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ABSOLUTE CONTINUITY OF SOME VECTOR FUNCTIONS AND MEASURES

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Introduction. In the theory of vector valued functions there is a theorem which states that if a function from a compact interval I into a normed linear space X is of weak bounded variation, then it is of bounded variation. The proof uses in a straightforward way the Uniform Boundedness Principle (see [**2**, p. 60]). The present paper grew from the question of whether an analogous theorem holds for absolutely continuous functions. The answer is in the negative, and an example will be given (Theorem 7). But it will also be shown that if X is weakly sequentially complete (e.g. an L_p space, $1 \leqq p < \infty$), then a weakly absolutely continuous point function from I into X *is* absolutely continuous. The method of proof involves the construction of a countably additive set function in the standard Lebesgue-Stieltjes fashion.

The paper is divided into three parts. In Section 1 extensions of finitely additive, absolutely continuous set functions are carried out in an abstract setting. Section 2 applies this to vector valued (point) functions on the real line. This in turn is applied in Section 3 to the theory of random differential equations [**4**] to give sufficient conditions for a sample path solution to be a linear space solution as well.

The notation to be used here is standard. If A and B are sets, the symbols $A \backslash B$ denote the complement of B relative to A, even when B is not a subset of A. The symbols $A \triangle B$ denote the set $(A\backslash B) \cup (B \backslash A)$. The notation $f : A \rightarrow \mathbf{R}$ means that f is a function from A into the real numbers $\mathbf{R}$. We write "a.e." for "almost everywhere".

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1. Extension of absolutely continuous set functions. In this section we assume throughout that $\mathscr{S}$ is a ring of sets and that λ is a countably additive, non-negative measure on $\mathscr{S}$. We assume that $\mathscr{T}$ is a subring of $\mathscr{S}$ which is dense with respect to λ; that is, the metric function $\rho(A, B) = \lambda(A \triangle B)$ makes $\mathscr{T}$ a dense subset of $\mathscr{S}$. Finally we assume that μ is a finitely additive function on $\mathscr{T}$ with values in a Banach space X.

A function ν on $\mathscr{S}$ (or $\mathscr{T}$) with values in X is called λ-*absolutely continuous* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for each A in $\mathscr{S}$ (or $\mathscr{T}$), $\lambda(A) < \delta$ implies $||\nu(A)|| < \epsilon$. The function ν is called *strongly* λ-*absolutely continuous*

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if for each $\epsilon > 0$ there is a $\delta > 0$ such that for any finite disjoint collection $\{A_1, \ldots, A_n\}$ of elements of $\mathscr{S}$ satisfying $\sum\lambda(A_i) < \delta$, the further inequality $\sum||\nu(A_i)|| < \epsilon$ holds. Finally, ν is *weakly* λ*-absolutely continuous* if for each x^* in X^* (the topological dual of X) the composite function $x^*\nu$ is λ-absolutely continuous as a function from $\mathscr{S}$ into the scalar field of X.

THEOREM 1. *If μ is λ-absolutely continuous on $\mathscr{T}$, then μ has a countably additive extension $\bar{\mu}$ to $\mathscr{S}$ which is λ-absolutely continuous on $\mathscr{S}$. Moreover, if η is any function from $\mathscr{S}$ into X which is λ-absolutely continuous on $\mathscr{S}$ and which coincides with μ on $\mathscr{T}$, then $\eta = \bar{\mu}$.*

Proof. Since μ is finitely additive and λ-absolutely continuous, μ is uniformly continuous on $\mathscr{T}$ relative to the metric function $\rho(A, B) = \lambda(A \triangle B)$. This follows from the fact that for each $\epsilon > 0$ there is a $\delta > 0$ such that for all A and B in $\mathscr{T}$, if $\rho(A, B) < \delta$ then $||\mu(A) - \mu(B)|| = ||\mu(A \triangle B)|| < \epsilon$. Thus by the well-known extension theorem for uniformly continuous functions into complete spaces (see [**1**, p. 23]), there exists a unique continuous extension $\bar{\mu}$ of μ to the ring $\mathscr{S}$. Moreover, $\bar{\mu}$ is uniformly continuous on $\mathscr{S}$ relative to ρ, which is to say λ-absolutely continuous. And if η is a λ-absolutely continuous extension of μ to $\mathscr{S}$, then $\eta = \bar{\mu}$.

