Canad. J. Math. Vol. **60** (4), 2008 pp. 758–789

On the Hyperinvariant Subspace Problem. IV

H. Bercovici, C. Foias, and C. Pearcy

Abstract. This paper is a continuation of three recent articles concerning the structure of hyperinvariant subspace lattices of operators on a (separable, infinite dimensional) Hilbert space $\mathcal{H}$. We show herein, in particular, that there exists a "universal" fixed block-diagonal operator B on $\mathcal{H}$ such that if $\varepsilon > 0$ is given and T is an arbitrary nonalgebraic operator on $\mathcal{H}$, then there exists a compact operator K of norm less than ε such that (i) $\operatorname{Hlat}(T)$ is isomorphic as a complete lattice to $\operatorname{Hlat}(B+K)$ and (ii) $B+K$ is a quasidiagonal, C_{00}, (BCP)-operator with spectrum and left essential spectrum the unit disc. In the last four sections of the paper, we investigate the possible structures of the hyperlattice of an arbitrary algebraic operator. Contrary to existing conjectures, $\operatorname{Hlat}(T)$ need not be generated by the ranges and kernels of the powers of T in the nilpotent case. In fact, this lattice can be infinite.

1 Introduction

As the title indicates, this paper is a continuation of our study, begun in [15] and continued in [13, 17], of properties of the hyperinvariant subspace lattices (usually called hyperlattices in the literature) of operators on Hilbert space. Let us begin by reviewing some rather standard notation and terminology that will be used below and is entirely consistent with that employed in [13, 15, 17].

Throughout this paper $\mathcal{H}$ will denote a separable, infinite dimensional, complex, Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The ideal of compact operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathbf{K}$ or $\mathbf{K}(\mathcal{H})$, and π will denote the quotient map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbf{K}$. The symbols $\mathbb{N}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$, will designate the sets of positive integers, complex numbers, $\zeta \in \mathbb{C}$ such that $|\zeta| < 1$, and $\partial\mathbb{D}$, respectively. The spectrum of an operator T in $\mathcal{L}(\mathcal{H})$ will be written as $\sigma(T)$, and the left and right essential (*i.e.,* Calkin) spectrum of T by $\sigma_{\rm le}(T)$ and $\sigma_{\rm re}(T)$. Moreover $\sigma_e(T) = \sigma_{\rm le}(T) \cup \sigma_{\rm re}(T)$. The (complete) lattices of invariant and hyperinvariant subspaces of an operator T will be denoted, respectively by $\operatorname{Lat}(T)$ and $\operatorname{Hlat}(T)$. Furthermore, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete lattices, we write $\mathcal{L}_1 \equiv \mathcal{L}_2$ to signify that there is a (complete) lattice isomorphism of one onto the other.

Recall that, by definition, a (BCP)-operator is a completely nonunitary contraction T in $\mathcal{L}(\mathcal{H})$ such that $\sigma_e(T) \cap \mathbb{D}$ is a dominating set for $\mathbb{T}$ (meaning that almost every point of $\mathbb{T}$ (with respect to arclength measure) is a nontangential limit of a sequence of points from $\sigma_e(T) \cap \mathbb{D}$). The class of (BCP)-operators played a significant role in the theory of dual algebras (see [9] for more information about this theory), and (BCP)-operators have several nice properties. In the first place, the class (BCP) is a subset of the larger class $\mathbb{A}_{\aleph_0}$ which also plays a central role in the theory of dual

Received by the editors July 25, 2005; revised October 12, 2005.
The first author was supported in part by a grant from the National Science Foundation.
AMS subject classification: 47A15.

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

algebras (See [9] for the definition of and more information about this class.) Thus the lattice Lat (T) of an arbitrary (BCP)-operator T contains a sublattice isomorphic to the lattice $\text{Lat}(\mathcal{H})$ of all subspaces of $\mathcal{H}$. Moreover (BCP)-operators are "universal dilations" in the sense that any direct sum of strict contractions can be realized, up to unitary equivalence, as the compression of every (BCP)-operator to some semi-invariant subspace [9]. Furthermore it is known [6] that every (BCP)-operator T in $\mathcal{L}(\mathcal{H})$ has the property that there exists a sequence $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of cyclic invariant subspaces for T such that

$$\mathcal{M}_j \cap \Big(\bigvee_{n\neq j} \mathcal{M}_n \Big) = \{0\}, \quad j \in \mathbb{N}.$$

Thus it is fair to say that a reasonable amount of information is known about properties of $\text{Lat}(T)$ when T is a (BCP)-operator. On the other hand, at present precisely nothing can be said about the hyperlattice of a general (BCP)-operator.

One of the main, and perhaps unexpected, results of this paper is that if T is an arbitrary nonalgebraic operator in $\mathcal{L}(\mathcal{H})$, then there is a (BCP)-operator $\widehat{T}$ with several additional nice properties such that $\text{Hlat}(T) \equiv \text{Hlat}(\widehat{T})$. Perhaps, in time, this result will be useful in settling the (open) hyperinvariant subspace problem for operators on Hilbert space.

After making some elementary observations and reductions in Section 2, we review briefly in Section 3 the definitions and results from the earlier papers [13, 17] that we shall need. Section 4 is devoted to obtaining the "universality" of hyperlattices of (BCP)-operators mentioned above, and in Sections 5–7 we study the hyperlattices of algebraic operators, which trivially reduces to the study of the hyperlattices of nilpotent operators (see Proposition 3.3). Section 5 contains general remarks, as well as the classification of $\text{Hlat}(T)$ when T is nilpotent of order at most two. In Section 6 we focus on nilpotents of order three. We answer, in the negative, some long-standing conjectures concerning those hyperlattices. Thus, the hyperlattice of such an operator is not generally generated by the kernels and ranges of T and T^2. In fact we find that many such operators have infinite hyperlattices, some of them totally (even well-) ordered. We conclude in Sections 7 and 8 with results on general nilpotents, though some of the results yield insights into the structure of the hyperlattice of an arbitrary operator.

2 Preliminaries

In this section we present some preliminary remarks. First, we make some observations pertaining to the general question: what can be said about (the structure of) $\text{Hlat}(T)$ for various T in $\mathcal{L}(\mathcal{H})$? As usual, for T in $\mathcal{L}(\mathcal{H})$ we write

$$\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : ST = TS\}$$

for the commutant of T, $W^*(T)$ for the (unital) von Neumann algebra generated by T, and $P_{\mathcal{M}}$ for the (orthogonal) projection in $\mathcal{L}(\mathcal{H})$ whose range is the subspace $\mathcal{M}$ of $\mathcal{H}$. (Recall that a subspace $(0) \neq \mathcal{M} \subsetneqq \mathcal{H}$ is a nontrivial hyperinvariant subspace for an operator T in $\mathcal{L}(\mathcal{H})$ if $S\mathcal{M} \subset \mathcal{M}$ for every S in $\{T\}'$.) The following results are well known, but since their proofs are short, we give them.

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

Proposition 2.1 *For every T in $\mathcal{L}(\mathcal{H})$ and for every $\mathcal{M} \in \mathrm{Hlat}(T)$, $P_{\mathcal{M}}$ belongs to $W^*(T)$.*

Proof By the double commutant theorem, it suffices to show that if

$$Q = Q^2 = Q^* \in W^*(T)' = \{T\}' \cap \{T^*\}',$$

then $P_{\mathcal{M}}Q = QP_{\mathcal{M}}$ or, equivalently, that $Q\mathcal{M} \subset \mathcal{M}$. Since $Q \in \{T\}'$ and $\mathcal{M} \in \mathrm{Hlat}(T)$, this is immediate. ■

Proposition 2.2 *Let T be a normal operator in $\mathcal{L}(\mathcal{H})$. Then*

$$\mathrm{Hlat}(T) = \{\mathcal{M} \subset \mathcal{H} : P_{\mathcal{M}} \in W^*(T)\}.$$

Proof By Proposition 2.1, if $\mathcal{M} \in \mathrm{Hlat}(T)$, then $P_{\mathcal{M}} \in W^*(T)$. On the other hand, by Fuglede's theorem, $\{T\}' = \{T\}' \cap \{T^*\}' = W^*(T)'$. Thus if $P_{\mathcal{M}} \in W^*(T)'' = W^*(T)$ and $S \in \{T\}'$, then $P_{\mathcal{N}}S = SP_{\mathcal{N}}$ so $S\mathcal{N} \subset \mathcal{N}$ and $\mathcal{N} \in \mathrm{Hlat}(T)$. ■

In the following statement we write 2^n for the lattice of all subsets of a set with n elements.

Proposition 2.3 *Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator with countable spectrum whose eigenvectors span $\mathcal{H}$, and let n be the cardinal number of the set of distinct eigenvalues of T $(1 \leq n \leq \aleph_0)$. Then* $\mathrm{Hlat}(T) \equiv 2^n$.

Proof Let $\{\lambda_i\}_{0 \leq i < n}$ be the distinct eigenvalues of T and let $\widehat{T}$ be a normal operator with the same set of eigenvalues, each having multiplicity 1. Then, as is well known, the abelian von Neumann algebras $W^*(T)$ and $W^*(\widehat{T})$ are isomorphic as C^*-algebras, so by Proposition 2.2, $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(\widehat{T})$. Since the lattice of projections in $W^*(\widehat{T})$ is lattice isomorphic to the lattice of subsets of the initial segment of $\mathbb{N}$ containing n elements, the proof is complete. ■

Proposition 2.4 *If $T \in \mathcal{L}(\mathcal{H})$ has countably infinite spectrum, then T is similar to a direct sum $T_1 \oplus T_2 \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, where $\sigma(T_1)$ has exactly one accumulation point and $\sigma(T_1) \cap \sigma(T_2) = \varnothing$.*

Proof Observe first that $\sigma(T)$ must have at least one accumulation point λ_0 which is isolated in $\sigma(T)^d$, the derived set of $\sigma(T)$, for otherwise $\sigma(T)^d$ would be a nonempty countable perfect set in $\mathbb{C}$. Let $\mathcal{U}$ be a bounded open set in $\mathbb{C}$ such that $\sigma(T)^d \cap \mathcal{U} = \{\lambda_0\}$ and $\mathrm{card}(\sigma(T) \cap \mathcal{U}) = \aleph_0$. Let E be the Riesz idempotent corresponding to the separated subset $\sigma(T) \cap \mathcal{U}$ of $\sigma(T)$, and set $T_1 = T|_{E\mathcal{H}}$ and $T_2 = T|_{(1-E)\mathcal{H}}$. Then $T = T_1 \dotplus T_2$ and one knows that $T_1 \dotplus T_2$ is similar to $T_1 \oplus T_2$ and that $\sigma(T_1) \cap \sigma(T_2) = \varnothing$. ■

3 A Review

For the reader's convenience, we now review some pertinent definitions and results. For any ordinal number n satisfying $1 \leq n \leq \omega$, we denote by $\mathcal{H}^{(n)}$ the direct sum

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

of n copies of $\mathcal{H}$ indexed by the appropriate initial segment of $\mathbb{N}$, and for T in $\mathcal{L}(\mathcal{H})$ we write $T^{(n)}$ for the direct sum (ampliation) of n copies of T acting on $\mathcal{H}^{(n)}$ in the obvious fashion.

A new equivalence relation on $\mathcal{L}(\mathcal{H})$, called *hyperquasisimilarity* (notation: $\overset{h}{\sim}$) was introduced in [13], which is strictly stronger than quasisimilarity and has the property that if $T_1 \overset{h}{\sim} T_2$, then $\operatorname{Hlat}(T_1) \equiv \operatorname{Hlat}(T_2)$. More precisely, the operators T_1 and T_2 are said to be hyperquasisimilar if there exist quasiaffinities X, Y satisfying $XT_1 = T_2X, T_1Y = YT_2$, and the additional conditions that $(YX\mathcal{M}_1)^- = \mathcal{M}_1, (XY\mathcal{M}_2)^- = \mathcal{M}_2$ for every $\mathcal{M}_1 \in \operatorname{Hlat}(T_1)$ and $\mathcal{M}_2 \in \operatorname{Hlat}(T_2)$. If X and Y are such quasiaffinities, the maps $\mathcal{M}_1 \mapsto (X\mathcal{M}_1)^-, \mathcal{M}_2 \mapsto (Y\mathcal{M}_2)^-$ are bijections between $\operatorname{Hlat}(T_1)$ and $\operatorname{Hlat}(T_2)$, and they are inverse to each other. Concerning this equivalence relation, we will also need the following [13, Theorem 2.8] .

Lemma 3.1 *Suppose that $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ are bounded sequences of operators in $\mathcal{L}(\mathcal{H})$, and $\widehat{S} := \bigoplus_{n\in\mathbb{N}} S_n$ and $\widehat{T} := \bigoplus_{n\in\mathbb{N}} T_n$. Suppose, moreover, that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of invertible operators such that $X_n^{-1}S_nX_n = T_n$, $n \in \mathbb{N}$. Then $\widehat{S} \overset{h}{\sim} \widehat{T}$ and consequently* $\operatorname{Hlat}(\widehat{S}) \equiv \operatorname{Hlat}(\widehat{T})$.

We next review briefly another of the main concepts and results from [13]. If $T \in \mathcal{L}(\mathcal{H})$ and λ is an isolated point of $\sigma(T)$ then, consistently throughout what follows, we denote by $\mathcal{M}_\lambda = \mathcal{M}_\lambda(T)$ the range of the Riesz idempotent associated with the separated subset $\{\lambda\}$ of $\sigma(T)$. An operator T in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T)$ is either uncountable or contains an isolated point μ such that $(T - \mu 1_{\mathcal{H}})|_{\mathcal{M}_\mu}$ is not nilpotent is said to have property (AHV). The following is a condensed version of [13, Theorem 4.4].

Theorem 3.2 *Let $0 \le \theta < 1$ be arbitrarily given, set $\mathbb{A}_\theta = \{\zeta \in \mathbb{C} : \theta \le |\zeta| \le 1\}$, and let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})$ with property (AHV). Then there exists a (BCP)-operator $\widehat{T} \in C_{00}$* (i.e., *$\widehat{T}^n \to 0$ and $\widehat{T}^{*n} \to 0$ in the strong operator topology such that:*

(i) *there exist $\delta, \gamma \in \mathbb{C}$ such that $\widetilde{T} := \delta(T + \gamma 1_{\mathcal{H}})$ satisfies $\widetilde{T}^{(\omega)} \overset{h}{\sim} \widehat{T}$; so, in particular,* $\operatorname{Hlat}(T) \equiv \operatorname{Hlat}(\widehat{T})$*;*

(ii) $\|\widehat{T}^{-1}\| = 1/\theta$ *provided that* $\theta > 0$*;*

(iii) $\sigma(\widehat{T}) = \sigma_{le}(\widehat{T}) = \mathbb{A}_\theta$.

