DISCRETE STRUCTURE SPACES OF f-RINGS

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Birkhoff and Pierce [2] introduced the concept of an f-ring and showed that an l-ring is an f-ring if and only if it is a subdirect product of totally-ordered rings. An l-ideal of an f-ring R is an algebraic ideal which is at the same time a lattice ideal of R. Structure spaces (i.e. sets of prime ideals endowed with the so-called hull-kernel or Stone topology) for ordinary rings have been studied by many authors. In this paper we consider certain analogues for f-rings, and give characterisations of f-rings for which these structure spaces are discrete.

DEFINITION 1. A proper l-ideal I of an f-ring R is said to be *l-prime* if it satisfies the condition $a \wedge b \in I$ implies $a \in I$ or $b \in I$. We shall write I is an *lp-ideal*, following Pierce [6], if this condition is satisfied; the set of all lp-ideals of R will be denoted by $LP(R)$, or simply LP if no confusion is likely.

DEFINITION 2. A proper l-ideal P of an f-ring R is said to be a *P-ideal* if it satisfies the condition $ab \in P$ implies $a \in P$ or $b \in P$, i.e. if it is an (algebraic) prime ideal; the set of all P-ideals will be denoted by $AP(R)$, or simply AP.

We now give some characterisations of lp-ideals and P-ideals, which will be used without reference in this paper.

LEMMA 1. *If I is an l-ideal of an f-ring R then the following conditions are equivalent*:

(1) *I is an lp-ideal*;
(2) *if A, B are l-ideals and $I \supseteq A \cap B$ then $I \supseteq A$ or $I \supseteq B$*;
(3) *if A, B are l-ideals and $I \subset A$ and $I \subset B$ then $I \subset A \cap B$*;
(4) *if $a, b \in R^+ \backslash I$ then $a \wedge b \in R^+ \backslash I$*;
(5) *if $a, b \in R^+ \backslash I$ then $a \wedge b > 0$*;
(6) *R/I is totally-ordered*;
(7) *the l-ideals containing I form a chain*;
(8) *$a \wedge b = 0$ implies $a \in I$ or $b \in I$*;
(9) *$a_1 \wedge a_2 \wedge \cdots \wedge a_n = 0$ implies $a_i \in I$ for some i*;
(10) *$a_1 \wedge a_2 \wedge \cdots \wedge a_n \in I$ implies $a_i \in I$ for some i*;

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PROOF. Conrad [3] proves the equivalence of (1)–(7) for l-groups while Subramanian [7] does likewise for (8)–(9). The proofs for f-rings are identical.

The following result, characterising P-ideals, appears in Johnson [4].

PROPOSITION 1. *If I is an l-ideal of an f-ring R then the following conditions are equivalent*:

(1) $ab \in I$ *implies* $a \in I$ *or* $b \in I$;

(2) *if* A, B *are* (l-)*ideals of* R, $AB \subseteq I$ *implies* $A \subseteq I$ *or* $B \subseteq I$;

(3) R/I *is totally-ordered and has no divisors of zero.*

LEMMA 2. *In an f-ring R, any P-ideal I is an lp-ideal.*

PROOF. This may be deduced from part (3) of proposition 1 and part (6) of lemma 1, or deduced directly as follows: suppose $a \wedge b = 0$; then since R is an f-ring, $ab = 0 \in I$, so $a \in I$ or $b \in I$.

It is now clear that both $LP(R)$ and $AP(R)$ can be given the well-known hull-kernel topology — see, for example, Kist [5] for details of this topology. The following result is then immediate.

LEMMA 3. *The inclusion mapping $i : AP \to LP$ is continuous.*

NOTATION. If $\Sigma \subseteq LP(R)$ we define the *kernel* of Σ, denoted by $k(\Sigma)$, as

$$k(\Sigma) = \cap\{P : P \in \Sigma\}.$$

If $A \subseteq R$ we define the *hull* of A, denoted by $h(A)$, as

$$h(A) = \{J \in LP : J \supseteq A\}.$$

If $\Sigma \subseteq LP(R)$,

$$\Sigma_a = \{P \in \Sigma : a \notin P\}$$

for each $a \in R$. It is well-known that the sets $(LP)_a$ (where $a \in R^+$) form a basis for the open sets of LP. In this paper R will always denote an f-ring.

DEFINITION 3. For a non-empty subset $A \subseteq R$ put

$$A^\perp = \{x \in R : |x| \wedge |y| = 0 \text{ for all } y \in A\}.$$

We write $a^\perp$ for $\{a\}^\perp$. An l-ideal I is said to be *a polar* if $I = I^{\perp\perp}$, where $I^{\perp\perp}$ stands for $(I^\perp)^\perp$.

Two preliminary results which will be used in the sequel have their analogues proved in Kist [5], and are stated here for ease of reference.

PROPOSITION 2. *If Σ is a dense subset of $LP(R)$, (i.e. if $k(\Sigma) = (0)$), then or any non-empty set $A \subseteq R$, $A^\perp = k(\Sigma \setminus h(A))$. In particular, for $a \in R$, $a^\perp = k(\Sigma_a)$.*

PROPOSITION 3. *Suppose* Σ *is a dense subset of* $LP(R)$ *and that any proper* l*-ideal of* R *is contained in some element of* Σ. *Then an* l*-ideal* I *of* R *is a direct summand if and only if* $h(I)$ *is open-closed in* Σ.

Since any direct summand is obviously a polar, we may deduce the following result.

LEMMA 4. *Every polar of* R *is a direct summand if and only if* $LP(R)$ *is extremally disconnected.*

PROOF. It follows from proposition 2 that the polars are precisely those l-ideals which are kernels of open subsets of $LP(R)$. If each polar is a direct summand then for each open subset $\Gamma \subseteq LP(R)$, $h(k(\Gamma))$ is open-closed by proposition 3, and this means $LP(R)$ is extremally disconnected. The converse is obvious.