To see that $\bar{\mu}$ is countably additive on S, let $\{A_n\}$ be a disjoint sequence in $\mathscr{S}$ whose union is also in $\mathscr{S}$. Then since λ is countably additive, $\lambda(\bigcup_{n=m+1}^{\infty} A_n) \to 0$ as $m \to \infty$, whence

$$\left\| \bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n \right) - \sum_{n=1}^{m} \bar{\mu}(A_n) \right\| = \left\| \bar{\mu}\left(\bigcup_{n=m+1}^{\infty} A_n \right) \right\| \to 0.$$

It is worth noting that considerably weaker requirements may be imposed on λ, since in the proof above we used only the fact that $\lambda(A \triangle B)$ defines a metric on $\mathscr{S}$ and that λ is continuous at $\emptyset$ (i.e. if $\{B_n\}$ is a nested sequence in $\mathscr{S}$ with empty intersection, then $\lambda(B_n) \to 0$). Thus it suffices to require that λ be a non-negative function on $\mathscr{S}$ which is continuous at $\emptyset$ and which satisfies $\lambda(A) \leqq \lambda(B) + \lambda(A \triangle B)$ for all A and B in $\mathscr{S}$. (The triangle inequality follows from the last inequality.)

THEOREM 2. *If μ is strongly λ-absolutely continuous on $\mathscr{T}$, then the extension $\bar{\mu}$ of μ to $\mathscr{S}$ (Theorem 1) is strongly λ-absolutely continuous on $\mathscr{S}$.*

Proof. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that for every finite disjoint collection $\{J_1, \ldots, J_m\}$ in $\mathscr{T}$, if $\sum\lambda(J_i) < \delta$ then $\sum||\mu(J_i)|| < \epsilon$. We shall now show that if $\{A_1, \ldots, A_n\}$ is a finite disjoint collection in $\mathscr{S}$, and if $\sum\lambda(A_i) < \delta$, then $\sum||\bar{\mu}(A_i)|| \leqq \epsilon$. Fix $A_1, \ldots, A_n$. Take any $\alpha > 0$. We show there exists a disjoint collection $\{J_1, \ldots, J_n\}$ in $\mathscr{T}$ such that

$$\sum\lambda(J_i) < \delta \quad \text{and} \quad \sum||\bar{\mu}(A_i) - \mu(J_i)|| < 2\alpha,$$

whence $\sum||\bar{\mu}(A_i)|| < 2\alpha + \epsilon$, and since α is arbitrary, $\sum||\bar{\mu}(A_i)|| \leqq \epsilon$. To construct the sets J_i we find $\beta > 0$ such that when $J \in \mathscr{T}$ and $\lambda(J) < \beta$,

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then $||\mu(J)|| < \alpha/n$. The proof of Theorem 1 shows that we may choose sets $B_1, \ldots, B_n$ from $\mathscr{T}$ such that $\lambda(B_i \triangle A_i) < \beta/n$ and $||\mu(B_i) - \bar{\mu}(A_i)|| < \alpha/n$. Let $B = \bigcup_{i \neq j}(B_i \cap B_j)$. Since the A_i's are mutually disjoint, $B \subset \bigcup (A_i \triangle B_i)$ and thus $\lambda(B) < \beta$. Now let $J_i = B_i \backslash B$. Note that $J_i \in \mathscr{T}$ and that the J_i's are disjoint. Moreover, $\lambda(J_i \triangle B_i) = \lambda(B_i \cap B) < \beta$, so that $||\mu(J_i) - \mu(B_i)|| < \alpha/n$, whence $\sum ||\mu(J_i) - \bar{\mu}(A_i)|| < 2\alpha$.