Note that the operators in $\mathcal{L}(\mathcal{H})$ to which Theorem 3.2 does not apply either have countably infinite spectrum or are algebraic. (Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be algebraic if $p(T) = 0$ for some nonzero polynomial p and that, if T is algebraic, there exists a unique monic polynomial m_T of minimal degree such that $m_T(T) = 0$.) In what follows, the set of all algebraic operators in $\mathcal{L}(\mathcal{H})$ will be denoted by (A). The following fact is well known and needs no proof.

Proposition 3.3 *If $T \in (A)$ and $\lambda_1, \dots, \lambda_k$ are the distinct roots of m_T, then T is similar to $T|_{\mathcal{M}_{\lambda_1}} \oplus \cdots \oplus T|_{\mathcal{M}_{\lambda_k}}$. Moreover,*

$$\operatorname{Hlat}(T) \equiv \operatorname{Hlat}\big((T - \lambda_1 1_{\mathcal{H}})|_{\mathcal{M}_{\lambda_1}}\big) \oplus \cdots \oplus \operatorname{Hlat}\big((T - \lambda_k 1_{\mathcal{H}})|_{\mathcal{M}_{\lambda_k}}\big),$$

and for each $i = 1, \dots, k$, $(T - \lambda_i 1_{\mathcal{H}})|_{\mathcal{M}_{\lambda_i}}$ is a nilpotent operator.

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

Thus it is obvious that for any $T \in (A)$, the structure of $\operatorname{Hlat}(T)$ is determined by the properties of $\operatorname{Hlat}(J)$ where J runs through a finite set of nilpotent operators (some of which may act on finite dimensional spaces).

The case of operators with countably infinite spectrum is treated in Section 4, while a study of hyperlattices of nilpotent operators will be undertaken in Section 5.

Finally, we shall need in Section 4 two theorems from the deep theory of Apostol–Herrero–Voiculescu [2,19] which characterizes the (norm) closure $\mathcal{S}(T)^-$ of the similarity orbit $\mathcal{S}(T) = \{X^{-1}TX : X \text{ is invertible in } \mathcal{L}(\mathcal{H})\}$ of an (almost) arbitrary $T \in \mathcal{L}(\mathcal{H})$. The first is [19, Proposition 5.13].

Theorem 3.4 **(Herrero)** *Let $M, T \in \mathcal{L}(\mathcal{H})$ with M normal and $\sigma(M) = \sigma(T)$. Suppose, moreover, that for every isolated point $\mu \in \sigma(T)$, $\dim \mathcal{M}_\mu(T) = \dim \mathcal{M}_\mu(M)$. Then $M \in \mathcal{S}(T)^-$.*

The next is a very special case of the main theorem of [4].

Theorem 3.5 **(Barria–Herrero)** *Suppose M and N are normal operators in $\mathcal{L}(\mathcal{H})$ such that*

(i) $\sigma_e(M) = \sigma(M) \subset \sigma(N)$,
(ii) $\sigma(M)$ *is infinite,*
(iii) $\sigma(N)$ *is perfect and connected (so $\sigma(N) = \sigma_e(N)$ and $\sigma(N)$ is the connected component of each of its points).*

Then $N \in \mathcal{S}(M)^-$.

We also record here for future use the easily proved fact that if $T_2 \in \mathcal{S}(T_1)^-$ and $T_3 \in \mathcal{S}(T_2)^-$, then $T_3 \in \mathcal{S}(T_1)^-$. This allows us in Section 4 to combine Theorems 3.4 and 3.5, and thus to begin with an operator T_1 and proceed from T_1 to T_3.

4 Nonalgebraic Operators

In this section, by utilizing the results of Section 3 and the constructions from [13, 15], we will first establish the following theorem which improves [13, Theorem 4.4].

Theorem 4.1 *Fix $\theta \in [0, 1)$, and suppose $T \in \mathcal{L}(\mathcal{H}) \backslash (A)$. Then there exists a C_{00}, (BCP)-operator $\widehat{T} = \widehat{T}(\theta, T)$ such that*

(i) *there exist $\delta, \gamma \in \mathbb{C}$ such that $\widetilde{T} = \delta(T + \gamma 1_{\mathcal{H}})$ satisfies $\widetilde{T}^{(\omega)} \overset{h}{\sim} \widehat{T}$, and thus* $\operatorname{Hlat}(\widehat{T}) \equiv \operatorname{Hlat}(T)$,
(ii) $\theta \widehat{T}^{-1}$ *is a C_{00} (BCP)-operator whenever $\theta > 0$,*
(iii) $\sigma(\widehat{T}) = \sigma_{le}(\widehat{T}) = \mathbb{A}_\theta$.

Proof If T has property (AHV), then the desired conclusions are all given by [13, Theorem 4.4], except for the fact that $\theta \widehat{T}^{-1}$ belongs to C_{00}. But a close look at [15, (8), (10)] shows that $\theta \|S_n T^{-1} S_n^{-1}\| < 1$ for each $n \in \mathbb{N}$ and that θT^{-1} is a C_{00}-contraction follows immediately. Thus we may suppose that $\sigma(T)$ is either finite or countably infinite and that for every isolated point $\lambda \in \sigma(T)$, $(T - \lambda 1_{\mathcal{H}})|_{\mathcal{M}_\lambda}$ is nilpotent. Clearly then, if $\sigma(T)$ were a finite set, T would be an algebraic operator, contrary to hypothesis, so $\sigma(T)$ must be countably infinite.

We consider first the case in which $\sigma(T)$ has exactly one accumulation point, say μ. We observe first that the normalization carried out in [13, Theorem 4.4] consisting of replacing T by $\widetilde{T} = \delta(T + \gamma 1_{\mathcal{H}})$ so that $\delta(\mu + \gamma) = (1 + \theta)/2$ and

$$\sigma(\widetilde{T}) \subset D := D_{\frac{1+\theta}{2}, \frac{1-\theta}{4}}$$

can still be done and, of course, $\operatorname{Hlat}(T^{(\omega)}) \equiv \operatorname{Hlat}(T) \equiv \operatorname{Hlat}(\widetilde{T}) \equiv \operatorname{Hlat}(\widetilde{T}^{(\omega)})$. Next, let M be a diagonal normal operator in $\mathcal{L}(\mathcal{H})$ (relative to some orthonormal basis for $\mathcal{H}$) whose eigenvalues are exactly the (isolated) points

$$\alpha \in \sigma(\widetilde{T}) \setminus \{(1 + \theta)/2\},$$

each having multiplicity $\dim \mathcal{M}_\alpha(T)$. Then, by Theorem 3.5, $M \in \mathcal{S}(\widetilde{T})^-$, and consequently, $M^{(\omega)} \in \mathcal{S}(\widetilde{T}^{(\omega)})^-$. Note next that since $M^{(\omega)}$ is normal and has uniform infinite multiplicity, we have $\sigma(M^{(\omega)}) = \sigma_e(M^{(\omega)}) (= \sigma(\widetilde{T}))$. A casual perusal of the proof of [15, Theorem 1.1] now shows that every member of the sequence $\{N_n\}$ of normal operators constructed there is such that $\sigma(N_n) = \sigma_e(N_n)$ is a perfect, connected set containing $\sigma(M^{(\omega)})$. Thus Theorem 3.5 can be applied to show that for $n \in \mathbb{N}$, $N_n \in \mathcal{S}(M^{(\omega)})^-$ and thus, by what was already proved, that $N_n \in \mathcal{S}(\widetilde{T}^{(\omega)})^-$. With $\widehat{T}$ defined as in the proof of [15, Theorem 1.1] (with $\widetilde{T}^{(\omega)}$ replacing $\widetilde{T}$), the remainder of that proof goes through unchanged, and gives (via Lemma 3.1) that

$$\left(\widetilde{T}^{(\omega)}\right)^{(\omega)} \overset{h}{\sim} \widehat{T}$$

and, since $\left(\widetilde{T}^{(\omega)}\right)^{(\omega)}$ is unitarily equivalent to $\widetilde{T}^{(\omega)}$, that $\widehat{T}$ has all the required properties follows as above.

To complete the proof, we need to consider the case in which $\sigma(T)$ is an arbitrary countably infinite set (with more than one point of accumulation by virtue of the discussion above). By Proposition 2.4 we may suppose that $T = T_1 \oplus T_2$, where $\sigma(T_1)$ has exactly one accumulation point, say μ, and $\sigma(T_1) \cap \sigma(T_2) = \varnothing$. We apply the normalization procedure (to T) as before, replacing T by $\widetilde{T} = \widetilde{T}_1 \oplus \widetilde{T}_2 = \delta(T + \gamma 1_{\mathcal{H} \oplus \mathcal{H}})$ in such a way that $(1 + \theta)/2 = \delta(\mu + \gamma)$ and

$$\sigma(\widetilde{T}) \subset D_{\frac{1+\theta}{2}, \frac{1-\theta}{4}}.$$

As seen above, there exists a C_{00}, (BCP)-operator $\widehat{T}_1$ such that $\sigma(\widehat{T}_1) = \sigma_{le}(\widehat{T}_1) = \mathbb{A}_\theta$, $\|\widehat{T}_1^{-1}\| = 1/\theta$, and $\widehat{T}_1 \overset{h}{\sim} \widetilde{T}_1^{(\omega)}$. Moreover, since $\sigma(\widetilde{T}_2) \subset D_{\frac{1+\theta}{2}, \frac{1-\theta}{4}} \subset \mathbb{A}_\theta^\circ$, it follows from [21, Theorem 8.13] that $\widetilde{T}_2$ is similar to an operator $\widehat{T}_2$ satisfying $\|\widehat{T}_2\| < 1$ and $\|\widehat{T}_2^{-1}\| < 1/\theta$. Thus $\widehat{T} := \widehat{T}_1 \oplus \widehat{T}_2 \overset{h}{\sim} \widetilde{T}_1 \oplus \widetilde{T}_2 = \widetilde{T}$, and since $\widehat{T}$ clearly has all the desired properties as above, the proof is complete. ■

We now generalize [13, Theorem 5.3], and for this purpose we recall the definition of a "universal" block diagonal operator B which is given as follows. For each $n \in \mathbb{N}$ choose a dense sequence $\{B_j^{(n)}\}_{j=1}^{\infty}$ in open unit ball of $\mathcal{L}(\mathbb{C}^n)$, and set $B_\theta = \bigoplus_{n=1}^{\infty} \bigoplus_{j=1}^{\infty} B_j^{(n)}$. It follows easily that B is a C_{00}, (BCP)-contraction satisfying $\sigma(B) = \sigma_{le}(B) = \mathbb{D}^-$.

Theorem 4.2 *Let B be a fixed universal block diagonal operator as defined above, let T be an arbitrary operator in $\mathcal{L}(\mathcal{H}) \setminus (A)$, and let ε be an arbitrary positive number. Then there exists a compact operator $K = K(T, \varepsilon)$ such that:*

(i) $\|K\| < \varepsilon$,
(ii) $B + K$ *is quasidiagonal,*
(iii) $B + K$ *is a* C_{00}, *(BCP)-operator,*
(iv) $\sigma(B + K) = \sigma_{le}(B + K) = \mathbb{D}^-$,
(v) $\mathrm{Hlat}(T) \equiv \mathrm{Hlat}(B + K)$.

Proof If T has property (AHV), the result is [13, Theorem 5.3]. Moreover, a careful reading of that proof shows that if we employ Theorem 4.1 in place of [13, Theorem 4.4] at the beginning of the proof to yield an operator $T_1 \in C_{00} \cap$ (BCP) such that $\sigma(T_1) = \sigma_{le}(T_1) = \mathbb{D}^-$ and $[\delta(T + \gamma 1_{\mathcal{H}})]^{(\omega)} \overset{h}{\sim} T_1$, then the remainder of that proof goes through unchanged, so nothing more need be said. ■

Remark 4.3 Since θ in Theorem 4.1 can be chosen arbitrarily near to 1, a careful reading of the construction involved in the proof of this theorem (and in the proofs of its predecessors in [13, 15, 17]), together with some (BDF)-theory, shows that the folowing peculiar result is also valid.

Corollary 4.4 *Let B be an unweighted bilateral shift of multiplicity one, and let $\varepsilon > 0$ be given arbitrarily. Then for every $T \in \mathcal{L}(\mathcal{H}) \setminus (A)$, $\mathrm{Hlat}(T)$ is isomorphic to an element of the set of hyperlattices $\{\mathrm{Hlat}(B + P) : \|P\| < \varepsilon\}$.*

Remark 4.5 Another careful examination of the construction used in the proof of Theorem 4.1 shows that the annuli $\mathbb{A}_\theta$ appearing there are somewhat arbitrary, and could be replaced by many different dominating open subsets of $\mathbb{D}$ suitable for other purposes.

Remark 4.6 While Theorems 4.1 and 4.2 show that the hyperlattices of certain (BCP)-operators serve as universal models for hyperlattices of nonalgebraic operators, the analogous statement for invariant subspace lattices fails miserably. For example, it follows from well-known facts in the theory of (BCP)-operators that no linearly ordered lattice can be the invariant subspace lattice of any (BCP)-operator. What special class of operators would be a good candidate for producing all (or almost all) invariant subspace lattices of operators in $\mathcal{L}(\mathcal{H})$? In this connection see [13].

5 Nilpotent Operators: Preliminaries

In this section we will collect most of the known facts about the hyperinvariant subspaces of operators in the class (A) considered before, that is, algebraic operators. As already noted earlier, this case reduces immediately to studying the lattice $\mathrm{Hlat}(T)$, where $T \in \mathcal{L}(\mathcal{H})$ satisfies $T^n = 0$ for some integer $n \geq 1$.

