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BERNOULLI MAPS OF A LEBESGUE SPACE

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ABSTRACT. A collection of measure preserving mappings having Bernoulli generators is considered. Only three conditions are required to be satisfied, and they are quite easy to check.

Introduction. Dynamical systems generated by noninvertible maps of intervals have been extensively studied of late. The study of the well-known Lorenz differential equation (see Williams [15]), and certain other problems in ergodic theory leads to the study of maps of intervals. In this paper, we study noninvertible measure preserving mappings f of a probability space $(I, \mathbf{B}, \mu)$ into itself. Three conditions which persist under iteration and which are sufficient for f to be Bernoulli are given. We do not require the restriction of f to any of the atoms of the generating partition $\mathbf{C}$ to be continuous. We only insist that the restriction of f to $\Delta(\Delta \in \mathbf{C})$ should be one-to-one. Starting with this partition $\mathbf{C}$ we define the BR-cylinders which satisfy a kind of Markov condition, and require in condition (c) that there should be sufficiently many of them so that their collection is dense in $\mathbf{B}$. The behaviour of f on the BR-cylinders is quite simple, and helps one get a picture of what the whole system $(I, \mathbf{B}, \mu, f)$ looks like.

1. **Definitions and the class ^{S}B.** Let $(I, \mathbf{B}, \nu)$ be a probability space, that is, I is a set, $\mathbf{B}$ is a σ-algebra of subsets of I and ν is a complete normalised measure on $\mathbf{B}$. We consider a class ^{S}B of measurable nonsingular mappings f of $(I, \mathbf{B}, \nu)$ onto itself with $f^{-1}I = I$ having properties (a), (b), and (c), defined as follows:

(a) There is a finite measurable partition $\mathbf{C}_1 = \{\Delta(i): i \in J\}$ of I which generates $\mathbf{B}$ such that the restriction of f to each $\Delta(i) \in \mathbf{C}_1$ is one-to-one. Let $\mathbf{C}_n$ denote the atoms of $\vee_{i=0}^{n-1} f^{-i}\mathbf{C}_1$. We shall write $\Delta(i_1, \ldots, i_n)$ for $\cap_{m=1}^{n} f^{-(m-1)}\Delta(i_m)$. We write $\Delta(I_n K_m)$ for $\Delta(i_1, \ldots, i_n, k_1, \ldots, k_m) \in \mathbf{C}_{n+m}$. An atom of $\mathbf{C}_n$ is called a cylinder of order n. Given an integer $r \geqq 1$, we call an atom $\Delta(i_1, \ldots, i_n) \in \mathbf{C}_n$, a B-cylinder if

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$$f^m\Delta(i_1, \dots, i_n) = \Delta(i_{m+1}, \dots, i_n)$$

whenever $1 \leqq m \leqq n - r$. (For atoms of order 1, we call $\Delta(k) \in \mathbf{C}_1$ a B-cylinder if $f\Delta(k) = I$.) The set of all B-cylinders with integer r will de denoted by $B(r, f)$. Since the restriction of f^n on $\Delta(I_n) \in \mathbf{C}_n$ is one-to-one, its inverse, $V(I_n)$, exists. There exists

$$(d\nu \vee (I_n)/d\nu)(x) = H(I_n)(x)$$

on $f^n\Delta(I_n)$. Given a constant $c > 1$, we call an atom $\Delta(I_n) \in \mathbf{C}_n$, an R-cylinder if it satisfies the condition:

$$(\text{ess. sup } H(I_n)(x))/(\text{ess. inf } H(I_n)(x)) \leqq c.$$

The set of all R-cylinders with constant c will be denoted by $R(c, f)$.

(b) There are constants $c > 1$ and $r \geqq 1$, and a non-empty subset $I(c, r)$ of $B(r, f) \cap R(c, f)$ such that if $\Delta(I_n) \in I(c, r)$ and $\Delta(J_m) \in I(c, r)$, then $\Delta(I_n) \cap f^{-n}\Delta(J_m) \in I(c, r)$. The elements of $I(c, r)$ will be called BR-cylinders. Set

$$\mathbf{D}_n = \{\Delta(k_1, \dots, k_n) \in \mathbf{C}_n : \Delta(k_1, \dots, k_i) \in \mathbf{C}_i \backslash I(c, r), \quad \text{for} \quad 1 \leqq i \leqq n\}.$$

(c) The series

$$\sum_{n=1}^{\infty} \sum_{\Delta(I_n) \in \mathbf{D}_n} \nu(\Delta(I_n))$$

is convergent.