As in the proof of Theorem 1, the foregoing proof requires less than countable additivity of λ. It suffices to assume, in addition to the continuity of λ at $\emptyset$, that λ is monotone and finitely subadditive (which implies that $\lambda(A) \leqq \lambda(B) + \lambda(A \triangle B)$ for all A and B in $\mathscr{S}$).

THEOREM 3. *If X is weakly sequentially complete and if μ is weakly λ-absolutely continuous on $\mathscr{T}$, then μ can be extended to a countably additive, λ-absolutely continuous function $\bar{\mu}$ on $\mathscr{S}$.*

Proof. Let $\mathscr{U}$ be the σ-ring generated by $\mathscr{S}$. Then λ can be extended to a countably additive $\bar{\lambda}$ on $\mathscr{U}$, with $\mathscr{T}$ dense in $\mathscr{U}$, by the usual outer measure procedure. Let $\bar{\rho}(A, B) = \bar{\lambda}(A \triangle B)$ be the corresponding metric on $\mathscr{U}$. For each $A \in \mathscr{U}$, let $\{J_n\}$ be a sequence in $\mathscr{T}$ converging under $\bar{\rho}$ to A. Then $\{J_n\}$ is a Cauchy sequence in $\mathscr{T}$. For each $x^* \in X^*$ the composite function $x^*\mu$ is $\bar{\lambda}$-absolutely continuous and thus $\bar{\rho}$-uniformly continuous, which implies that $x^*\mu(J_n)$ is a Cauchy scalar sequence. Thus $\mu(J_n)$ is weakly Cauchy. Since X is weakly sequentially complete, the proof technique for the extension of uniformly continuous functions (see [**1**, p. 23]) can be used to obtain an extension of μ to $\bar{\mu}$ on $\mathscr{U}$. Since $x^*\bar{\mu}$ obviously coincides with the extension of $x^*\mu$ to $\mathscr{U}$ obtained by applying Theorem 1 to $x^*\mu$, we see that $x^*\bar{\mu}$ is countably additive and $\bar{\lambda}$-absolutely continuous on the σ-ring $\mathscr{U}$. By the well-known theorem of Pettis (see [**1**, p. 318]), $\bar{\mu}$ is countably additive on $\mathscr{U}$. To see that $\bar{\mu}$ is $\bar{\lambda}$-absolutely continuous on $\mathscr{U}$ we note that if $A \in \mathscr{U}$ and $\bar{\lambda}(A) = 0$, then for each $x^* \in X^*$, $x^*\bar{\mu}(A) = 0$ because $x^*\bar{\mu}$ is $\bar{\lambda}$-absolutely continuous. Thus $\bar{\mu}(A) = 0$, which implies (see [**1**, p. 318]) that $\bar{\mu}$ is $\bar{\lambda}$-absolutely continuous on $\mathscr{U}$. Now $\bar{\mu}$ can be restricted to $\mathscr{S}$ to obtain the conclusion of the theorem.

The author is unable to see any way of weakening the requirement of countable additivity on λ without losing the λ-absolute continuity and (strong) countable additivity of $\bar{\mu}$. It is important to observe that the hypothesis of weak sequential completeness for X cannot be omitted entirely. This will be seen as a consequence of Theorem 7 in the next section.