We first review the finite-dimensional situation, in which case T is similar to an operator of the form $J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_k}$, where $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ are

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

integers, and J_n denotes the standard $n \times n$ nilpotent Jordan cell, satisfying $J_n e_1 = 0$, $J_n e_j = e_{j-1}$, $1 < j \le n$, in the usual basis of $\mathbb{C}^n$. Any hyperinvariant subspace of an operator of this form has the form $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k$, where each $\mathcal{M}_j$ is invariant for J_{n_j}. Since the invariant subspaces of J_n are totally ordered by inclusion, $\mathcal{M}_j$ is determined by its dimension m_j, which is an arbitrary integer such that $0 \le m_j \le n_j$. More precisely, $\mathcal{M}_j = \ker J_{n_j}^{m_j} = \operatorname{ran} J_{n_j}^{n_j - m_j}$. The space $\mathcal{M}$ is hyperinvariant if and only if $m_i \ge m_j$ and $n_i - m_i \ge n_j - m_j$ whenever $i \le j$. Observe in particular that we must have $m_i = m_j$ whenever $n_i = n_j$ so that (spatial) multiplicity does not affect the labeling of the hyperinvariant subspaces. This result was proved in [11]; a related result about characteristic subgroups in p-groups was proved by Kaplansky [20] (see also [16, Theorem 67.1] and the pertinent remarks in [10]).

It may be worth noting for future reference that this description of hyperinvariant subspaces is a particular case of the following easily verified result.

Lemma 5.1 *Let $(T_i)_{i \in I}$ be a norm-bounded family of operators acting on Hilbert spaces, and let $(\mathcal{M}_i)_{i \in I}$ be a family of closed spaces such that $\mathcal{M}_i$ is invariant for T_i for $i \in I$. Then the space $\bigoplus_{i \in I} \mathcal{M}_i$ is hyperinvariant for $\bigoplus_{i \in I} T_i$ if and only if $\mathcal{M}_i \oplus \mathcal{M}_j$ is hyperinvariant for $T_i \oplus T_j$ for all $i, j \in I$.*

The lattice $\operatorname{Hlat}(T)$ can also be described completely for operators of class C_0 with finite multiplicity, and we will have more to say about this later.

The hyperinvariant subspace $\mathcal{M}$ associated with the integers $m_j \le n_j$ can also be written as

$$\mathcal{M} = \bigvee_{j=1}^{k} (\ker T^{m_j} \cap \operatorname{ran} T^{n_j - m_j}) = \bigcap_{j=1}^{k} (\ker T^{m_j} \vee \operatorname{ran} T^{n_j - m_j}).$$

This fact, also noted in [11], shows that $\operatorname{Hlat}(T)$ is generated as a lattice by the spaces $\ker T^m$ and $\operatorname{ran} T^m$, $m = 0, 1, \dots, n$. The above formula also implies that

$$\operatorname{ran} T^{n_1 - m_1} \subset \mathcal{M} \subset \ker T^{m_1}.$$

In particular, $\operatorname{ran} T^{n-1}$ (resp., $\ker T^{n-1}$) is the smallest (resp., largest) nontrivial hyperinvariant subspace for T, where $n = n_1$ is the order of nilpotency of T. When $n = 1$, we have $T = 0$, and the only hyperinvariant subspaces are the trivial ones. When $n = 2$, there are two possible lattices

- $T = J_2 : \{0\} \subset \ker T = \operatorname{ran} T \subset \mathcal{H}$,
- $T = J_2 \oplus J_1$: $\{0\} \subset \ker T \subset \operatorname{ran} T \subset \mathcal{H}$;

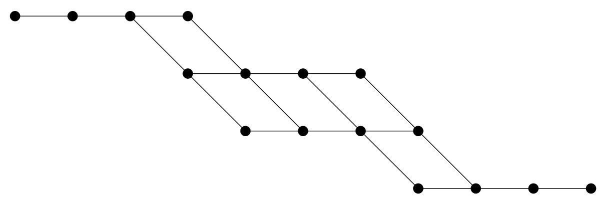
while for $n = 3$ there are four possibilities:

- $T = J_3$: $\{0\} \subset \operatorname{ran} T^2 = \ker T \subset \operatorname{ran} T = \ker T^2 \subset \mathcal{H}$,
- $T = J_3 \oplus J_2$: $\{0\} \subset \operatorname{ran} T^2 \subset \ker T \subset \operatorname{ran} T \subset \ker T^2 \subset \mathcal{H}$,
- $T = J_3 \oplus J_1$: $\{0\} \subset \operatorname{ran} T^2 = \operatorname{ran} T \cap \ker T \subset \operatorname{ran} T$,
  $\ker T \subset \operatorname{ran} T \vee \ker T = \ker T^2 \subset \mathcal{H}$,

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

- $T = J_3 \oplus J_2 \oplus J_1$: $\{0\} \subset \operatorname{ran} T^2 \subset \operatorname{ran} T \cap \ker T \subset \operatorname{ran} T$,
  $\ker T \subset \operatorname{ran} T \vee \ker T \subset \ker T^2 \subset \mathcal{H}$.

Thus we have two totally ordered lattices with four and six elements, respectively, and two which are not totally ordered with six and eight elements. When $n = 4$, there are eight possible lattices. We will illustrate just the largest one, corresponding with $T = J_4 \oplus J_3 \oplus J_2 \oplus J_1$. In the diagram below, and in all later diagrams, the smaller spaces are on the left, and the larger ones on the right.



In general, the operator $J_n \oplus J_{n-1} \oplus \cdots \oplus J_1$ has 2^n hyperinvariant subspaces.

Nilpotent operators on infinite-dimensional Hilbert spaces were completely classified up to quasisimilarity in [1]. In the separable case, the result states that every nilpotent operator is quasisimilar to a direct sum of the form $J_{n_1} \oplus J_{n_2} \oplus \cdots$, where $n_1 \geq n_2 \geq \cdots$; operators of this form will be referred to as Jordan nilpotent operators. The decreasing sequence n_j must, of course, be stationary, so that $n_k = n_{k+1} = \cdots$ for sufficiently large k. The hyperlattice of a Jordan operator is then isomorphic to the lattice of the finite-dimensional operator $J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_k}$, and is therefore completely understood.

It is sometimes useful to deal with more general sums of Jordan cells. A *generalized Jordan operator* is an operator of the form

$$(J_1 \otimes I_{N_1}) \oplus (J_2 \otimes I_{N_2}) \oplus \cdots \oplus (J_n \otimes I_{N_n}),$$

where I_N denotes the identity operator on a Hilbert space of dimension N, with $0 \leq N \leq \aleph_0$. Unlike Jordan operators, generalized Jordan operators are allowed to have more than one cell of infinite multiplicity. While it is possible to have distinct quasisimilar generalized Jordan operators, the situation is different for hyperquasisimilarity.

Proposition 5.2 *If two generalized Jordan operators are hyperquasisimilar, then they are unitarily equivalent.*

Proof Consider two generalized Jordan operators

$$T = (J_1 \otimes I_{N_1}) \oplus (J_2 \oplus I_{N_2}) \oplus \cdots \oplus (J_n \otimes I_{N_n})$$
$$T' = (J_1 \otimes I_{N_1'}) \oplus (J_2 \oplus I_{N_2'}) \oplus \cdots \oplus (J_n \otimes I_{N_n'}),$$

and quasiaffinities X, Y as in the definition of hyperquasisimilarity. We will show first that

$$(X(\ker T \cap \operatorname{ran} T'^{m}))^- = (\ker T' \cap \operatorname{ran} T'^{m})^-$$

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

for every positive integer m. Indeed, it is clear that $Y((\ker T' \cap \operatorname{ran} T'^m))^- \subset ((\ker T \cap \operatorname{ran} T^m))^-$, and therefore

$$\begin{aligned}(\ker T' \cap \operatorname{ran} T'^m)^- &= (XY(\ker T' \cap \operatorname{ran} T'^m)^-)^- \\ &\subset (X(\ker T \cap \operatorname{ran} T^m))^- \\ &\subset (\ker T' \cap \operatorname{ran} T'^m)^-,\end{aligned}$$

where we used the basic property of hyperquasisimilarity. Since X also maps $\mathcal{M}_{m+1} = (\ker T \cap \operatorname{ran} T^{m+1})^-$ into $\mathcal{M}'_{m+1} = (\ker T' \cap \operatorname{ran} T'^{m+1})^-$, X induces an operator with dense range from $\mathcal{M}_m \ominus \mathcal{M}_{m+1}$ to $\mathcal{M}'_m \ominus \mathcal{M}'_{m+1}$. The space $\mathcal{M}_m \ominus \mathcal{M}_{m+1}$ has dimension precisely N_m, so that necessarily $N_m \geq N'_m$ for every m. The proposition follows by symmetry. ∎

D. Herrero [18] showed that the structure of the hyperlattice is not preserved under quasisimilarity. He exhibited an operator T such that $T^3 = 0$ and $\operatorname{Hlat}(T)$ consists of five elements. This provides an example, since the hyperlattices of all Jordan nilpotent operators of order three have 4, 6, or 8 elements. Actually, the simplest example of non-invariance under quasisimilarity is obtained by considering the quasisimilar operators

$$T = J_1 \oplus J_2 \oplus J_2 \oplus \cdots, \quad T' = J_2 \oplus J_2 \oplus \cdots.$$

In this case, $\operatorname{Hlat}(T)$ has four elements and $\operatorname{Hlat}(T')$ has three elements. Note that T is not a Jordan operator according to the definition adopted above, but it is a generalized Jordan operator. However, Herrero's example actually exhibits a hyperlattice which is not realized by a finite-dimensional nilpotent operator of order three. We will see other such lattices later.

The hyperlattice of Herrero's operator is still generated by the subspaces $\ker T^m$ and $(\operatorname{ran} T^m)^-$ for $m = 0, 1, 2, 3$. In fact Herrero conjectured (see especially his review of [3]) that $\operatorname{Hlat}(T)$ is always generated by $\ker T^m$ and $(\operatorname{ran} T^m)^-$ if T is nilpotent and therefore T has a smallest and a largest nontrivial hyperinvariant subspace. The latter statement is in fact true, as shown by the following result of Barraa [3].

Proposition 5.3 **(Barraa)** *Let $T \in \mathcal{L}(\mathcal{H})$ be a nilpotent operator of order n, and $\mathcal{M}$ a hyperinvariant subspace for T. There exists a unique integer k, $0 \leq k \leq n$, such that* $(\operatorname{ran} T^{n-k})^- \subset \mathcal{M} \subset \ker T^k$.

More precisely, assume that T and $\mathcal{M}$ are as above, and choose for each $i = 1, 2, \dots, n$ an integer m_i such that $T^{i-1}\mathcal{M} \subset (\operatorname{ran} T^{m_i})^-$ and $T^{i-1}\mathcal{M} \not\subset (\operatorname{ran} T^{m_i+1})^-$.

Then Barraa proved the following result.

Theorem 5.4 **(Barraa)** *With the preceding notation, we have*

$$\bigvee_{i=1}^{n} (\ker T^i \cap \operatorname{ran} T^{m_i - i+1}) \subset \mathcal{M}.$$

Moreover, the inclusion is an equality in case $\operatorname{ran} T^i$ *is a closed space for all* $i = 1, 2, \dots, n-1$.

The last assertion of the preceding theorem can also be deduced from the following result of Williams [25]. Note however that Barraa's result is actually proved for nilpotent operators on a Banach space, while Williams's result applies only to Hilbert space.

Theorem 5.5 **(Williams)** *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a nilpotent operator such that* T^m *has closed range for every* m*. Then* T *is similar to an orthogonal sum of Jordan cells (which need not be a Jordan operator).*

Corollary 5.6 **(Williams)** *Under the hypotheses of Theorem* 5.5*,* $\mathrm{Hlat}(T)$ *is isomorphic to the hyperlattice of a finite-dimensional nilpotent operator.*

Thus Herrero's full conjecture is verified for operators whose powers have closed ranges. Alas, we will soon see that this conjecture is not generally true, though the following result verifies it when the degree of nilpotency is two.

Theorem 5.7 *Let* $T \in \mathcal{L}(\mathcal{H})$ *be a nilpotent operator of order two. The only nontrivial hyperinvariant subspaces of* T *are* $(\mathrm{ran}\, T)^-$ *and* $\ker T$.

Proof Using the decomposition

$$\mathcal{H} = (\mathrm{ran}\, T)^- \oplus [\ker T \ominus (\mathrm{ran}\, T)^-] \oplus [\mathcal{H} \ominus \ker T],$$

we can write

$$T = \begin{bmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where A is one-to-one and has dense range. Every operator $X \in \{T\}'$ then has the form

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix}$$

relative to this decomposition, where the entries X_{12}, X_{13}, X_{22} and X_{23} are arbitrary bounded operators and $AX_{33} = X_{11}A$.

Consider a nontrivial invariant subspace $\mathcal{M}$ for T. By Proposition 5.3, we have

$$(\mathrm{ran}\, T)^- \subset \mathcal{M} \subset \ker T.$$

Thus we can write $\mathcal{M} = (\mathrm{ran}\, T)^- \oplus (\mathcal{M} \ominus (\mathrm{ran}\, T)^-)$. If the second summand in this orthogonal sum contains a nonzero vector, by applying to this vector operators of the form

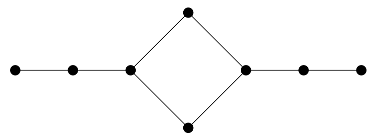
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \{T\}',$$

we obtain all vectors in $\ker T \ominus (\mathrm{ran}\, T)^-$. It follows that $\mathcal{M} \ominus (\mathrm{ran}\, T)^-$ is either zero, or $\ker T \ominus (\mathrm{ran}\, T)^-$, as desired. ■

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

6 Nilpotents of Order Three

Let $T \in \mathcal{L}(\mathcal{H})$ be a nilpotent operator of order three. Herrero [18] conjectured that Hlat(T) consists of the eight subspaces $\{0\}$, $(\operatorname{ran} T^2)^-$, $\ker T \cap (\operatorname{ran} T)^-$, $\ker T$, $(\operatorname{ran} T)^-$, $\ker T \vee \operatorname{ran} T$, $\ker T^2$, and $\mathcal{H}$. When these spaces are distinct, this lattice has the diagram below, with $\ker T$ and $(\operatorname{ran} T)^-$ not comparable.



Observe that the space $(\ker \mathrm{T} \cap \operatorname{ran} \mathrm{T})^-$ does not appear on this list, and neither does its dual version $(\ker T^* \cap \operatorname{ran} T^*)^\perp$. The following result indicates that these two subspaces can in fact be different from the other eight.