2. f **is Bernoulli.** It is known [6] that f preserves a probability measure μ equivalent to ν such that

$$c^{-2}q \leqq \frac{d\mu}{d\nu}(x) \leqq q^{-1}c \text{ a.e.} \quad \dots \quad \dots \quad \dots \quad (1)$$

where $q = \min\{\nu(\Delta(I_r)) : \Delta(I_r) \in \mathbf{C}_r\}$, and c and r are the constants of (b). Moreover, f is exact [6]. Define

$$D(P', P'') = \sum_{\Delta' \in P'} \sum_{\Delta'' \in P''} |\mu(\Delta' \cap \Delta'') - \mu(\Delta')\mu(\Delta'')|$$

for any measurable partitions P' and P'' of $(I, \mathbf{B}, \mu)$. A measurable partition P of a Lebesgue space $(I, \mathbf{B}, \mu)$ is said to be Bernoulli for f if for each $\epsilon > 0$ there is an $N(\epsilon)$ such that for $n = 0, 1, 2, \dots$, we have

$$D(\vee_{i=n+N(\epsilon)}^{2n+N(\epsilon)} f^{-i}P, \vee_{i=0}^{n} f^{-i}P) < \epsilon.$$

We shall prove that $\mathbf{C}_1$ is Bernoulli for f.

THEOREM. f is Bernoulli.

The proof is preceded by some Lemmas.

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LEMMA 1. *For any* $\epsilon > 0$ *there is* $N_0 = N_0(\epsilon)$ *such that every* $\Delta(I_n) \in \mathbf{C}_n$, $1 \leqq n < \infty$, *can be filled to within a set of* ν*-measure less than* ϵ *with disjoint BR-cylinders of order between* n *and* $n + N_0$.

The proof of this lemma follows the lines of argument of the Lemma 1 in [5].

LEMMA 2. *For any* $\epsilon > 0$, *there is* $N_1 = N_1(\epsilon)$ *such that for* $n > N_1(\epsilon)$, *the inequality*

$$|\mu(A) - \nu(f^{-n}A)| \leqq \epsilon\mu(A)$$

holds for any $A \in \mathbf{B}$.

PROOF. Since f is exact the tail sets are trivial and so

$$\int_{f^{-n}A} E\left(\frac{d\nu}{d\mu} | f^{-n}\mathbf{B}\right) d\mu \rightarrow \mu(A)$$

as $n \rightarrow \infty$, by martingale convergence theorem.
But

$$\int_{f^{-n}A} E\left(\frac{d\nu}{d\mu} | f^{-n}\mathbf{B}\right) d\mu = \nu(f^{-n}A)$$

hence

$$\mu(A) = \lim_{n \rightarrow \infty} \nu(f^{-n}A)$$

for each $A \in \mathbf{B}$, and on applying (1), the lemma follows.

LEMMA 3. *Given* $\epsilon > 0$, *there exists* $N_2(\epsilon)$ *such that for* $n > N_2(\epsilon)$, *the inequality*

$$\sum_{\Delta(I_m) \in \mathbf{C}_m} |\mu(\Delta(I_s) \cap f^{-(n+s)}\Delta(I_m)) - \mu(\Delta(I_m))\mu(\Delta(I_s))| < \epsilon$$

holds for each $\Delta(I_s) \in \mathbf{C}_s$.

PROOF. Let $\mathbf{F}_k$ be the σ-algebra generated by $\cup_{i=k}^{\infty} f^{-i}\mathbf{C}_1$. By the use of martingale convergence theorem and the fact that $\cap_{k=1}^{\infty} \mathbf{F}_k$ is trivial,

$$\mu(\Delta(I_s) | \mathbf{F}_k)(x) \rightarrow \mu(\Delta(I_s)) \text{ a.e.}$$

as $k \rightarrow \infty$. Hence, by Egoroff theorem, for any $\epsilon' > 0$ there is a set A with $\mu(A) < \epsilon'$ such that on $I \backslash A$, $\mu(\Delta(I_s) | \mathbf{F}_k)(x)$ converges uniformly. That is, we can find $N_2(\epsilon')$ such that for $k > N_2(\epsilon')$, we have

$$|\mu(\Delta(I_s) | \mathbf{F}_k)(x) - \mu(\Delta(I_s))| < \epsilon' \text{ if } x \in I \backslash A.$$