2. Absolutely continuous functions on an interval. We now apply the preceding theorems to point functions on the real line. Let I be a compact interval in the real numbers **R**. A (point) function ξ on I with values in a Banach space X is called *absolutely continuous* on I if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all finite disjoint collections $\{(a_i, b_i)\}_{i=1}^n$ of sub-intervals of I, $\sum (b_i - a_i) < \delta$ implies

$$||\sum[\xi(b_i) - \xi(a_i)]|| < \epsilon.$$

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The function ξ is called *strongly absolutely continuous* on I if the latter inequality in the implication can be replaced by $\sum\|\xi(b_i) - \xi(a_i)\| < \epsilon$. Finally, ξ is called *weakly absolutely continuous* on I if for each $x^* \in X^*$, the scalar valued composite function $x^*\xi$ is absolutely continuous on I. When X is finite dimensional all three forms of absolute continuity are equivalent. But when infinite dimensional spaces are admitted it is easy to find examples of functions ξ which are absolutely continuous but not strongly so (see [**2**, p. 60]). As Corollary 6 will show, it is more difficult to construct a weakly absolutely continuous function which is not absolutely continuous.

THEOREM 4. *If a point function $\xi : I \to X$ is absolutely continuous on I, then ξ generates a unique, absolutely continuous, countably additive set function ν on the Lebesgue measurable subsets of I with the property*

$$\nu((s, t]) = \xi(t) - \xi(s),$$

for $s < t$ in I. If ξ is strongly absolutely continuous, then ν is strongly absolutely continuous.

Proof. (The absolute continuity of ν is, of course, with respect to Lebesgue measure.) Let $\mathscr{T}$ denote the ring of all finite disjoint unions of subintervals of the form $(s, t]$. Then $\mathscr{T}$ is dense with respect to Lebesgue measure λ in the σ-ring $\mathscr{S}$ of Lebesgue measurable sets. Define $\mu((s, t])$ to be $\xi(t) - \xi(s)$, and extend μ in the obvious finitely additive way to $\mathscr{T}$. Clearly μ is λ-absolutely continuous on $\mathscr{T}$. Use Theorem 1 to extend μ to ν. The strong absolute continuity makes use of Theorem 2.

THEOREM 5. *Let X be weakly sequentially complete. If a point function $\xi : I \to X$ is weakly absolutely continuous, then ξ generates a unique, absolutely continuous, countably additive set function ν on the Lebesgue measurable subsets of I with the property*

$$\nu((s, t]) = \xi(t) - \xi(s)$$

for $s < t$ in I.

Proof. The proof is similar to that of Theorem 4, but we use Theorem 3 in place of Theorem 1.

COROLLARY 6. *A weakly absolutely continuous function on a compact real interval having values in a weakly sequentially complete space X is absolutely continuous.*

Proof. Let ξ be the function, $[a, b]$ the interval. Use Theorem 5 to generate the absolutely continuous set function ν. Then $\xi(t) = \xi(a) + \nu((a, t])$. Since ν is absolutely continuous, clearly so is ξ.

THEOREM 7. *There exist weakly absolutely continuous point functions, defined on compact real intervals and taking values in a Banach space, which are not absolutely continuous.*

Proof. We construct an example. Let c_0 be the B-space of all scalar sequences converging to zero; the norm of such a sequence is the maximum of the absolute values of its terms. It is well-known that c_0 is not weakly sequentially complete (see [**1**, pp. 239, 339]). Let $I = [0, 1]$. Let $\xi : I \to c_0$ be defined as follows: $\xi(0)$ is the zero vector in c_0; if $0 < t \leqq 1$, then $\xi(t) = \{\xi_n(t)\}_{n=1}^{\infty}$, where

$$\xi_n(t) = \int_0^t f_n(s)ds$$

and

$$f_n(s) = n, \text{ for } 0 \leqq s \leqq 1/2n,$$
$$f_n(s) = -n, \text{ for } 1/2n < s \leqq 1/n,$$
$$f_n(s) = 0, \text{ for } 1/n < s \leqq 1.$$

It is easy to see that ξ is not continuous at zero because $||\xi(1/2k)|| = 1/2$ for $k = 1, 2, 3, \ldots$. Hence ξ is certainly not absolutely continuous on I.