Theorem 6.1 *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $T^3 = 0$. Define spaces*

$$\mathcal{H}_1 = (\operatorname{ran} T^2)^-, \quad \mathcal{H}_2 = (\ker \mathrm{T} \cap \operatorname{ran} \mathrm{T})^- \ominus (\operatorname{ran} T^2)^-,$$

$$\mathcal{H}_3 = (\ker T \cap (\operatorname{ran} T)^-) \ominus (\ker \mathrm{T} \cap \operatorname{ran} \mathrm{T})^-, \quad \mathcal{H}_4 = \ker T \ominus (\ker T \cap (\operatorname{ran} T)^-),$$

$$\mathcal{H}_5 = (\ker T \vee \operatorname{ran} T) \ominus \ker T, \quad \mathcal{H}_6 = (\ker T^* \cap \operatorname{ran} T^*)^\perp \ominus (\ker T \vee \operatorname{ran} T,$$

$$\mathcal{H}_7 = \ker T^2 \ominus (\ker T^* \cap \operatorname{ran} T^*)^\perp, \quad \mathcal{H}_8 = \mathcal{H} \ominus \ker T^2$$

The matrix of the operator T relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_8$ has the form

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & T_{15} & T_{16} & T_{17} & T_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & T_{27} & T_{28} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{38} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{48} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where T_{15}, T_{58} and T_{27} are quasiaffinities,

$$T_{16}^{-1}(\operatorname{ran} T_{15}) = \{0\}, \quad T_{38}^{*-1}(\operatorname{ran} T_{58}^* + \operatorname{ran} T_{48}^*) = \{0\},$$

and $T_{48}^{*-1}(\operatorname{ran} T_{58}^*)$ is dense in $\mathcal{H}_4$.

Conversely, assume that $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_8$ are Hilbert spaces, and $T_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ are bounded linear operators satisfying these conditions. Then the 8×8 matrix T written *above satisfies* $T^3 = 0$, *and*

$$(\operatorname{ran} T^2)^- = \mathcal{H}_1, \quad (\ker \mathrm{T} \cap \operatorname{ran} \mathrm{T})^- = \mathcal{H}_1 \oplus \mathcal{H}_2,$$
$$\ker T \cap (\operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \quad \ker T = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$
$$\ker T \vee \operatorname{ran} T = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5,$$
$$(\ker T^* \cap \operatorname{ran} T^*)^\perp = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 \oplus \mathcal{H}_6,$$
$$\ker T^2 = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 \oplus \mathcal{H}_6 \oplus \mathcal{H}_7.$$

When T is given by such a matrix, we have

$$(\operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \left(\operatorname{ran} \begin{bmatrix} T_{48} \\ T_{58} \end{bmatrix}\right)^- \oplus \{0\} \oplus \{0\} \oplus \{0\}.$$

Proof The first four columns of the matrix of T are clearly zero as they correspond with the restriction of T to its kernel. Similarly, the last three rows are zero because the range of T is contained in $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$. Next we see that

$$T\mathcal{H}_5 \subset T((\operatorname{ran} T)^-) \subset (\operatorname{ran} T^2)^- = \mathcal{H}_1,$$

so that $T_{i5} = 0$ for $i \geq 2$. We also have $T^*(\ker T^{*2}) \subset \ker T^* \cap \operatorname{ran} T^*$, and by passing to orthogonal complements we see that

$$T((\ker T^* \cap \operatorname{ran} T^*)^\perp) \subset (\ker T^{*2})^\perp = (\operatorname{ran} T^2)^- = \mathcal{H}_1.$$

Since $\mathcal{H}_6 \subset (\ker T^* \cap \operatorname{ran} T^*)^\perp$, we must have $T_{i6} = 0$ for $i \geq 2$ as well. Similarly,

$$T\mathcal{H}_7 \subset T(\ker T^2) \subset \ker T \cap \operatorname{ran} T \subset \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and thus $T_{i7} = 0$ for $i \geq 2$.

We have established that the required 55 entries in the matrix of T are equal to zero, and we now focus on the remaining nine entries. To begin, we note that any vector $h_5 \in \mathcal{H}_5$ such that $T_{15}h_5 = 0$ will belong to $\ker T = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$, and must therefore be zero. Also, note that T^2 has only one nonzero entry, namely

$$T_{15}T_{58} = T^2|(\mathcal{H} \ominus \ker T^2) \to (\operatorname{ran} T^2)^-,$$

which must be one-to-one with dense range. We conclude that T_{15} has dense range and T_{58} is one-to-one. To see that T_{58} is a quasiaffinity as well, observe that

$$\operatorname{ran} T \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \operatorname{ran} T_{58},$$

so that

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 = \ker T \vee \operatorname{ran} T \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus (\operatorname{ran} T_{58})^-.$$

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

Let us note at this point that every matrix of the given form, with T_{15} and T_{58} quasiaffinities, represents an operator T such that $T^3 = 0$, $(\operatorname{ran} T^2)^- = \mathcal{H}_1$, and $\ker T^2 = \mathcal{H}_8^\perp$. To complete the proof, we will assume from this point on just that T is such a matrix, with T_{15}, T_{58} quasiaffinities, and we will derive necessary and sufficient conditions on the matrix entries so that the remaining five hyperinvariant subspaces are given by the correct formulas.

We begin with $(\ker T \cap \operatorname{ran} T)^-$. Consider a vector $h = Tk \in \ker T$. Thus we have the following equations:

$$h_1 = T_{15}k_5 + T_{16}k_6 + T_{17}k_7 + T_{18}k_8, \quad h_2 = T_{27}k_7 + T_{28}k_8,$$
$$h_3 = T_{38}k_8, \quad h_4 = T_{48}k_8, \quad h_5 = T_{58}k_8,$$
$$h_6 = 0, \quad h_7 = 0, \quad h_8 = 0,$$

and

$$T_{15}h_5 + T_{16}h_6 + T_{17}h_7 + T_{18}h_8 = 0, \quad T_{27}h_7 + T_{28}h_8 = 0,$$
$$T_{38}h_8 = 0, \quad T_{48}h_8 = 0, \quad T_{58}h_8 = 0.$$

The last five equations reduce to $T_{15}h_5 = 0$, so that $h_5 = 0$ as well. Then $T_{58}k_8 = h_5 = 0$ implies $k_8 = 0$, and the formulas now give

$$h_1 = T_{15}k_5 + T_{16}k_6 + T_{17}k_7, \quad h_2 = T_{27}k_7,$$
$$h_3 = 0, \quad h_4 = 0, \quad h_5 = 0 \quad h_6 = 0, \quad h_7 = 0, h_8 = 0,$$

where k_5, k_6, k_7 are now arbitrary. Since T_{15} has dense range, we see that

$$(\ker T \cap \operatorname{ran} T)^- = \mathcal{H}_1 \oplus (\operatorname{ran} T_{27})^-.$$

We conclude that the equality $(\ker T \cap \operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2$ holds if and only if T_{27} has dense range. We next focus on $\ker T$, which consists of vectors h satisfying

$$T_{15}h_5 + T_{16}h_6 + T_{17}h_7 + T_{18}h_8 = 0, \quad T_{27}h_7 + T_{28}h_8 = 0,$$
$$T_{38}h_8 = 0, \quad T_{48}h_8 = 0, \quad T_{58}h_8 = 0.$$

Since T_{58} is one-to-one, these equations reduce to

$$T_{15}h_5 + T_{16}h_6 + T_{17}h_7 = 0, \quad T_{27}h_7 = 0, h_8 = 0.$$

Thus the equality $\ker T = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ occurs if and only if the operator

$$\begin{bmatrix} T_{15} & T_{16} & T_{17} \\ 0 & 0 & T_{27} \end{bmatrix}$$

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

is one-to-one. From this point on we shall assume that this matrix is one-to-one, and that T_{27} has dense range. Under these additional assumptions it is clear that

$$(\operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \left(\operatorname{ran}\begin{bmatrix} T_{38} \\ T_{48} \\ T_{58}\end{bmatrix}\right)^-,$$

so that

$$\ker T \cap (\operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \left[(\mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \{0\}) \cap \left(\operatorname{ran}\begin{bmatrix} T_{38} \\ T_{48} \\ T_{58}\end{bmatrix}\right)^-\right].$$

We conclude that the equality $\ker T \cap (\operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ is equivalent to

$$(\mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \{0\}) \cap \left(\operatorname{ran}\begin{bmatrix} T_{38} \\ T_{48} \\ T_{58}\end{bmatrix}\right)^- = \mathcal{H}_3 \oplus \{0\} \oplus \{0\}.$$

Taking orthogonal complements, this condition is equivalent to saying that the space $\ker[T_{38}^*, T_{48}^*, T_{58}^*] + \{0\} \oplus \{0\} \oplus \mathcal{H}_5$ is a dense subspace of $\{0\} \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$. This amounts to two conditions. First, $\ker[T_{38}^*, T_{48}^*, T_{58}^*]$ must be contained in $\{0\} \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$, and this means precisely that $T_{38}^{*-1}(\operatorname{ran} T_{48}^* + \operatorname{ran} T_{58}^*) = \{0\}$. Second, the collection of vectors of the form $h_4 \oplus h_5$ such that $T_{48}^* h_4 + T_{58}^* h_5 = 0$ must be dense in $\mathcal{H}_4 \oplus \mathcal{H}_5$ modulo $\{0\} \oplus \mathcal{H}_5$, and this means precisely that $T_{48}^{*-1}(\operatorname{ran} T_{58}^*)$ is dense in $\mathcal{H}_4$.

From this point on we also assume that these last two conditions are satisfied. Note that we can calculate $(\operatorname{ran} T)^-$ at this point. Indeed, we already know that this space contains $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, so that the formula in the statement becomes obvious. It now follows easily that

$$\ker T \vee \operatorname{ran} T = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus (\operatorname{ran} T_{58})^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5.$$

To finish the job, we must find equivalent conditions for the equality

$$(\ker T^* \cap \operatorname{ran} T^*)^\perp = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 \oplus \mathcal{H}_6,$$

which is the same as $(\ker T^* \cap \operatorname{ran} T^*)^- = \mathcal{H}_7 \oplus \mathcal{H}_8$. Consider then a vector $h = T^* k \in \ker T^*$. Thus we have

$$h_1 = 0, \quad h_2 = 0, \quad h_3 = 0, \quad h_4 = 0, \quad h_5 = T_{15}^* k_1, \quad h_6 = T_{16}^* k_1,$$
$$h_7 = T_{17}^* k_1 + T_{27}^* k_2, \quad h_8 = T_{18}^* k_1 + T_{28}^* k_2 + T_{38}^* k_3 + T_{48}^* k_4 + T_{58}^* k_5,$$

and

$$T_{15}^* h_1 = 0, \quad T_{16}^* h_1 = 0, \quad T_{17}^* h_1 + T_{27}^* h_2 = 0,$$
$$T_{18}^* h_1 + T_{28}^* h_2 + T_{38}^* h_3 + T_{48}^* h_4 + T_{58}^* h_5 = 0.$$

Since h_1, h_2, h_3, h_4 are zero, the last equation yields $T_{58}^* h_5 = 0$, so that $h_5 = 0$ as well. Therefore $T_{15}^* k_1 = h_5 = 0$, and we deduce that $k_1 = 0$. Thus the general form of a vector $h \in \ker T^* \cap \operatorname{ran} T^*$ is

$$0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus (T_{27}^* k_2) \oplus (T_{28}^* k_2 + T_{38}^* k_3 + T_{48}^* k_4 + T_{58}^* k_5),$$

with k_2, k_3, k_4, k_5 arbitrary. Now, T_{58}^* has dense range, and therefore

$$(\ker T^* \cap \operatorname{ran} T^*)^- = (\operatorname{ran} T_{27}^*)^- \oplus \mathcal{H}_8.$$

Thus the condition $(\ker T^* \cap \operatorname{ran} T^*)^- = \mathcal{H}_7 \oplus \mathcal{H}_8$ is equivalent to the fact that T_{27} is one-to-one.

Finally, we need to to remark that, when T_{27} is one-to-one, the operator

$$\begin{bmatrix} T_{15} & T_{16} & T_{17} \\ 0 & 0 & T_{27} \end{bmatrix}$$

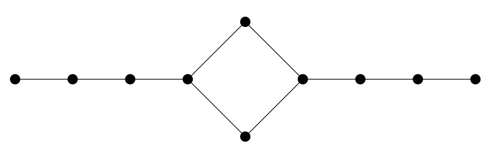
is one-to-one if and only if $[T_{15}, T_{16}]$ is one-to-one. This last condition is the same as $T_{16}^{-1}(\operatorname{ran} T_{15}) = \{0\}$ because T_{15} is also one-to-one. ■

The preceding statement takes a more symmetric form if we use a shorter chain of hyperinvariant subspaces, excluding the space $\ker T$. This amounts to regarding $\mathcal{H}_3 \oplus \mathcal{H}_4$ as a single space. The interested reader should have no difficulty formulating this result.

It is easy to see that operators T_{ij} as above can always be constructed if $\mathcal{H}_1$, $\mathcal{H}_5$, $\mathcal{H}_8$ have the same infinite dimension, $\mathcal{H}_2$, $\mathcal{H}_7$ have the same dimension (finite or infinite), and $\mathcal{H}_3$, $\mathcal{H}_6$ have smaller dimension than $\mathcal{H}_1$ (and $\mathcal{H}_8$). If one takes $T_{48} = 0$, then

$$(\operatorname{ran} T)^- = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \{0\} \oplus \mathcal{H}_5 \oplus \{0\} \oplus \{0\} \oplus \{0\}.$$

We see thus that it is possible for the lattice of T to contain these ten distinct, canonically defined, hyperinvariant subspaces. The lattice they form is pictured below.



One may ask whether a weaker form of Herrero's conjecture is true: are these ten spaces the only hyperinvariant subspaces of an operator T such that $T^3 = 0$? We will see that the answer to this question is also negative. This general 8×8 matrix is, however, not suitable for determining the commutant of T. We will instead focus on a simpler class of operators. Consider two operators $A, B \in \mathcal{L}(\mathcal{H})$ such that

(i) A and B are one-to-one;
(ii) AB has dense range;
(iii) $\mathcal{H} \ominus B\mathcal{H}$ has dimension one.