Now

$$\mu(\Delta(I_s) \cap f^{-(n+s)}\Delta(I_m)) = \int_{f^{-(n+s)}\Delta(I_m)} \mu(\Delta(I_s) \,|\mathbf{F}_{n+s})(x)d\mu.$$

Hence

$$\sum_{\Delta(I_m)\in \mathbf{C}_m} |\mu(\Delta(I_s) \cap f^{-(n+s)}\Delta(I_m)) - \mu(\Delta(I_s))\mu(\Delta(I_m))|$$

$$\leqq \sum_{\Delta(I_m)} \int_{f^{-(n+s)}\Delta(I_m)\cap I\setminus A} |\mu(\Delta(I_s) \,|\mathbf{F}_{n+s})(x) - \mu(\Delta(I_s))\,|d\mu$$

$$+ \sum_{\Delta(I_m)} \int_{f^{-(n+s)}\Delta(I_m)\cap A} |\mu(\Delta(I_s) \,|\mathbf{F}_{n+s})(x) - \mu(\Delta(I_s))\,|d\mu$$

$$\leqq \epsilon'\mu(I\setminus A) + 2\mu(A), \text{ for } n > N_2(\epsilon'), \leqq 3\epsilon',$$

and the result follows.

LEMMA 4. *There exist $\theta(i)$ ($\theta(i) \to 0$ as $i \to \infty$) and $r_0 > r$ such that for $i > r_0$ the inequality*

$$\left|\frac{\mu(\Delta(I_kJ_iM_n))}{\mu(\Delta(I_kJ_i))} - \frac{\mu(\Delta(J_iM_n))}{\mu(\Delta(J_i))}\right| < \theta(i)\left(\frac{\mu(\Delta(J_iM_n))}{\mu(\Delta(J_i))}\right)$$

holds for any BR-cylinders $\Delta(I_k)$ and $\Delta(J_i)$; and $\Delta(M_n) \in \mathbf{C}_n$.

PROOF. Now for each $A \in \Delta_k \cap f^{-k}\mathbf{B}$, $\Delta_k \in \mathbf{C}_k$,

$$\nu(f^kA) \to \nu(A) \text{ as } k \to \infty \text{ if } A \neq \Delta_k.$$

Hence, using (1), we have that for any $\epsilon > 0$ there is $N_3(\epsilon)$ such that $k \geqq N_3(\epsilon)$ implies

$$|\nu(f^kA) - \nu(A)| \leqq \epsilon\nu(A)$$

for each $A \in \Delta_k \cap f^{-k}\mathbf{B}$, $A \neq \Delta_k$. Therefore, for $k > N_3(\epsilon)$, the inequalities

$$|\nu(\Delta(J_iM_n)) - \nu(\Delta(I_kJ_iM_n))| < \epsilon\nu(\Delta(I_kJ_iM_n))$$

and

$$|\nu(\Delta(J_i)) - \nu(\Delta(I_kJ_i))| < \epsilon\nu(\Delta(I_kJ_i))$$

hold. Thus there exist $\beta(i)$ ($\beta(i) \to 0$ as $i \to \infty$) and $i_0 > N_1$ such that for $i > i_0$,

$$\left(\frac{\nu(\Delta(I_kJ_iM_n))}{\nu(\Delta(I_kJ_i))}\right)\left(\frac{\nu(\Delta(J_i))}{\nu(\Delta(J_iM_n))}\right) = 1 + t\beta(i),$$

where $|t| < 1$. Applying Lemma 2 the required result follows.

LEMMA 5. *For any $\epsilon > 0$, there is an $r_1 = r_1(\epsilon) > \sup (N_0(\epsilon), N_1(\epsilon), N_2(\epsilon), r_0)$ so that for each $n \geqq 0$ we can find a collection*

$$\mathbf{B}_{n+r_1} \subset \vee_{i=0}^{n+r_1} f^{-i}\mathbf{C}_1$$

of BR-cylinders such that

(i) $$\mu(\cup \mathbf{B}_{n+r_1}) > 1 - \epsilon,$$

and

(ii) $$(1 - \epsilon)\left(\frac{\mu(f^n B')}{\mu(f^n B)}\right) \leqq \frac{\mu(B')}{\mu(B)} \leqq (1 + \epsilon)\left(\frac{\mu(f^n B')}{\mu(f^n B)}\right)$$

for $B' \subset B$, $B \in \mathbf{B}_{n+r_1}$.