To show that ξ is weakly absolutely continuous on I we suppose $x^* \in c_0^*$. It is well-known [**1**, p. 74] that there corresponds to x^* an absolutely convergent series $\sum_{n=1}^{\infty}\lambda_n$ such that for all $z = \{\zeta_n\}_{n=1}^{\infty}$ belonging to c_0, $x^*z = \sum_{n=1}^{\infty}\lambda_n\zeta_n$. Now

$$x^*\xi(t) = \sum_{n=1}^{\infty} \int_0^t \lambda_n f_n(s)ds.$$

Since

$$\sum_{n=1}^{\infty} \int_0^t |\lambda_n f_n(s)|ds \leqq \sum_{n=1}^{\infty} |\lambda_n|,$$

by the Fubini-Tonelli theorem we have

$$x^*\xi(t) = \int_0^t \left[\sum_{n=1}^{\infty} \lambda_n f_n(s) \right] ds,$$

which implies that $x^*\xi$ is absolutely continuous.

The preceding example shows that the hypothesis of weak sequential completeness in Theorem 3 cannot be omitted entirely; for if it could be omitted, then the proof of Corollary 6 would render every weakly absolutely continuous point function absolutely continuous.

3. Absolutely continuous stochastic processes. We now apply the results of Section 2 to stochastic processes and to the theory of random ordinary differential equations as discussed in [**4**]. A *stochastic process* on an interval I is a function, call it x, from I into the set of random variables on a probability space $(\Omega, \mathscr{F}, P)$. We will write $x(t, \omega)$ for the value of x at $t \in I$ and $\omega \in \Omega$. A sample path of x is a function $x(\cdot, \omega)$ on I for some fixed $\omega \in \Omega$. If a sample path has a derivative at $t \in I$, that derivative is denoted by $x'(t, \omega)$.

A random variable z on Ω has a p-mean if

$$\int_{\Omega} |z(\omega)|^p P(d\omega) < \infty .$$

When z has a p-mean it will be convenient to use the notation $z^\wedge$ to denote the equivalence class of random variables which coincide with z a.e. on Ω. (Thus $z^\wedge \in L_p(\Omega)$.) If x is a process with a p-mean at each point of I, then $x^\wedge$ denotes the obvious function from I into $L_p(\Omega)$. We shall say that a process x is L_p-*absolutely continuous* on I if x has p-means everywhere on I and $x^\wedge$ is strongly absolutely continuous (with respect to Lebesgue measure). We call x W_p-*absolutely continuous* on I if x has p-means everywhere on I and $x^\wedge$ is absolutely continuous. Since the L_p spaces, $1 \leqq p < \infty$, are known to be weakly sequentially complete (see [**1**, pp. 69, 290]), it follows from Corollary 7 that if $x^\wedge$ is weakly absolutely continuous, then x is W_p-absolutely continuous.

It is of interest to find connections between various types of absolute continuity for a stochastic process. In Theorems 8 and 9 we relate W_p- and L_p-absolute continuity to the absolute continuity of the sample paths. We note that if x is a real-valued process on $I = [a, b]$, then each absolutely continuous sample path, $x(\cdot, \omega)$, has a derivative a.e. on I, and

$$x(t, \omega) = x(a, \omega) + \int_a^t x'(\tau, \omega) d\tau .$$

(The integral is Lebesgue's.)

THEOREM 8. *If x is a real stochastic process on the compact interval $I = [a, b]$, if almost every sample path of x is monotone and absolutely continuous, and if $x(a,\cdot)$ and $x(b, \cdot)$ have p-means for some finite $p \geqq 1$, then x is W_p-absolutely continuous on I.*