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

One possible choice is $A = I + S^*$, $B = S$, where S is a unilateral shift of multiplicity one. The operator $T_0 \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H})$ defined by the matrix

$$T_0 = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}$$

is nilpotent of order three. To identify the 8×8 matrix for T_0, one should note that A plays the role of $[T_{15}, T_{16}]$, and B plays the role of $\left[\begin{smallmatrix} T_{48} \\ T_{58} \end{smallmatrix}\right]$, with $T_{48} = 0$, and several of the spaces $\mathcal{H}_j$ are equal to zero.

Lemma 6.2 *The only nontrivial hyperinvariant subspaces of T_0 are*

$$\ker T_0 = (\operatorname{ran} T_0^2)^- = \mathcal{H} \oplus \{0\} \oplus \{0\},$$
$$(\operatorname{ran} T_0)^- = \mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\}, \quad \ker T_0^2 = \mathcal{H} \oplus \mathcal{H} \oplus \{0\}.$$

Proof Let $\mathcal{M}$ be a nontrivial hyperinvariant subspace for T_0. By Proposition 5.3 we have either $(\operatorname{ran} T_0)^- \subset \mathcal{M} \subset \ker T_0^2$, or $(\operatorname{ran} T_0^2)^- \subset \mathcal{M} \subset \ker T_0$. In the first case we have either $\mathcal{M} = \ker T_0^2$, or $\mathcal{M} = (\operatorname{ran} T_0)^-$, because these are the only linear spaces satisfying these inclusions. In the second case we have $\mathcal{M} = (\operatorname{ran} T_0^2)^- = \ker T_0$. ■

It may be worth noting at this point that a finite-dimensional operator

$$T = J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_k}$$

such that $n_1 \geq n_2 \geq \cdots \geq n_k$, satisfying an equation of the form $\operatorname{ran} T^m = \ker T^{n_1 - m}$ with $1 \leq m \leq n_1 - 1$ is quite special. Indeed, in this case one sees immediately that $n_1 = n_2 = \cdots = n_k$, so the hyperlattice of T consists of the $n_1 + 1$ elements $\{0\}, \ker T^{n_1 - 1}, \ker T^{n_1 - 2}, \ldots, \ker T, \mathcal{H}$.

Our operator T_0 exhibits another interesting property.

Lemma 6.3 *An operator $X\colon \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \to \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ satisfies the equation $XT_0 = J_2 X$ if and only if*

$$X = \begin{bmatrix} 0 & X_{12} & X_{13} \\ 0 & 0 & X_{23} \end{bmatrix},$$

with $X_{12}B = X_{23}$.

Proof If we write

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix},$$

the commutation relation in the statement amounts to

$$\begin{bmatrix} 0 & X_{11}A & X_{12}B \\ 0 & X_{21}A & X_{22}B \end{bmatrix} = \begin{bmatrix} X_{21} & X_{22} & X_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly then $X_{21} = 0$ and $X_{11}AB = X_{22}B = 0$. Since AB has dense range, $X_{11} = 0$, and then $X_{22} = X_{11}A = 0$ as well. ■

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

In the following result we denote by e_1, e_2 the standard basis in $\mathbb{C}^2$, so that $J_2 e_1 = 0$ and $J_2 e_2 = e_1$.

Proposition 6.4 *The only nontrivial hyperinvariant subspaces of the operator $T_1 = T_0 \oplus J_2$ are, in increasing order,*

$$(\operatorname{ran} T_1^2)^- = (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \{0\},$$
$$\ker T_1 = (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$(\operatorname{ran} T_1)^- = (H \oplus (B\mathcal{H})^- \oplus \{0\}) \oplus \mathbb{C}e_1, (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$\ker T_1^2 = (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}^2.$$

In particular, $\operatorname{Hlat}(T_1)$ *contains one space which is not among the ten canonical hyperinvariant ones.*

Proof Let $\mathcal{M}$ be a nontrivial hyperinvariant subspace for T_1, so that either

$$(\operatorname{ran} T_1^2)^- \subset \mathcal{M} \subset \ker T_1, \quad \text{or} \quad (\operatorname{ran} T_1)^- \subset M \subset \ker T_1^2.$$

The first possibility yields only two subspaces because $\ker T_1 \ominus (\operatorname{ran} T_1^2)^-$ has dimension one. In the second case we must have $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\} \subset \mathcal{M}_1 \subset \mathcal{H} \oplus \mathcal{H} \oplus \{0\}$ and $\mathbb{C}e_1 \subset \mathcal{M}_2 \subset \mathbb{C}^2$, which gives the following four possibilities for $\mathcal{M}$:

$$(\operatorname{ran} T_1)^- = (\mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$(\mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\}) \oplus \mathbb{C}^2,$$
$$(\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$\ker T_1^2 = (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}^2.$$

It is easy to see that $(\mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\}) \oplus \mathbb{C}^2$ is not hyperinvariant by noting that operators of the form

$$Y = \begin{bmatrix} AY_{22} & 0 \\ 0 & Y_{22} \\ 0 & 0 \end{bmatrix} : \mathbb{C}^2 \to \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$$

satisfy the equation $YJ_2 = T_0 Y$. On the other hand, $(\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}e_1$ is hyperinvariant. Indeed, any operator X such that $XT_0 = J_2 X$ satisfies $X(\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \subset \mathbb{C}e_1$ by the preceding lemma, and any operator Y such that $YJ_2 = T_0 Y$ satisfies

$$Ye_1 \in Y \ker J_2 \subset \ker T_0 \subset \mathcal{H} \oplus \mathcal{H} \oplus \{0\}. \qquad \blacksquare$$

As we noted earlier, a finite-dimensional nilpotent operator of order three has at most eight hyperinvariant subspaces.

Proposition 6.5 *The operator* $T_2 = T_0 \oplus J_2 \oplus J_1$ *has exactly nine hyperinvariant subspaces, and* $\operatorname{Hlat}(T_2)$ *is not self-dual.*

Proof The hyperinvariant subspaces of T_2 are of the form $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, with $\mathcal{M}_1 \in \operatorname{Hlat}(T_1)$ and $\mathcal{M}_2 \in \{\{0\}, \mathbb{C}\}$. We must also have either

$$\begin{aligned}(\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \{0\} \oplus \{0\} = (\operatorname{ran} T_2^2)^- \subset \mathcal{M} \subset \ker T_2 \\ = (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \mathbb{C},\end{aligned}$$

or

$$\begin{aligned}(\mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \{0\} = (\operatorname{ran} T_2)^- \subset \mathcal{M} \subset \ker T_2^2 \\ = (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}^2 \oplus \mathbb{C}.\end{aligned}$$

Besides the two extremes $(\operatorname{ran} T_2^2)^-$ and $\ker T_2$, the first set of inclusions is also satisfied by

$$(\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \{0\} = (\operatorname{ran} T_2)^- \cap \ker T_2,$$

and $(\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}$ which is not hyperinvariant because $\{0\} \oplus \mathbb{C}$ is not hyperinvariant for $J_2 \oplus J_1$. The second set of inclusions yields the four possibilities

$$\begin{gathered}(\operatorname{ran} T_2)^-, \quad \ker T_2^2, \\ (\mathcal{H} \oplus (B\mathcal{H})^- \oplus 0) \oplus \mathbb{C}e_1 \oplus \mathbb{C} = (\ker T_2) \vee (\operatorname{ran} T_2)^-, \\ (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \mathbb{C}.\end{gathered}$$

It is easy to see that this last space is in fact hyperinvariant. ∎

Little effort is now required to exhibit one more lattice in our collection.

Proposition 6.6 *The operator* $T_3 = T_0^* \oplus T_0 \oplus J_2 \oplus J_1$ *has exactly ten hyperinvariant subspaces, and* $\operatorname{Hlat}(T_3)$ *is self-dual.*

Proof We use, as before, the fact that a nontrivial hyperinvariant subspace $\mathcal{M}$ for T_3 satisfies either $(\operatorname{ran} T_3^2)^- \subset \mathcal{M} \subset \ker T_3$, or $(\operatorname{ran} T_3)^- \subset \mathcal{M} \subset \ker T_3^2$, and it must also be a sum $\mathcal{M}_1 \oplus \mathcal{M}_2$, with $\mathcal{M}_1 \in \operatorname{Hlat}(T_0^*)$ and $\mathcal{M}_2 \in \operatorname{Hlat}(T_2)$. Thus $\mathcal{M}_1$ must be one of the five spaces

$$\{0\}\oplus\{0\}\oplus\{0\} \subset \{0\}\oplus\{0\}\oplus\mathcal{H} \subset \{0\}\oplus(B\mathcal{H})^\perp\oplus\mathcal{H} \subset \{0\}\oplus\mathcal{H}\oplus\mathcal{H} \subset \mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H},$$

while $\mathcal{M}_2$ must be among the nine subspaces described in the preceding proposition. Now,

$$\begin{aligned}(\operatorname{ran} T_3^2)^- &= (\{0\} \oplus \{0\} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \{0\} \oplus \{0\}, \\ \ker T_3 &= (\{0\} \oplus (B\mathcal{H})^\perp \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \mathbb{C},\end{aligned}$$

leaving the following four possibilities for the intermediate spaces:

$$(\{0\} \oplus (B\mathcal{H})^{\perp} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \{0\} = (\ker T_3) \vee \operatorname{ran} T_3,$$
$$(\{0\} \oplus \{0\} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \{0\},$$
$$(\{0\} \oplus \{0\} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \mathbb{C},$$
$$(\{0\} \oplus (B\mathcal{H})^{\perp} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \{0\} \oplus \{0\}.$$

The third of these two spaces is not hyperinvariant because there is an operator Z so that $ZJ_1 = T_0^* J$ and the range of Z generates $\{0\} \oplus (B\mathcal{H})^{\perp} \oplus \{0\}$. The fourth is not hyperinvariant because there is W such that $WT_0^* = J_2 W$ and the range of W is $\mathbb{C}e_1$. A similar analysis (or an appeal to the self-duality of $\operatorname{Hlat}(T_3)$) yields the remaining four nontrivial hyperinvariant subspaces

$$(\operatorname{ran} T_3)^{-} = (\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus (B\mathcal{H})^{-} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \{0\},$$
$$\ker T_3 \vee \operatorname{ran} T_3 = (\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus (B\mathcal{H})^{-} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \mathbb{C},$$
$$(\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}e_1 \oplus \mathbb{C},$$
$$\ker T_3^2 = (\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}^2 \oplus \mathbb{C}.$$

The hyperlattice is obviously self-dual because T_3 and T_3^* are unitarily equivalent. ∎

Let us note here that the hyperlattice of T_3 contains two noncanonical spaces. Also, the interested reader may verify that $\operatorname{Hlat}(T_0^* \oplus T_0 \oplus J_2)$ is a totally ordered lattice of eight elements: $\{0\}$,

$$(\operatorname{ran}(T_0^* \oplus T_0 \oplus J_2)^2)^{-} = (\{0\} \oplus \{0\} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \{0\},$$
$$(\{0\} \oplus \{0\} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$\ker(T_0^* \oplus T_0 \oplus J_2) = (\{0\} \oplus (B\mathcal{H})^{\perp} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \{0\} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$(\operatorname{ran}(T_0^* \oplus T_0 \oplus J_2))^{-} = (\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus (B\mathcal{H})^{-} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$(\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}e_1,$$
$$\ker(T_0^* \oplus T_0 \oplus J_2)^2 = (\{0\} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H} \oplus \{0\}) \oplus \mathbb{C}^2,$$

and the whole space $(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}) \oplus \mathbb{C}^2$.

The record number of hyperinvariant subspaces for a nilpotent of order three is ten so far. In order to surpass this record we need to be more specific about the operators A and B used in the construction of T_0. Assume therefore that the Hilbert space $\mathcal{H}$ is ℓ^2, $(e_j)_{j=0}^{\infty}$ denotes the standard orthonormal basis of ℓ^2, and $\alpha \in (1, 2)$. We define operators A_α, B on ℓ^2 by requiring that $Be_j = e_{j+1}$ for all j, $A_\alpha e_j = (1/2^j)e_{j-1}$ for $j \geq 1$, and $A_\alpha e_0 = x_\alpha$, where $x_\alpha = (1, \alpha^{-1}, \alpha^{-2}, \dots) = \sum_{j=0}^{\infty} \alpha^{-j} e_j$. It is easy to verify that A_α, B are one-to-one, $A_\alpha B$ has dense range, and B has range of

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

codimension one. Therefore the operator

$$T_\alpha = \begin{bmatrix} 0 & A_\alpha & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}$$

has just five hyperinvariant subspaces:

$$\{0\}, \quad (\operatorname{ran} T_\alpha^2)^- = \ker T_\alpha,$$
$$\mathcal{K}_0 = (\operatorname{ran} T_\alpha)^- = \ell^2 \oplus B\ell^2 \oplus \{0\},$$
$$\mathcal{K}_1 = \ker T_\alpha^2 = \ell^2 \oplus \ell^2 \oplus \{0\},$$

and the whole space $\ell^2 \oplus \ell^2 \oplus \ell^2$.

In order to study the hyperlattices of direct sums of operators of the form T_α, we need an interpolation result. For each $\alpha \in (1,2]$ denote by $\mathcal{H}_\alpha$ the dense linear manifold in ℓ^2 consisting of those sequences $(\xi_j)_{j=0}^\infty$ such that $\sum_{j=0}^\infty \alpha^{2j}|\xi_j|^2 < \infty$. More generally, assume that $\varphi\colon [0,1] \to [0,\infty)$ is a continuous, increasing, concave function such that $\varphi(0) = 0$. We denote by $\mathcal{H}_\varphi$ the collection of those sequences $(\xi_j)_{j=0}^\infty$ such that $\sum_{j=1}^\infty \varphi(2^{-2j})^{-1}|\xi_j|^2 < \infty$. The linear manifold $\mathcal{H}_\alpha$ corresponds with the concave function $\varphi_\alpha(t) = t^{\log_2 \alpha}$. The following result was proved by Peetre [22,23] (see also[12,14] for related results and norm estimates).