Proof. This follows directly from Lemmas 1, 2 and 4. We shall need the following inequality:

$$D(P', P'') \leqq 2(2 - \mu(\cup P_1') - \mu(\cup P_2'')) + D(P_1', P_2''),$$

where P_1', P_2'' are any collections of atoms of measurable partitions P', P''.

We can now prove the theorem.

Proof of the Theorem. Choose $\epsilon' > 0$ and $r_1 = r_1(\epsilon')$ as in Lemma 5. Let $\alpha(\mu) = D(\vee_{i=0}^{n+r_1} f^{-i}\mathbf{C}_1, \vee_{i=n+r_1+N}^{2n+2r_1+N} f^{-i}\mathbf{C}_1)$, N will be chosen later. Now

$$\alpha(\mu) \leqq 2\epsilon' + D(\mathbf{B}_{n+r_1}, \vee_{i=n+r_1+N}^{2n+2r_1+N} f^{-i}\mathbf{C}_1).$$

For $B \in \mathbf{B}_{n+r_1}$, and $A \in \vee_{i=n+r_1+N}^{2n+2r_1+N} f^{-i}\mathbf{C}_1$, we have

$$(1 - \epsilon')\left(\frac{\mu(f^n B \cap f^n A)}{\mu(f^n B)}\right) \leqq \frac{\mu(A \cap B)}{\mu(B)} \leqq (1 + \epsilon')\left(\frac{\mu(f^n B \cap f^n A)}{\mu(f^n B)}\right).$$

Thus

$$\begin{aligned} &D(\mathbf{B}_{n+r_1}, \vee_{i=n+r_1+N}^{2n+2r_1+N} f^{-i}\mathbf{C}_1) \\ &\leqq \sum_B \mu(B) \sum_A \left(\left|\frac{\mu(A \cap B)}{\mu(B)} - \frac{\mu(f^n A \cap f^n B)}{\mu(f^n B)}\right| \right. \\ &\left. + \left|\frac{\mu(f^n B \cap f^n A)}{\mu(f^n B)} - \mu(f^n A)\right| + |\mu(f^n A) - \mu(A)|\right) \\ &\leqq \epsilon' + D(\vee_{i=0}^{r_1} f^{-i}\mathbf{C}_1, \vee_{i=r_1+N}^{2r_1+n+N} f^{-i}\mathbf{C}_1). \end{aligned}$$

Hence

$$\begin{aligned} \alpha(\mu) &\leqq 3\epsilon' + D(\vee_{i=0}^{r_1} f^{-i}\mathbf{C}_1, \vee_{i=r_1+N}^{2r_1+n+N} f^{-i}\mathbf{C}_1) \\ &\leqq (3 + M)\epsilon' \text{ for } N > N_2(\epsilon'), \end{aligned}$$

where M is the number of the atoms of $\mathbf{C}_{r_1+1}$ (using Lemma 3). Given $\epsilon > 0$, by choosing ϵ' very small and N accordingly, the inequality

$$D(\vee_{i=0}^{n} f^{-i}\mathbf{C}_1, \vee_{i=n+N(\epsilon)}^{2n+N(\epsilon)} f^{-i}\mathbf{C}_1) < \epsilon$$

holds for all $n \geqq 0$. Therefore, $\mathbf{C}_1$ is a Bernoulli generator for f.

3. Remarks.

(i) If the weaker condition: $\lim_{n\to\infty} \sum_{\Delta(I_n)\in\mathbf{D}_n} \nu(\Delta(I_n)) = 0$ is assumed in place of (c), then we can only conclude [6] that the maps are "Loosely Bernoulli" (See Feldman [9] for the definition).

(ii) If the partition $\mathbf{C}_1$ is assumed countable but not necessarily finite, the conclusions of the Theorem are still true provided that the set of BR-cylinders $\Delta(I_n)$, $n \geqq 1$ with $f^n\Delta(I_n) = I$ is dense in $\mathbf{B}$ (e.g. in the class W of [3]).

(iii) The maps considered in Adler [1], [2], Alufohai [4], [5], Schweiger [13], Bowen [7], Ratner [11], Waterman [14] are special families of the maps in S_B.

(iv) The maps in Fischer [10] (whose exactness he showed there) are all Bernoulli; and the maps of Schweiger [12], Bowen [8] and those in the class W of [3] are loosely Bernoulli.

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