Proof. Assume without loss of generality that every sample path of x is absolutely continuous and monotone. (Some paths may be increasing, the rest decreasing.) The inequality $|x(t, \omega)| \leqq |x(a, \omega)| + |x(b, \omega)|$ shows that x has a p-mean at each point of I. We shall show that $x^\wedge$ is weakly absolutely continuous on I. (As remarked above, this will show x is W_p-absolutely continuous.) Let x^* be a continuous linear functional on $L_q(\Omega)$. As is well-known, there exists a function φ on Ω with $\varphi^\wedge \in L_p(\Omega)$, $(1/p) + (1/q) = 1$, such that for all functions y on Ω with $y^\wedge \in L_p(\Omega)$,

$$x^*(y^\wedge) = \int_{\Omega} \varphi(\omega) y(\omega) P(d\omega).$$

Let $E = \{\omega \in \Omega\colon x(a, \omega) < x(b, \omega)\}$, $F = \Omega \backslash E$, $G = \{\omega \in \Omega : \varphi(\omega) > 0\}$, and $H = \Omega \backslash G$. Next let $D_1 = E \cap G$, $D_2 = F \cap G$, $D_3 = E \cap H$, and $D_4 = F \cap H$. The absolute continuity of the sample paths of x implies that for each $t \in I$,

$$x(t, \omega) - x(a, \omega) = \int_a^t x'(\tau, \omega) d\tau .$$

Thus

$$x^*(x^\wedge(t) - x^\wedge(a)) = \int_\Omega \varphi(\omega)[x(t,\omega) - x(a,\omega)]P(d\omega)$$

$$= \sum_{i=1}^{4} \int_{D_i} \int_a^t \varphi(\omega)x'(\tau,\omega)d\tau P(d\omega).$$

Now each of the four integrals is finite, and each has an integrand of constant sign. Moreover, each integrand is product measurable because $\varphi(\omega)x'(\tau, \omega)$ is product measurable. (Clearly it suffices to show that $x'(\tau, \omega)$ is product measurable; this is not difficult since x has continuous sample paths and for each t, $x(t, \cdot)$ is measurable on Ω, whence $x(t, \omega)$ is measurable in the pair (t, ω). Now $x'(\tau, \omega)$ is the limit a.e. on $I \times \Omega$ of (product measurable) linear combinations of $x(t, \omega)$.) Applying the Fubini-Tonelli theorem to each double integral separately and summing we get

$$x^*(x^\wedge(t) - x^\wedge(a)) = \int_a^t d\tau \int_\Omega \varphi(\omega)x'(\tau,\omega)P(d\omega).$$

This shows that the function

$$\alpha(t) = \int_\Omega \varphi(\omega)x'(t,\omega)P(d\omega)$$

is integrable on I and that

$$x^*x^\wedge(t) = x^*x^\wedge(a) + \int_a^t \alpha(\tau)d\tau.$$

By classical theorems the real function on the right side of this equation is absolutely continuous on I. Hence $x^\wedge$ is weakly absolutely continuous.

The processes of Theorem 8 are actually L_1-absolutely continuous and in fact have L_1 derivatives a.e. on I, as we shall show below. An L_p *derivative* at a point $t \in I$ of a process x with p-means in a neighborhood of t is the strong derivative of $x^\wedge$, that is, an element $x^{\wedge\prime}(t)$ of $L_p(\Omega)$ for which

$$\lim_{k\to 0}||(x^\wedge(t+k) - x^\wedge(t))/k - x^{\wedge\prime}(t)|| = 0.$$

(See [**2**, p. 59].)

Note that while $x^{\wedge\prime}(t)$ is to be distinguished from $x'(t, \cdot)^\wedge$ (either may exist without the existence of the other), when $x^{\wedge\prime}(t)$ exists and $x'(t, \omega)$ exists for almost all $\omega \in \Omega$, then $x'(t, \cdot)$ has a p-mean and $x'(t, \cdot)^\wedge = x^{\wedge\prime}(t)$. This is because limits in p-norm and pointwise limits on Ω imply existence of limits in measure (P), and such limits are unique a.e. on Ω.