Lemma 6.7 *Let $X \in \mathcal{L}(\ell^2)$ be an operator such that $X\mathcal{H}_2 \subset \mathcal{H}_2$. Then we also have $X\mathcal{H}_\varphi \subset \mathcal{H}_\varphi$ for all continuous, increasing, concave functions $\varphi\colon [0,1] \to [0,\infty)$ such that $\varphi(0) = 0$. In particular, $X\mathcal{H}_\alpha \subset \mathcal{H}_\alpha$ for $\alpha \in (1,2]$.*

Proposition 6.8 *Fix two numbers $\alpha, \alpha' \in (1,2)$ such that $\alpha < \alpha'$. The nontrivial hyperinvariant subspaces of the operator $T = T_\alpha \oplus T_{\alpha'}$ are, in increasing order, $(\operatorname{ran} T^2)^- = \ker T$, $(\operatorname{ran} T)^- = \mathcal{K}_0 \oplus \mathcal{K}_0$, $\mathcal{K}_0 \oplus \mathcal{K}_1$, and $\ker T^2 = \mathcal{K}_1 \oplus \mathcal{K}_1$.*

Proof Since $(\operatorname{ran} T^2)^- = \ker T$, there are no other nontrivial hyperinvariant subspaces contained in $\ker T$. The remaining nontrivial spaces $\mathcal{M}$ in $\operatorname{Hlat}(T)$ must satisfy

$$(\operatorname{ran} T)^- = \mathcal{K}_0 \oplus \mathcal{K}_0 \subset M \subset \ker T^2 = \mathcal{K}_1 \oplus \mathcal{K}_1,$$

which yields the two possibilities $\mathcal{K}_0 \oplus \mathcal{K}_1$ and $\mathcal{K}_1 \oplus \mathcal{K}_0$. To show that the second of these is not hyperinvariant it suffices to construct an operator X such that $XT_\alpha = T_{\alpha'}X$ and $X\mathcal{K}_1 \not\subset \mathcal{K}_0$. One such operator is given by

$$X = \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{bmatrix},$$

where $X_{11}e_j = X_{33}e_j = (\alpha/\alpha')^j e_j$, for all $j \geq 0$, and $X_{22}e_j = (\alpha/\alpha')^{j-1}e_j$ for $j \geq 1$, and $X_{22}e_0 = e_0$. (Observe that $X_{11}x_\alpha = x_{\alpha'}$.) Finally, to show that $\mathcal{K}_0 \oplus \mathcal{K}_1$

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

is hyperinvariant, it suffices to show that $X\mathcal{K}_1 \subset \mathcal{K}_0$ for any bounded operator X satisfying $XT_{\alpha'} = T_\alpha X$. The operators X satisfying this equation have the form

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix},$$

where $X_{11}A_{\alpha'} = A_\alpha X_{22}$, $X_{22}B = BX_{33}$, $X_{12}B = BX_{23}$, and X_{13} is an arbitrary bounded operator on ℓ^2. These equations imply that $X_{11}A_{\alpha'}B = A_\alpha X_{22}B = A_\alpha BX_{33}$, so that X_{11} must leave invariant the common range of $A_\alpha B$ and $A_{\alpha'}B$; this range is precisely the space $\mathcal{H}_2$ considered above. Since $\alpha' > \alpha$, the vector $x_{\alpha'}$ belongs to $\mathcal{H}_\alpha$, and the preceding lemma implies that $X_{11}x_{\alpha'}$ belongs to $\mathcal{H}_\alpha$ as well. On the other hand, the equality $X_{11}x_{\alpha'} = X_{11}A_{\alpha'}e_0 = A_\alpha X_{11}e_0$ shows that $X_{11}x_{\alpha'}$ belongs to the range of A_α, and this range is the linear space generated by $\mathcal{H}_2$ and the vector x_α. In other words, we can write $X_{11}x_{\alpha'} = v_1 + \lambda x_\alpha$ with $v_1 \in \mathcal{H}_2$ and $\lambda \in \mathbb{C}$. If the scalar λ were not zero, we would conclude that x_α itself belongs to $\mathcal{H}_\alpha$, and this is clearly not the case. We conclude that $X_{11}x_{\alpha'}$ belongs to $\mathcal{H}_2$, and therefore $X_{11}x_{\alpha'} = A_\alpha By$ for some $y \in \ell^2$. Thus $A_\alpha X_{22}e_0 = X_{11}A_{\alpha'}e_0 = X_{11}x_{\alpha'} = A_\alpha By$, showing that $X_{22}e_0 = By$ belongs to the range of B. Now, ℓ^2 is the linear span of $B\ell^2$ and e_0, and the equation $X_{22}B = BX_{33}$ shows that X_{22} leaves the range of B invariant. We conclude that $X_{22}\ell^2$ is contained in $B\ell^2$. This implies immediately the desired conclusion that $X\mathcal{K}_1 \subset \mathcal{K}_0$. ■

We can now construct nilpotent operators of order three with rather large hyperlattices. Consider indeed an arbitrary set $S \subset (1.5, 2)$, and construct the operator $T_S = \bigoplus_{\alpha\in S} T_\alpha$. Observe that the norms of the direct summands are bounded since α does not come too close to 1. Given a function $f\colon S \to \{0, 1\}$, we will consider the space $\mathcal{K}_f = \bigoplus_{\alpha\in S} \mathcal{K}_{f(\alpha)}$, where the spaces $\mathcal{K}_0, \mathcal{K}_1$ are the hyperinvariant subspaces of T_α considered above. Observe that $\mathcal{K}_f = (\operatorname{ran} T_S)^-$ if $f = 0$, and $\mathcal{K}_f = \ker T_S^2$ if $f = 1$. We can now state the following result, whose proof is an immediate consequence of the preceding proposition and of Lemma 5.1.

Theorem 6.9 *Let $S \subset (1.5, 2)$ be an arbitrary set, and define $T_S = \bigoplus_{\alpha\in S} T_\alpha$. The only nontrivial hyperinvariant subspaces of T_S are $(\operatorname{ran} T_S^2)^- = \ker T_S$ and the spaces $\mathcal{K}_f$, where $f\colon S \to \{0, 1\}$ is a nondecreasing function. The lattice* Hlat(T) *is totally ordered.*

One can now get many examples of lattices by choosing various sets S. If S is a finite set with n elements, there are precisely $n + 1$ nondecreasing functions $f\colon S \to \{0, 1\}$, so that Hlat(T) will have precisely $n + 4$ elements (counting the trivial ones). If S is well-ordered, we obtain countably many hyperinvariant subspaces. Since every countable well-ordering type can be realized as a subset of $(1.5, 1)$, it follows that we obtain $\aleph_1$ mutually nonisomorphic lattices of this type. We obtain another $\aleph_1$ lattices by considering sets S which are well-ordered relative to reverse inequality. If S is dense in $(1.5, 2)$, we obtain a continuum of hyperinvariant subspaces. Still more lattices can be obtained by considering direct sums of operators of the form T_S, T_S^*, J_2, and J_1. The interested reader will be able to determine these lattices using techniques illustrated earlier in this section.

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

We will explore a few more examples which yield hyperlattices with more complicated structure. We start by constructing a family of concave functions.

Lemma 6.10 *Construct a function* $\varphi\colon [0,1/2] \to [0,\infty)$ *such that* $\varphi(0) = 0$, $t^{1/2} \leq \varphi(t) \leq t^{1/4}$ *for* $t = 2^{-4^n}$, $n = 0, 1, 2 \dots$, *and* φ *is obtained by linear interpolation at all other points. Then* φ *is concave.*

Proof Setting $t_n = 2^{-4^n}$, it suffices to verify the inequalities

$$\frac{t_n^{1/4} - t_{n+1}^{1/2}}{t_n - t_{n+1}} \leq \frac{t_{n+1}^{1/2} - t_{n+2}^{1/4}}{t_{n+1} - t_{n+2}},$$

and this is easily done. ■

Now we construct a sequence of functions φ_n in the following way. Consider a sequence of pairwise disjoint infinite sets S_n of natural numbers, and set $\varphi_n(t_k) = t_k^{1/2}$ for $k \in S_n$, $\varphi_n(t_k) = t_k^{1/4}$ for $k \notin S_n$. Extend φ_n by linear interpolation to $[0,1/2]$, and extend it as a concave function on $[0,1]$ making it, for instance, constant on $[1/2,1]$. Also define vectors $x_n = (\xi_j^{(n)})_{j=1}^{\infty} \in \ell^2$ by setting $\xi_j^{(n)} = \varphi_n(2^{-2j})$ if $2j = 2^{4^n}$ for some $n \in S_n$, and $\xi_j^{(n)} = 0$ for all other values of j. The important property of these vectors is that x_n belongs to $\mathcal{H}_{\varphi_m}$ if and only if $n = m$.

Now define operators A_n, B on ℓ^2 by requiring that $Be_j = e_{j+1}$ for all j, $A_n e_j = 2^{-j} e_j$ for $j \geq 1$, and $A_n e_0 = x_n$. As seen before, the only nontrivial hyperinvariant subspaces of the operator

$$T_n = \begin{bmatrix} 0 & A_n & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}$$

are $\ker T_n$, $\mathcal{K}_0 = \ell^2 \oplus B\ell^2 \oplus \{0\}$, and $\mathcal{K}_1 = \ell^2 \oplus \ell^2 \oplus \{0\}$. The techniques used earlier will easily yield the following result.

Theorem 6.11 *Given a subset S of the natural numbers, consider the operator*

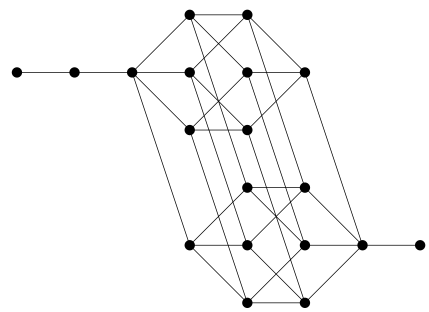
$$T_S = \bigoplus_{n \in S} T_n.$$

The only nontrivial hyperinvariant subspaces of T_S *are* $\ker T_S$ *and the subspaces*

$$\mathcal{K}_f = \bigoplus_{n \in S} \mathcal{K}_{f(n)},$$

where $f\colon S \to \{0,1\}$ *is an arbitrary function.*

Thus, if S contains a finite number n of elements, the hyperlattice of T_S contains precisely $2^n + 3$ spaces. If S is infinite, there is a continuum of hyperinvariant subspaces. The following diagram represents the case $n = 4$.



As mentioned earlier in this section, it was conjectured that the lattice $\mathrm{Hlat}(T)$ of a nilpotent operator is generated by all the spaces $\ker T^j$ and $(\operatorname{ran} T^k)^-$. In particular, such lattices would be finite. The preceding results reveal a completely different reality, which we summarize below.

Theorem 6.12 *There exist operators $T \in \mathcal{L}(\mathcal{H})$ such that $T^3 = 0$ and $\mathrm{Hlat}(T)$ is totally ordered and contains $2^{\aleph_0}$ elements.*

The analysis of canonical hyperinvariant subspaces seems to be much more difficult for nilpotent operators of order four. To see why, consider the following six spaces:

$$(\operatorname{ran} T \cap (\operatorname{ran} T^2 \vee \ker T))^-, \quad (\operatorname{ran} T)^- \cap (\operatorname{ran} T^2 \vee \ker T),$$
$$(\operatorname{ran} T \cap (\ker T^{*2} \cap \operatorname{ran} T^*)^\perp)^-, \quad (\operatorname{ran} T)^- \cap (\ker T^{*2} \cap \operatorname{ran} T^*)^\perp,$$
$$(\operatorname{ran} T \cap \ker T)^- \vee \operatorname{ran} T^2, \quad ((\operatorname{ran} T)^- \cap \ker T) \vee \operatorname{ran} T^2.$$

These six spaces coincide when T acts on a finite-dimensional space, but are most likely distinct in general. In addition, one can expect quite a variety of hyperlattices based on the preceding constructions.

The examples in this section indicate that the classification of the lattices $\mathrm{Hlat}(T)$, for T a nilpotent operator, will be a difficult task.

To conclude this section, let us note that the nilpotent operators studied here are quasisimilar to $J_3 \otimes I_{\aleph_0}$, but many of them are not mutually hyperquasisimilar as they exhibit different lattices of hyperinvariant subspaces. On the other hand, due to their special structure, these operators generate the same closed similarity orbit by a fundamental result of Apostol, Fialkow, Herrero, and Voiculescu [2].

7 Coarse Structure for Hyperinvariant Subspaces

We conclude this section with some general remarks on the structure of the lattice of hyperinvariant subspaces for general nilpotent operators. To start with, consider two arbitrary operators T, T'. We can always construct maps

$$\Phi_{T,T'}, \Psi_{T,T'} \colon \mathrm{Hlat}(T) \to \mathrm{Hlat}(T')$$

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

as follows:

$$\Phi_{T,T'}(\mathcal{M}) = \bigvee\{X\mathcal{M} : XT = T'X\}, \quad \mathcal{M} \in \mathrm{Hlat}(T),$$
$$\Psi_{T,T'}(\mathcal{M}) = \bigcap\{Y^{-1}\mathcal{M} : YT' = TY\}, \quad \mathcal{M} \in \mathrm{Hlat}(T),$$

where, of course, X, Y represent bounded linear operators between the relevant Hilbert spaces. It is easy to verify that

$$\Phi_{T,T'}(\mathcal{M}) \subset \Psi_{T,T'}(\mathcal{M}), \quad \Psi_{T,T'}(\mathcal{M}) = (\Phi_{T^*,T'^*}(\mathcal{M}^\perp))^\perp$$

and

$$(\dagger) \qquad \Phi_{T',T}(\Psi_{T,T'}(\mathcal{M})) \subset \mathcal{M} \subset \Psi_{T',T}(\Phi_{T,T'}(\mathcal{M}))$$

for all $\mathcal{M} \in \mathrm{Hlat}(T)$. When T is an operator of class C_0, and T' is the Jordan model of T, it is known [5, 7, see Theorem III.2.8] that $\Phi_{T,T'}(\Phi_{T',T}(\mathcal{M}')) = \mathcal{M}'$ for all $\mathcal{M}' \in \mathrm{Hlat}(T')$. It follows that the hyperinvariant subspaces of T can be classified using $\mathrm{Hlat}(T')$. More precisely, for every $\mathcal{M} \in \mathrm{Hlat}(T)$ there exists a unique $\mathcal{M}' \in \mathrm{Hlat}(T')$ (namely, $\Phi_{T,T'}(\mathcal{M})$) such that $\Phi_{T',T}(\mathcal{M}') \subset \mathcal{M} \subset \Psi_{T',T}(\mathcal{M}')$. If the operator T has finite cyclic multiplicity (or, more generally, if T has the finiteness property (P); see [5, 7]), then $\Phi_{T,T'} = \Psi_{T,T'}$, and this map is an isomorphism between the hyperlattices of T and T'.

