j}e^{\iota(\lambda_j-\lambda_n)t}+a_{1j}te^{\iota(\lambda_j-\lambda_n)t}+\cdots+a_{m_j j}t^{m_j}e^{\iota(\lambda_j-\lambda_n)t}\right)+p_n(t)=0,$$

where $p_n(t)=a_{n0}+a_{n1}t+\cdots+a_{nm_n}t^{m_n}=a_{nm_n}(t-\beta_1)(t-\beta_2)\cdots(t-\beta_{n_m})$, for some $\beta_1,\beta_2,\ldots,\beta_{n_m}\in\mathbb{C}$. Now as $t\to-\infty$, $t^k e^{\iota(\lambda_j-\lambda_n)}\to 0$ for every $j\neq n$. This implies $a_{nm_n}=0$, since as $t\to-\infty$, $p_n(t)\not\to 0$ if $a_{nm_n}\neq 0$. Similarly by repeating the same argument one can easily show that $a_{ij}=0$ for all i,j.

(ii) Case (i): $\lambda_1,\lambda_2,\ldots,\lambda_n$ are distinct. For $a_{lj}\in\mathbb{C}$,

$$\sum_{l=1}^{n}\sum_{j=1}^{n}\left(a_{lj}t_1^{\alpha_{lj}}t_2^{\beta_{lj}}e^{\iota(\lambda_l t_1+\eta_l t_2)}\right)=0 \quad\Longrightarrow\quad \sum_{l=1}^{n}\sum_{j=1}^{n}\left((a_{lj}t_2^{\beta_{lj}}e^{\iota\eta_l t_2})t_1^{\alpha_{lj}}e^{\iota\lambda_l t_1}\right)=0.$$

By (i), this implies $a_{ij}=0$ for all i,j.

Case (ii): $\lambda_i=\lambda_j$ for some i and j. In this case rearrange the terms of the above expression by collecting the distinct exponential monomials in t_1. By the hypothesis, the coefficients of the exponential monomial in t_1 are finite linear combination of exponential monomials in t_2. By applying (i) twice, namely, first t_1 variable and then t_2 variable we get $a_{ij}=0$ for all i,j. □

THEOREM 4.6. *Let V be a closed translation invariant subspace of $C(\mathbb{R}^2)$ satisfying any one of the following conditions*:

(i) *V is finite dimensional.*
(ii) *V is rotation invariant.*
(iii) *$V=\tau_\mu:=\{f\in C(\mathbb{R}^2): f\star\mu=0\}$ for some $\mu\in M_c(\mathbb{R}^2)$.*

Then V contains an exponential.

PROOF. Case (i): V is finite dimensional. Let $f\in V$ and $f\neq 0$. By Theorem 1.9, f is of the form $f=\sum_{j=1}^{m}p_j(t_1,t_2)e^{\iota(\lambda_j t_1+\eta_j t_2)}$, where p_j is a non-zero polynomial in t_1,t_2 and $(\lambda_j,\eta_j)\neq(\lambda_k,\eta_k)$ for $j\neq k$. Let $\mu\in M_c(\mathbb{R}^2)$ be such that $\mu(V)=\{0\}$. We show that $\mu(e^{\iota(\lambda_j t_1+\eta_j t_2)})=0$. Since V is translation invariant, $f\star\mu=0$. Write f as a linear combination of elements in $\{t_1^{\alpha_{lj}}t_2^{\beta_{lj}}e^{\iota(\lambda_l t_1+\eta_l t_2)}:1\le l,j\le n\}$. Let $c_{k_1}t_1^{\alpha_0}t_2^{\beta_{k_1}}e^{\iota(\lambda_j t_1+\eta_j t_2)}, c_{k_2}t_1^{\alpha_0}t_2^{\beta_{k_2}}e^{\iota(\lambda_j t_1+\eta_j t_2)},\ldots,c_{k_m}t_1^{\alpha_0}t_2^{\beta_{k_m}}e^{\iota(\lambda_j t_1+\eta_j t_2)}$ be the terms containing $e^{\iota(\lambda_j t_1+\eta_j t_2)}$ and the largest degree term of t_1, namely $t_1^{\alpha_0}$, where $c_{k_1},c_{k_2},\ldots,c_{k_m}$ are non-zero scalars. Also, $f\star\mu$ has the same representation and the terms containing $t^{\alpha_0}e^{\iota(\lambda_j t_1+\eta_j t_2)}$ are $c_{k_1}\hat{\mu}(\lambda_j,\eta_j)t_1^{\alpha_0}t_2^{\beta_{k_1}}e^{\iota(\lambda_j t_1+\eta_j t_2)}$, $c_{k_2}\hat{\mu}(\lambda_j,\eta_j)t_1^{\alpha_0}t_2^{\beta_{k_2}}e^{\iota(\lambda_j t_1+\eta_j t_2)},\ldots,$

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$c_{k_m}\hat{\mu}(\lambda_j, \eta_j)t_1^{\alpha_0}t_2^{\beta_{k_m}}e^{\iota(\lambda_j t_1+\eta_j t_2)}$. Since $f \star \mu = 0$ and $c_{k_j} \neq 0$, $\hat{\mu}(\lambda_j, \eta_j) = 0$, by Lemma 4.5. Therefore $\mu(e^{\iota(\lambda_j t_1+\eta_j t_2)}) = 0$. Thus $e^{\iota(\lambda_j t_1+\eta_j t_2)} \in V$.

Cases (ii) and (iii): V is rotation invariant, or $V = \tau_\mu$. By Theorem 1.7 and Theorem 1.8, V contains an exponential polynomial. It follows easily from the proof of (i) that V contains an exponential. □

References

[1] P. M. Anselone and J. Korevaar, 'Translation invariant subspaces of finite dimension', *Proc. Amer. Math. Soc.* **15** (1964), 747–752.