THEOREM 9. *If x is a real stochastic process on the compact interval $I = [a, b]$, if almost every sample path of x is monotone and absolutely continuous, and if $x(a, \cdot)$ and $x(b, \cdot)$ have 1-means, then x is L_1-absolutely continuous on I and has L_1 derivative $x'(t, \cdot)^\wedge$ for almost every $t \in I$.*

Proof. As in Theorem 8 we assume every sample path is monotone. As before, x has a 1-mean at each point of I. The absolute continuity of the sample paths yields

$$|x(b, \omega) - x(a, \omega)| = \left| \int_a^b x'(t, \omega)dt \right|.$$

The monotonicity yields

$$\left| \int_a^b x'(t, \omega)dt \right| = \int_a^b |x'(t, \omega)|dt.$$

Thus

$$(1) \quad ||x^\wedge(b) - x^\wedge(a)||_1 = \int_\Omega \int_a^b |x'(t, \omega)|dtP(d\omega) = \int_a^b dt \int_\Omega |x'(t, \omega)|P(d\omega).$$

This shows that the function

$$\alpha(t) = \int_\Omega |x'(t, \omega)|P(d\omega) = ||x'(t, \cdot)^\wedge||_1$$

is integrable on I; hence (cf. proof of Theorem 8) $x^\wedge$ is strongly absolutely continuous on I. Note moreover that

$$\int_I ||x'(t, \cdot)^\wedge||_1 dt < \infty$$

follows from (1), and this in turn shows that $x'(t, \cdot)$ has a 1-mean for almost all $t \in I$.

We now show that $x'(t, \cdot)^\wedge$ is the L_1 derivative of x. As noted in the proof of Theorem 8, $x'(t, \omega)$ is product measurable on $I \times \Omega$. Thus [**1**, p. 196, Lemma 16(b)], $x'(t, \cdot)^\wedge$ is a strongly measurable function on I with values in $L_1(\Omega)$. Since

$$\int_I ||x'(t, \cdot)^\wedge||_1 dt < \infty,$$

$x'(t, \cdot)^\wedge$ is Bochner-integrable [**2**, pp. 79,80]. Thus [**1**, p. 196] for all $t \in I$,

$$\int_a^t x'(\tau, \cdot)d\tau = (B)\int_a^t x'(\tau, \cdot)^\wedge d\tau,$$

in the sense that the random variable on the left is a member of the equivalence class of random variables represented by the Bochner integral on the right. This means that

$$x^\wedge(t) = x^\wedge(a) + (B)\int_a^t x'(\tau, \cdot)^\wedge d\tau,$$

and thus [**2**, p. 88] $x^\wedge$ has strong (L_1) derivative $x'(t, \cdot)^\wedge$ a.e. on I.

Since the assumptions of Theorems 8 and 9 are rather strong, it is worth pointing out that Theorem 9 fails if 1 is replaced by $p > 1$. For example if $(\Omega, \mathscr{F}, P)$ is taken to be the interval $[0, 1]$ with Lebesgue measure P, and if $I = [0, 1]$, then the process x defined by $x(t, \omega) = (t - \omega)^{1/2}$ for $\omega \leqq t$ and $x(t, \omega) = 0$ for $\omega > t$ is bounded and has monotone, absolutely continuous sample paths, but is not L_2-absolutely continuous, nor even of L_2 (strong) bounded variation on I. For if $0 \leqq s < t \leqq 1$, then

$$E|x(t, \cdot) - x(s, \cdot)|^2 = (1/2)(\sqrt{t} - \sqrt{s})(t^{3/2} - s^{3/2}) + ((t-s)^2/4) \log ((\sqrt{t} + \sqrt{s})/(\sqrt{t} - \sqrt{s})) > ((t-s)^2/8) \log (t/(t-s)).$$

Thus if $0 \leqq a = t_0 < \ldots < t_n = b \leqq 1$ and $t_k - t_{k-1} = (b - a)/n$, then

$$\sum^n ||x(t_k, \cdot) - x(t_{k-1}, \cdot)||_2 > ((b-a)/n\sqrt{8}) \sum^n (\log k)^{1/2}$$

and the last expression is unbounded as $n \to \infty$.