Since nilpotent operators are essentially of class C_0 (they may not be contractions, but we can always multiply them by small scalars to decrease their norm), the same result applies, thereby providing a classification of the hyperinvariant subspaces. Unfortunately, this classification might put in the same class many distinct hyperinvariant subspaces from the lattice generated by the kernels and ranges of the powers of a nilpotent operator T [1, 7]. To see how this can occur, assume that $T = T' \oplus T''$, with $T' = J_n \oplus J_n \oplus \cdots$, and T'' a nilpotent of order at most n on a separable Hilbert space. In this case T' is the Jordan model of T, and $\mathrm{Hlat}(T')$ consists precisely of the $n+1$ spaces $\ker T'^m$, $m = 0, 1, \dots, n$. The reader will verify without difficulty that

$$\Phi_{T',T}(\ker T'^m) = (\mathrm{ran}\, T^{n-m})^-, \quad \Psi_{T',T}(\ker T'^m) = \ker T^m, \quad m = 0, 1, \dots, n,$$

so that the classification in this case amounts to the statement that every element $\mathcal{M} \in \mathrm{Hlat}(T)$ satisfies $(\mathrm{ran}\, T^{n-m})^- \subset \mathcal{M} \subset \ker T^m$ for a unique m. Note that this also follows from Proposition 5.2, and one can deduce from the argument above a proof of that proposition by noting that $\mathrm{Hlat}(T)$ is canonically isomorphic to $\mathrm{Hlat}(T \oplus T \oplus \cdots)$.

Assume for the moment that T is an arbitrary operator. One can still use the maps Φ and Ψ to compare $\mathrm{Hlat}(T)$ with $\mathrm{Hlat}(J_n)$ for some integer n.

Proposition 7.1 *For every integer $m \le n$, we have*

$$\Phi_{J_n,T}(\ker J_n^m) = (\ker T^m \cap \mathrm{ran}\, T^{n-m})^-$$

and

$$\Psi_{J_n,T}(\ker J_n^m)^\perp = (\mathrm{ran}\, T^{*m} \cap \ker T^{*n-m})^-.$$

Proof Observe that the second relation follows from the first upon replacing J_n, T, m by J_n^*, T^*, and $n-m$, respectively. Therefore we only prove the first equality. The fact that $\Phi_{J_n,T}(\ker J_n^m) \subset (\ker T^m \cap \operatorname{ran} T^{n-m})^-$ follows from the equality $\ker J_n^m = \operatorname{ran} J_n^{n-m}$. To conclude the proof. take a vector $x \in \operatorname{ran} T^m \cap \ker T^{n-m}$. Since $T^m x = 0$ and $x = T^{n-m}z$ for some z, we must have $T^n z = 0$. An operator X such that $XJ_n = TX$ can now be obtained by setting $Xe_1 = T^{n-1}z$, $Xe_2 = T^{n-2}z, \dots, Xe_m = x, \dots$, $Xe_n = z$. Thus x does indeed belong to $\Phi_{J_n,T}(\ker J_n^m)$. ■

The preceding result yields an interesting fact even when $n = 1$.

Corollary 7.2 *Given an arbitrary operator T, and a space $\mathcal{M} \in \operatorname{Hlat}(T)$, we either have $\mathcal{M} \supset \ker T$, or $\mathcal{M} \subset \operatorname{ran} T$.*

Proof The space $\Phi_{T,J_1}(\mathcal{M})$ can be either $\{0\}$ or $\mathbb{C}$. By virtue of (†), in the first case, $\mathcal{M} \subset \Psi_{J_1,T}(\{0\}) = (\operatorname{ran} T)^-$, while in the second case, $\mathcal{M} \supset \Phi_{J_1,T}(\mathbb{C}) = \ker T$. ■

We can apply Proposition 7.1, along with (†), for all values of n to obtain the following general result.

Theorem 7.3 *Let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary operator, $\mathcal{M} \in \operatorname{Hlat}(T)$, and define $p_n, q_n \le n$ by $\Phi_{T,J_n}(\mathcal{M}) = \ker J_n^{p_n}$, $\Psi_{T,J_n}(\mathcal{M}) = \ker J_n^{q_n}$, $n \ge 1$. We then have*

$$\bigvee_{n=1}^{\infty} (\ker T^{q_n} \cap \operatorname{ran} T^{n-q_n}) \subset \mathcal{M} \subset \Big(\bigvee_{n=1}^{\infty} (\operatorname{ran} T^{*p_n} \cap \ker T^{*n-p_n}) \Big)^{\perp}.$$

The numbers p_n and q_n can also be characterized by

$$T^{p_n}\mathcal{M} \subset (\operatorname{ran} T^n)^-, \quad T^{p_n-1}\mathcal{M} \not\subset (\operatorname{ran} T^n)^-,$$

and

$$\mathcal{M} \supset T^{n-q_n} \ker T^n, \quad M \not\supset T^{n-q_n-1} \ker T^n,$$

respectively. They satisfy the additional relations

$$0 \le p_{n+1} - p_n \le 1 \quad \textit{and} \quad 0 \le q_{n+1} - q_n \le 1$$

for every n. If $T^N = 0$ for some $N \ge 1$, we also have $p_n = p_N, q_n = q_N$,

$$(\ker T^{q_n} \cap \operatorname{ran} T^{n-q_n})^- \subset (\ker T^{q_N} \cap \operatorname{ran} T^{N-q_N})^-,$$

and

$$(\operatorname{ran} T^{*p_n} \cap \ker T^{*n-p_n})^- \subset (\operatorname{ran} T^{*p_N} \cap \ker T^{*N-p_N})^-$$

for all $n \ge N$. (Thus one only needs to use indices $n \le N$ in the above formulas for $\mathcal{M}$.)

Proof The second inclusion in the statement follows from the first upon replacing T and $\mathcal{M}$ by T^* and $\mathcal{M}^\perp$, respectively. The first inclusion follows then immediately from relation (†) and Proposition 7.1 since

$$(\ker T^{q_n} \cap \operatorname{ran} T^{n-q_n})^- = \Phi_{J_n,T}(\Psi_{T,J_n}(\mathcal{M})).$$

Next we consider the characterization of the number q_n. To do this, we need to show that, given an integer $q \le n$, the inclusion $\Psi_{T,J_n}(\mathcal{M}) \supset \ker J_n^q$ is equivalent to $T^{n-q} \ker T^n \subset \mathcal{M}$. Indeed, the first inclusion means that $Y^{-1}\mathcal{M} \supset \ker J_n^q = \operatorname{ran} J_n^{n-q}$ whenever $Y J_n = TY$. Equivalently,

$$T^{n-q}\Big(\bigvee_{Y J_n = TY} \operatorname{ran} Y\Big) = \bigvee_{Y J_n = TY} Y \operatorname{ran} J_n^{n-q} \subset \mathcal{M},$$

and clearly the first space above is exactly $T^{n-q}\Phi_{J_n,T}(\ker J_n^0) = T^{n-q}\ker T^n$ by Proposition 7.1. The characterization of p_n follows analogously by passing to adjoints.

For the other estimates on these numbers, it is easier to check that $0 \le p_{n+1} - p_n \le 1$. This follows from the fact that

$$\ker J_n^{p_n} \oplus \ker J_{n+1}^{p_{n+1}} = \Phi_{T,J_n}(\mathcal{M}) \oplus \Phi_{T,J_{n+1}}(\mathcal{M}) = \Phi_{T,J_n \oplus J_{n+1}}(\mathcal{M})$$

is a hyperinvariant subspace for $J_n \oplus J_{n+1}$ by applying the remarks at the beginning of Section 5 to this operator.

Let us assume now that T is nilpotent of order N and $n \ge N$. If Y satisfies $YT = J_n T$, we also have $J_n^N Y = YT^N = 0$, so that the range of Y is always contained in $\ker J_n^N$. Therefore

$$\ker J_n^{p_n} = \Phi_{T,J_n}(\mathcal{M}) = \Phi_{T,J_n|\ker J_n^N}(\mathcal{M}) = \ker(J_n|\ker J_n^N)^{p_N} = \ker J_n^{p_N}$$

because $J_n|\ker J_n^N$ is unitarily equivalent to J_N. This proves the equality $p_n = p_N$, and $q_n = q_N$ follows analogously. Finally, passing from Y to $Z = Y|\ker J_n^N$, we see that

$$\begin{aligned}(\ker T^{q_n} \cap \operatorname{ran} T^{n-q_n})^- &= \bigvee_{Y J_n = TY} Y \ker J_n^{q_n} \\ &\subset \bigvee_{Z(J_n|\ker J_n^N) = TZ} Z \ker(J_n|\ker J_n^N)^{q_n} = \bigvee_{W J_N = TW} W \ker J_N^{q_n},\end{aligned}$$

and the last space above is $(\ker T^{q_N} \cap \operatorname{ran} T^{N-q_N})^-$ because $q_n = q_N$. ■

The theorem above provides a coarse structure for $\operatorname{Hlat}(T)$, especially when this lattice is infinite. This result also extends the main theorem in [3], and it represents a slight improvement even in the nilpotent case.

Note that when 0 is not an eigenvalue of T, the first inclusion in the theorem is simply $\{0\} \subset \mathcal{M}$. The dual inclusion becomes $\mathcal{M} \subset \mathcal{H}$ in case 0 is not an eigenvalue for T^*. On the other hand, when T is nilpotent, one cannot refine this coarse structure by considering any Jordan cells with nonzero eigenvalues or, for that matter, any other operators acting on finite dimensional spaces.

The values of the integers p_n, q_n in the preceding result may not be obvious from the structure of the hyperinvariant subspace. Consider, for instance, the space $\mathcal{M} = \mathcal{H} \oplus \mathcal{H} \oplus \{0\} = \ker T_0^2$ (with T_0 as in Lemma 6.2), and set $J = J_1 \oplus J_2 \oplus J_3$. If we had to guess what $\Phi_{T_0,J}(\mathcal{M})$ is, $\ker J^2$ would first come to mind; this corresponds with the indices $(p_1, p_2, p_3) = (1, 2, 2)$. However, Lemma 6.3 shows that the space $\Phi_{T_0,J}(\mathcal{M})$

actually corresponds with $(1, 1, 2)$, so that $\Phi_{T_0,J}(\mathcal{M}) = \ker J \vee \operatorname{ran} J$. The preceding result produces the inclusions

$$\mathcal{H} \oplus (B\mathcal{H})^- \oplus \{0\} = \ker T_0 \vee \operatorname{ran} T_0 \subset \mathcal{M} \subset (\ker T_0^* \cap \operatorname{ran} T_0^*)^\perp = \mathcal{H} \oplus \mathcal{H} \oplus \{0\}.$$

Similarly, all the subspaces $\mathcal{K}_f$ in Theorems 6.9 and 6.11 are associated with the sequence $(1, 1, 2)$.

8 Generalized Jordan Operators as Invariants

There are still other ways to construct maps between hyperlattices. One such construction, which also leads to hyperquasisimilarity invariants for nilpotent operators, is as follows. Fix an operator X satisfying $XT' = TX$, and define $\Phi_X\colon \operatorname{Hlat}(T') \to \operatorname{Hlat}(T)$ by

$$\Phi_X(\mathcal{M}) = \bigvee\{AX\mathcal{M} : A \in \{T\}'\}, \quad \mathcal{M} \in \operatorname{Hlat}(T').$$

We focus on a particular operator X constructed in [1, Lemma 1], which is also a quasiaffinity. For the reader's convenience, we recall how that map is constructed. We start with a nilpotent operator $T \in \mathcal{L}(\mathcal{H})$ of order n, and we construct inductively the spaces

$$\mathcal{K}_n = \mathcal{H} \ominus \ker T^{n-1}, \qquad \mathcal{K}_{n-1} = \ker T^{n-1} \ominus (T\mathcal{K}_n \vee \ker T^{n-2}), \dots,$$
$$\mathcal{K}_{n-j} = \ker T^{n-j} \ominus (T^j\mathcal{K}_n \vee T^{j-1}\mathcal{K}_{n-1} \vee \cdots \vee T\mathcal{K}_{n-j+1} \vee \ker T^{n-j-1})^-, \dots,$$

and finally

$$\mathcal{K}_1 = \ker T \ominus (T^{n-1}\mathcal{K}_n \vee T^{n-2}\mathcal{K}_{n-1} \vee \cdots \vee T\mathcal{K}_2)^-.$$

The operator

$$T' = \bigoplus_{j=1}^{n} (J_j \otimes I_{\mathcal{K}_j}) \in \mathcal{L}(\mathcal{H}'), \quad \mathcal{H}' = \bigoplus_{j=1}^{n} (\mathbb{C}^j \otimes \mathcal{K}_j)$$

is nilpotent of order n, and there is a unique operator $X\colon \mathcal{H}' \to \mathcal{H}$ such that $XT' = TX$ and $X(e_j \otimes k_j) = k_j$ for $k_j \in \mathcal{K}_j$. The operator X is a quasiaffinity, and it is invertible in case all the powers of T have closed ranges (see also [25]). The map Φ_X may be useful because X is so intimately tied to the structure of T; for instance,

$$(X \operatorname{ran} T'^m)^- = (\operatorname{ran} T^m)^-, \quad (X \ker T'^m)^- = \ker T^m, \quad m = 0, 1, \dots, n.$$

The reader will have little difficulty verifying these equalities, the first of which is true for any intertwining quasiaffinity X. Of course, X is unitary if and only if T is already a generalized Jordan operator.