[2] S. C. Bagchi and A. Sitaram, 'Spherical mean-periodic functions on semi simple Lie groups', *Pacific J. Math.* **84** (1979), 241–250.

[3] C. A. Berenstein and B. A. Taylor, 'Mean-periodic functions', *Internat. J. Math. Math. Sci.* **3** (1980), 199–235.

[4] L. Brown, B. M. Schreiber and B. A. Taylor, 'Spectral synthesis and the Pompeiu problem', *Ann. Inst. Fourier (Grenoble)* **23** (1973), 125–154.

[5] J. Delsarte, 'Les fonctions moyenne-periodiques', *J. Math. Pures Appl.* **14** (1935), 403–453.

[6] P. Devaraj and I. K. Rana, 'Relation between Pompeiu groups and mean-periodic groups', preprint, 2000.

[7] D. G. Dickson, 'Analytic mean-periodic functions', *Trans. Amer. Math. Soc.* **14** (1972), 361–374.

[8] J. Diestel and J. J. Uhl Jr., *Vector measures*, Math. Surveys Monographs 15 (Amer. Math. Soc., Providence RI, 1977).

[9] J. Dugunji, *Topology* (Prentice-Hall, New Delhi, 1975).

[10] L. Ehrenpreis, 'Appendix to the paper 'Mean-periodic functions I'', *Amer. J. Math.* **77** (1955), 731–733.

[11] ——, 'Mean-periodic functions, Part I. Varieties whose annihilator ideals are principal', *Amer. J. Math.* **77** (1955), 293–328.

[12] R. J. Elliott, 'Some results in spectral synthesis', *Proc. Camb. Phil. Soc.* **61** (1965), 395–424.

[13] ——, 'Two notes on spectral synthesis for discrete abelian groups', *Proc. Camb. Phil. Soc.* **61** (1965), 617–620.

[14] M. Engert, 'Finite dimensional translation invariant subspaces', *Pacific J. Math.* **32** (1970), 333–343.

[15] J. E. Gilbert, 'Spectral synthesis problems for invariant subspaces on groups II', in: *Proc. Int. Sym. on Function Algebras at Tulane Univ.* (1965) pp. 257–264.

[16] ——, 'Spectral synthesis problems for invariant subspaces on groups I', *Amer. J. Math.* **88** (1966), 626–635.

[17] D. L. Gurevich, 'Counter examples to a problem of L. Schwartz', *Funct. Anal. Appl.* **9** (1975), 116–120.

[18] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. 21 (Amer. Math. Soc., Providence, RI, 1957).

[19] J. P. Kahane, *Lectures on mean-periodic functions* (Tata Institute, 1957).

[20] P. Koosis, 'On functions which are mean-periodic on a half-line', *Comm. Pure Appl. Math.* **10** (1957), 133–149.

[21] G. E. Ladas and V. Lakshmikantham, *Differential equations in abstract spaces* (Academic Press, New York, 1972).

[22] P. G. Laird, 'Some properties of mean-periodic functions', *J. Austral. Math. Soc.* **14** (1972), 424–432.
[23] ———, 'Functional differential equations and continuous mean-periodic functions', *J. Math. Anal. Appl.* **47** (1974), 406–423.
[24] ———, 'Entire mean-periodic functions', *Canad. J. Math.* **17** (1975), 805–818.
[25] ———, 'On characterisations of exponential polynomials', *Pacific J. Math.* **80** (1979), 503–507.
[26] M. Lefranc, 'L' analysis harmonique dans $\mathbb{Z}^n$', *C. R. Acad. Sci. Paris* **246** (1958), 1951–1953.
[27] B. M. Levitan and V. V. Zhikov, *Almost periodic functions and differential equations* (Cambridge University Press, Cambridge, 1982).
[28] B. Malgrange, 'Sur quelques propriètès des equations des convolution', *C. R. Acad. Sci. Paris* **238** (1954), 2219–2221.
[29] A. Meril, 'Analytic functions with unbounded carriers and mean-periodic functions', *Trans. Amer. Math. Soc.* **278** (1983), 115–136.
[30] Y. Meyer, 'Harmonic analysis of mean-periodic functions', in: *Studies in harmonic analysis*, MAA Stud. Math. 13 (Math. Assoc. Amer., Washington D.C., 1976) pp. 151–160.
[31] E. Novak and I. K. Rana, 'On the unsmoothing of functions on the real line', *Proc. Nede. Acad. Sci. Ser. A* **89** (1986), 201–207.
[32] I. K. Rana, 'Unsmoothing over balls via plane wave decomposition', *Rend. Circ. Mat. Palermo (2)* **34** (1990), 217–234.
[33] I. K. Rana and N. Gowri, 'Integrable mean-periodic functions on locally compact abelian groups', *Proc. Amer. Math. Soc.* **117** (1993), 405–410.
[34] J. Schmets, *Spaces of vector valued continuous functions*, Lecture Notes in Math. 1003 (Springer, 1983).
[35] L. Schwartz, 'Theorie generale des fonctions moyenne-periodiques', *Ann. of Math. (2)* **48** (1947), 857–929.
[36] H. S. Shapiro, 'The expansions of mean-periodic functions in series of exponentials', *Comm. Pure Appl. Math.* **11** (1958), 1–21.
[37] L. Székelyhidi, 'The Fourier transform of mean-periodic functions', *Utilitas Math.* **29** (1986), 43–48.
[38] ———, *Convolution type functional equations on topological abelian groups* (World Scientific Publishing, Singapore, 1991).
[39] Y. Weit, 'On Schwartz theorem for the motion group', *Ann. Inst. Fourier (Grenoble)* **30** (1980), 91–107.

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