However, if the sample path derivative of the process x has p-means a.e. on I, then even though x may not have an L_p derivative, it does have what we shall call a *W_p derivative.* A W_p derivative of x is a pseudo-derivative, that is, a function y, defined a.e. on I and having values in $L_p(\Omega)$, such that for each continuous linear functional x^* on $L_p(\Omega)$, the scalar valued composite function $x^*x\hat{}$ has derivative x^*y a.e. on I (see [**5**, p. 300]).

THEOREM 10. *Let a real process x satisfy the hypotheses of Theorem 8. If in addition the (sample path) derivative process x' has p-means a.e. on I, then $x'\hat{}$ is a W_p derivative for x on I.*

Proof. Using the notation of the proof of Theorem 8, we recall that

$$x^*x\hat{}(t) = x^*x\hat{}(a) + \int_a^t \alpha(\tau)d\tau.$$

By the classical theorems of Lebesgue integration, the derivative of the right side of the equation exists for almost all $t \in I$ and is equal to

$$\alpha(t) = \int_\Omega \varphi(\omega)x'(\tau, \omega)P(d\omega) = x^*(x'(t, \cdot)\hat{}).$$

Theorems 9 and 10 may be applied to the theory of random ordinary differential equations to produce relationships between the various types of solutions, as discussed in [4]. Let I be the compact interval $[a, b]$, and for each $\omega \in \Omega$ let $S(\omega)$ be a subset of $I \times \mathbf{R}$. ($\mathbf{R}$ is the set of real numbers.) Let $f(\cdot, \cdot, \omega)$ be a function from $S(\omega)$ into $\mathbf{R}$. Let x_0 be a random variable such that for almost all $\omega \in \Omega$, $(a, x_0(\omega)) \in S(\omega)$. We consider the random differential equation

$$x' = f(t, x, \omega), \quad x(a, \omega) = x_0(\omega). \tag{2}$$

A function $x : I \times \Omega \to \mathbf{R}$ will be called a *sample path* (*S.P.*) *solution* of (2) if x is a stochastic process such that for almost every $\omega \in \Omega$ the following are true:

(a) $x(a, \omega) = x_0(\omega)$;

(b) for almost all $t \in I$, $(t, x(t, \omega)) \in S(\omega)$;

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(c) $x(\cdot, \omega)$ is absolutely continuous on I;

(d) for almost all $t \in I$, $x'(t, \omega) = f(t, x(t, \omega), \omega)$.

Other types of solutions for (2) can also be defined. In particular, an L_p-*solution* is a stochastic process x with p-means everywhere on I satisfying (a) and (b) above and also

(c′) x is L_p-absolutely continuous on I;

(d′) $x^{\wedge\prime}(t) = f(t, x(t, \cdot), \cdot)^{\wedge}$ for almost all $t \in I$, where $x^{\wedge\prime}(t)$ denotes the L_p derivative of x at t.

If we require only that x be W_p-absolutely continuous on I and if we substitute the W_p derivative for the L_p derivative in (d′), the process x is called a W_p-*solution* of (2).

COROLLARY 11. *Let x be a sample path solution of equation* (2) *on $I = [a, b]$, and suppose that almost every sample path of x is monotone. If $x(a, \cdot)$ and $x(b, \cdot)$ have* 1*-means, then x is an L_1-solution of* (2).

Proof. Since an S.P. solution has absolutely continuous sample paths, this follows immediately from Theorem 9.

COROLLARY 12. *Let x be a sample path solution of equation* (2) *on $I = [a, b]$ and suppose that almost every sample path of x is monotone. If, for some finite $p \geqq 1$, $x(a, \cdot)$ and $x(b, \cdot)$ have p-means, and if, for almost every $t \in I$, $f(t, x(t, \cdot), \cdot)$ has a p-mean, then x is a W_p-solution of* (2).

Proof. This follows from Theorems 8 and 10.

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