Proposition 8.1 *If T and T' are hyperquasisimilar nilpotent operators, then J_T (respectively J_{T^*}) and $J_{T'}$ (respectively $J_{T'^*}$) are unitarily equivalent.*

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

Proof Let $\mathcal{K}_j, \mathcal{K}'_j$ be the spaces associated with T, T' by the process described above, and let A, B be quasiaffinities which satisfy the definition of hyperquasisimilarity, and $AT = T'A, TB = BT'$. As in the proof of Proposition 5.2, we can verify that A induces an operator with dense range from $\mathcal{K}_j$ to $\mathcal{K}'_j$ and B induces an operator with dense range from $\mathcal{K}'_j$ to $\mathcal{K}_j$. This follows as in Proposition 5.2 by observing the following alternative description of the space $\mathcal{K}_{n-j}$ for $j \geq 1$:

$$\mathcal{K}_{n-j} = \ker T^{n-j} \ominus (\operatorname{ran} T^j + T^{j-1} \ker T^{n-1} + \cdots + T \ker T^{n-j+1} + \ker T^{n-j-1})^-.$$

We deduce that $\mathcal{K}_j$ and $\mathcal{K}'_j$ have the same dimension, thus verifying the unitary equivalence of J_T and $J_{T'}$.

In order to deal with J_{T^*}, let us denote by $\mathcal{K}^*_{n-j}, \mathcal{K}'^*_{n-j}$ the spaces analogous to $\mathcal{K}_{n-j}, \mathcal{K}'_{n-j}$, but with T^*, T'^* in place of T, T'. The preceding formula can also be used to write $\mathcal{K}^*_{n-j}$ as an orthogonal difference of two spaces in $\operatorname{Hlat}(T)$. Namely,

$$\mathcal{K}^*_{n-j} = \Big[\ker T^{j+1} \cap \Big(\bigcap_{\ell=1}^{j} (T^{\ell-1})^{-1}(\operatorname{ran} T^{n-\ell})^-\Big)\Big] \ominus (\operatorname{ran} T^{n-j})^-, \quad j \geq 1.$$

Arguing as in the proof of Proposition 5.2, we conclude that $\mathcal{K}^*_{n-j}$ and $\mathcal{K}'^*_{n-j}$ have the same dimension, and this yields the second unitary equivalence in the statement. ∎

The discussion above associates with each nilpotent operator T of order n a canonical model T' which we will now denote J_T, and which is a hyperquasisimilarity invariant for T. Proposition 8.1 also shows that J_{T^*} is a hyperquasisimilarity invariant. The following consequence of Proposition 8.1 is immediate.

Corollary 8.2 *If a nilpotent operator T is hyperquasisimilar to a generalized Jordan operator J, then J_T and J_{T^*} are both unitarily equivalent to J.*

These two invariants, which are certainly quasisimilar, may not be hyperquasisimilar. As an example, observe that for the operator T_0 of Lemma 6.2 we have

$$J_{T_0} = (J_3 \otimes I_{\mathcal{H}}) \oplus (J_2 \otimes I_{\mathcal{H}}), \quad J_{T_0^*} = (J_3 \otimes I_{\mathcal{H}}) \oplus (J_2 \otimes I_{\mathcal{H}}) \oplus J_1.$$

These two operators are clearly not hyperquasisimilar. Note however that the leading summand $J_3 \otimes I_{\mathcal{H}}$ is the same for the two operators; this is a general fact which follows because J_T and J_{T^*} are quasisimilar [1, 7]. Curiously, $T_0, J_{T_0}, J_{T_0^*}$ have 5, 6, and 8 distinct hyperinvariant subspaces, respectively. In particular we deduce that T_0 is not hyperquasisimilar to T_0^*. This can also be seen by examining $\operatorname{Hlat}(T_0)$; this lattice is isomorphic to $\operatorname{Hlat}(T_0^*)$, but the isomorphism does not respect the description of hyperinvariant subspaces in terms of kernels and ranges. Indeed, we have $\ker T_0 = (\operatorname{ran} T_0^2)^-$, but the analogous equality for T_0^* does not hold. In particular, T_0 and T_0^* are not even hyperquasisimilar. By contrast, it is an easy exercise that any nilpotent operator of order two is hyperquasisimilar to a generalized Jordan operator.

The reason we needed separate arguments for J_T and J_{T^*} in Proposition 8.1 is that hyperquasisimilarity may not be preserved by passing to adjoints. We have however the following result.

https://doi.org/10.4153/CJM-2008-034-2 Published online by Cambridge University Press

Proposition 8.3 *Let T and T' be two hyperquasisimilar operators, and let X, Y be quasiaffinities implementing the hyperquasisimilarity, so that $XT = T'X$ and $TY = YT'$. Then*

$$\Phi_{X^*}(\Phi_{Y^*}(\mathcal{N})) = \Phi_{X^*Y^*}(\mathcal{N}) = \mathcal{N} \quad \textit{for every } \mathcal{N} \in \mathrm{Hlat}(T),$$
$$\Phi_{Y^*}(\Phi_{X^*}(\mathcal{N}')) = \Phi_{Y^*X^*}(\mathcal{N}') = \mathcal{N}' \quad \textit{for every } \mathcal{N}' \in \mathrm{Hlat}(T').$$

Proof For reasons of symmetry, it will suffice to prove the first half of the conclusion. Fix $\mathcal{N} \in \mathrm{Hlat}(T)$, and observe that the inclusions

$$\Phi_{X^*Y^*}(\mathcal{N}) \subset \Phi_{X^*}(\Phi_{Y^*}(\mathcal{N})) \subset \mathcal{N}$$

are immediate. It is enough then to show that the space $\mathcal{M} = (\Phi_{X^*Y^*}(\mathcal{N}))^{\perp}$ is contained in $\mathcal{N}^{\perp}$. We have $\mathcal{M} = \{h : YXAh \in \mathcal{N}^{\perp}$ for every $A \in \{T\}'\}$, and considering the case when A is the identity operator, we see that $YX\mathcal{M} \subset \mathcal{N}^{\perp}$. However, $(YX\mathcal{M})^{-} = \mathcal{M}$ because X, Y implement a hyperquasisimilarity. Thus $\mathcal{M} \subset \mathcal{N}^{\perp}$, as desired. ■

A careful examination of the preceding argument shows that a stronger result is true.

Proposition 8.4 *Assume that T and T' are two operators on Hilbert spaces and X, Y intertwine them,* i.e., *$XT = T'X$ and $TY = YT'$. If Φ_X and Φ_Y are inverse bijections between* $\mathrm{Hlat}(T)$ *and* $\mathrm{Hlat}(T')$, *then Φ_{X^*} and Φ_{Y^*} are inverse bijections between* $\mathrm{Hlat}(T^*)$ *and* $\mathrm{Hlat}(T'^*)$.

Proof As in the preceding proof, we fix a space $\mathcal{N} \in \mathrm{Hlat}(T^*)$, and note that $\Phi_{X^*}(\Phi_{Y^*}(\mathcal{N})) \subset \mathcal{N}$. It suffices to show that $\mathcal{M} = (\Phi_{X^*}(\Phi_{Y^*}(\mathcal{N})))^{\perp}$ is contained in $\mathcal{N}^{\perp}$. We have $\mathcal{M} = \{h : YBXAh \in \mathcal{N}^{\perp}$ for every $A \in \{T\}'$ and $B \in \{T'\}'\}$, and by taking A to be the identity operator, we see that $YBX\mathcal{M} \subset \mathcal{N}^{\perp}$ for every $B \in \{T'\}'$. In other words, $Y(\Phi_X(\mathcal{M})) \subset \mathcal{N}^{\perp}$. Since $\mathcal{N}^{\perp}$ is hyperinvariant for T, we also have $\Phi_Y(\Phi_X(\mathcal{M})) \subset \mathcal{N}^{\perp}$, which concludes the proof because $\Phi_Y(\Phi_X(\mathcal{M})) = \mathcal{M}$. ■

It is now natural to introduce a new relation between operators, slightly weaker than hyperquasisimilarity. Let us say that T and T' are *structurally hyperquasisimilar* if there exists quasiaffinities X, Y such that $XT = T'X, TY = YT'$ and the maps Φ_X, Φ_Y are inverse bijections of $\mathrm{Hlat}(T)$ and $\mathrm{Hlat}(T')$. The preceding proposition implies that structural hyperquasisimilarity is preserved by passing to adjoints. When T is nilpotent, J_T will no longer be a structural hyperquasisimilarity invariant. Indeed, let us fix two numbers $N, N' \leq \aleph_0$, and consider the operators $T = (J_2 \otimes I_{\aleph_0}) \oplus (J_1 \otimes I_N) \oplus J_1$ and $T' = (J_2 \otimes I_{\aleph_0}) \oplus (J_1 \otimes I_{N'}) \oplus J_1$. These are generalized Jordan operators, and they are not hyperquasisimilar if $N \neq N'$. However, T and T' are structurally hyperquasisimilar. Indeed, the operators $(J_2 \otimes I_{\aleph_0}) \oplus (J_1 \otimes I_N)$ and $(J_2 \otimes I_{\aleph_0}) \oplus (J_1 \otimes I_{N'})$ are quasisimilar; fix quasiaffinities X, Y intertwining them. Then the quasiaffinities $A = X \oplus I_1, B = Y \oplus I_1$ intertwine T and T', and it is easy to see that they implement the structural hyperquasisimilarity of these operators.

It is however true that structural hyperquasisimilarity preserves some of the structure of J_T.

Proposition 8.5 *Consider two structurally quasisimilar nilpotent operators T and T' such that*

$$J_T = \bigoplus_{k=1}^{n} (J_k \otimes I_{N_k}), \qquad J_{T'} = \bigoplus_{k=1}^{n} (J_k \otimes I_{N'_k}).$$

We have $N_k = 0$ if and only if $N'_k = 0$ for $k = 1, 2, \dots, n$.

Proof Fix quasiaffinities X, Y implementing the structural hyperquasisimilarity of T and T', so that $XT = T'X$ and $TY = YT'$. As in the proof of Proposition 5.2, we can verify that $\Phi_X(\ker T^{n-j}) = \ker T'^{n-j}$, and $\Phi_X((T^{\ell-1} \ker T^{n-\ell})^-) = (T'^{\ell-1} \ker T'^{n-\ell})^-$. With the notation used in the proof of Proposition 8.1, we see that $\mathcal{K}_{n-j} = \{0\}$ if and only if $\mathcal{K}'_{n-j} = \{0\}$. ■

Corollary 8.6 *If a nilpotent Jordan operator T is structurally quasisimilar to a generalized Jordan operator J, then the operators J_T and J_{T^*} contain the same Jordan cells as direct summands (with possibly different multiplicities).*

The newly defined relation of structural quasisimilarity is quite obviously symmetric and reflexive, but may not be transitive.

The discussion started in Section 5 shows that there is more to hyperinvariant subspaces of nilpotent operators than meets the eye.

References

[1] C. Apostol, R. G. Douglas, and C. Foias, *Quasi-similar models for nilpotent operators.* Trans. Amer. Math. Soc. **224**(1976), no. 2, 407–415.

[2] C. Apostol, L. Fialkow, D. Herrero, and D. Voiculescu, *Approximation of Hilbert space operators.* Vol. II. Research Notes in Mathematics 102. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[3] M. Barraa, *Sous-espaces hyperinvariants d'un opérateur nilpotent sur un espace de Banach.* J. Operator Theory **21**(1989), no. 2, 315–321.

[4] J. Barria and D. Herrero, *Closure of similarity orbits of Hilbert space operators. IV. Normal operators.* J. London Math. Soc. **17**(1978), no. 3, 525–536.

[5] H. Bercovici, *C_0-Fredholm operators. II.* Acta Sci. Math. (Szeged) **42**(1980), no. 1-2, 3–42.

[6] ———, *A note on disjoint invariant subspaces*, Michigan Math. J. **34**(1987), no. 3, 435-439.

[7] ———, *Operator theory and arithmetic in H^∞*. Mathematical Surveys and Monographs 26, American Mathematical Society, Providence, RI, 1988.

[8] H. Bercovici, C. Foias, and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I.* Michigan Math. J. **30**(1983), no. 3, 335-354.

[9] ———, *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series in Mathematics 56, American Mathematical Society, Providence, RI, 1985.

[10] B. Charles, *Opérateurs linéaires sur un espace de Banach et modules sur un anneau principal.* In: Symposia Mathematica 23, Academic Press, London, 1979, pp. 121–143.

[11] P. A. Fillmore, D. A. Herrero, and W. E. Longstaff, *The hyperinvariant subspace lattice of a linear transformation.* Linear Algebra and Appl. **17**(1977), no. 2, 125–132.

[12] C. Foias, *Invariant para-closed subspaces.* Indiana Univ. Math. J. **21**(1971/72), 887–906.

[13] C. Foias, S. Hamid, C. Onica, and C. Pearcy, *On the hyperinvariant subspace problem. III.* J. Funct. Anal. **222**(2005), no. 1, 129–142.

[14] C. Foias, S.-C. Ong, and P. Rosenthal, *An interpolation theorem and operator ranges.* Inegral Equations Operator Theory **10**(1978), no. 6, 802–811.

[15] C. Foias and C. Pearcy, *On the hyperinvariant subspace problem.* J. Funct. Anal. **219**(2005), no. 1, 134–142.

[16] L. Fuchs, *Infinite abelian groups.* Pure and Applied Mathematics 36-II. Academic Press, New York, 1973.

[17] S. Hamid, C. Onica, and C. Pearcy, *On the hyperinvariant subspace problem. II.* Indiana Univ. Math. J. **54**(2005), no. 3, 743–754.

[18] D. A. Herrero, *Quasisimilarity does not preserve the hyperlattice.* Proc. Amer. Math. Soc. **65**(1977), no. 1, 80–84.
[19] ———, *Approximation of Hilbert space operators.* Second edition. Pitman Research Notes in Mathematics Series 224, John Wiley, New York, 1989.
[20] I. Kaplansky, *Infinite Abelian Groups.* University of Michigan Press, Ann Arbor, Michigan, 1954.
[21] V. Paulsen, *Completely bounded maps and dilations.* Pitman Research Notes in Mathematics Series 146, John Wiley, New York, 1986.
[22] J. Peetre, *On an interpolation theorem of Foias and Lions.* Acta Sci. Math. (Szeged) **25**(1964), 255–261.
[23] ———, *On interpolation functions*, Acta Sci. Math. (Szeged) **27**(1966), 167–171.
[24] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 77, Springer-Verlag, New York, 1973.
[25] L. Williams, *Similarity invariants for a class of nilpotent operators.* Acta Sci. Math. (Szeged) **38**(1976), no. 3-4, 423–428.

(H. Bercovici) *Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.*
e-mail: bercovic@indiana.edu

(C. Foias and C. Pearcy) *Department of Mathematics Texas A&M University College Station, TX 77843 U.S.A.*
e-mail: pearcy@math.tamu.edu