We now give a characterisation of those f-rings R for which $LP(R)$ is discrete, and subsequently this result will be sharpened.

LEMMA 5. $LP(R)$ *is discrete if and only if each of the following conditions holds*:

(1) *each* $P \in LP(R)$ *is a minimal* lp*-ideal*; *and*

(2) *each* $P \in LP(R)$ *is a direct summand.*

PROOF. Suppose, firstly, that $LP(R)$ is discrete. Then for each $P \in LP(R)$, $\{P\}$ is open in the hk-topology. Thus there exists $x \in R$ such that $P \in (LP)_x \subseteq \{P\}$, i.e. P is the unique lp-ideal not containing x. If M is any minimal lp-ideal contained in P — such an M exists by Zorn's lemma — then $x \notin M$ so by the uniqueness of P, $M = P$; so P minimal. Thus the lp-ideals are not comparable (under set inclusion) and hence $\{P\} = h(P)$ and proposition 3 implies P is a direct summand.

Conversely, suppose conditions (1) and (2) hold, and suppose $P \in LP(R)$. Then (1) implies $\{P\} = h(P)$ and (2) implies $h(P)$ is open. Thus $LP(R)$ is discrete.

It shall be shown shortly that, in fact, condition (2) implies condition (1), but firstly we give some properties of f-rings R for which $LP(R)$ is discrete.

(A) Condition (1) implies that each lp-ideal of R is a maximal l-ideal. Hence each totally ordered homomorphic image of R has no proper l-ideals.

(B) If R is any f-ring then $\mathrm{Max}_L(R)$ — the space of all maximal l-deals — is a subspace of $LP(R)$. If there exists $e \in R$ such that e is not contained in any maximal l-ideal (e.g. if e is a multiplicative identity or a strong order unit) then it can be shown that $\mathrm{Max}_L(R)$ is compact. Hence, if in addition R satisfies the conditions of lemma 5, R has only a finite number of maximal l-ideals.

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(C) If $LP(R)$ is discrete then for all $x \in R$, $\langle x \rangle$ — the smallest l-ideal containing x — is a direct summand. In fact each l-ideal is a direct summand since $h(I)$ is open. Thus $R = \langle x \rangle \oplus x^{\perp}$ for all $x \in R$, and this implies that $\langle x \rangle = x^{\perp\perp}$.

(D) The two conditions in (C) imply that R is a projectable (or Stone) f-ring, i.e. $x^{\perp\perp} \oplus x^{\perp} = R$ for all $x \in R$.

(E) The discreteness of $LP(R)$ is not related to the existence of nilpotent elements in R, as the following examples show.

(i) Consider $\mathbb{R}^3$ with the usual pointwise operations and order. Then the (minimal) lp-ideals are $(0) \times \mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times (0) \times \mathbb{R}$, and $\mathbb{R} \times \mathbb{R} \times (0)$. These are the only lp-ideals and each is a direct summand. There are no nilpotents.

(ii) Consider $\mathbb{R}^3$ with the usual pointwise order and addition and with multiplication given by $(a_1, a_2, a_3)(b_1, b_2, b_3) = (0, a_2b_2, a_3b_3)$. The lp-ideals are the same as before : $(1, 0, 0)$ is a non-zero nilpotent.

(iii) The ring $C(N)$ of continuous real-valued functions defined on the natural numbers can be shown to have a non-discrete structure space (making use of remark (B)) and yet it has no non-zero nilpotent elements.

The next result shows which rings with discrete structure spaces have no non-zero nilpotent elements.

LEMMA 6. *Suppose $LP(R)$ is discrete. Then R has no non-zero nilpotent elements if and only if $LP(R)$ equals $AP(R)$ (see definition 2).*

PROOF. If $LP(R) = AP(R)$ then $k(AP) = k(LP) = (0)$ and this implies R has no nilpotents (Johnson [4]).

Conversely, suppose R has no nilpotent elements, $P \in LP$ and $xy \in P$. Since $LP(R)$ is discrete, P is a direct summand, and hence P is a polar. Thus

$$P = P^{\perp\perp} \supseteq (xy)^{\perp\perp} = x^{\perp\perp} \cap y^{\perp\perp}.$$

Since P is an lp-ideal, $x^{\perp\perp} \subseteq P$ or $y^{\perp\perp} \subseteq P$, thus $x \in P$ or $y \in P$.

To improve Lemma 5, we shall use results concerning polars and lp-ideals which have some independent interest.

PROPOSITION 4. *If A is a non-zero l-ideal of an f-ring R, the following conditions are equivalent:*

(1) *$A^{\perp}$ is an lp-ideal;*
(2) *each $a \in A \backslash (0)$ has precisely one value;*
(3) *A is totally-ordered;*
(4) *$A^{\perp}$ is a minimal lp-ideal;*

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(5) $A^{\perp\perp}$ *is a minimal polar;*
(6) $A^{\perp}$ *is a maximal polar;*
(7) $A^{\perp} = a^{\perp}$, *for all* $a \in A \backslash (0)$;
(8) $A^{\perp\perp}$ *is a maximal totally-ordered l-ideal;*

PROOF. Conrad [3] has proved the equivalence of these conditions in the setting of l-groups. Since f-rings are characterised among the l-rings by the property $\langle a \wedge b \rangle = \langle a \rangle \cap \langle b \rangle$ for a, b positive (unpublished result of the author), the result for l-groups can be used to prove the analogue for f-rings.

As a corollary to this we have the following lemma which extends a result of Anderson [1, lemma 5], but the method of proof here is different.

LEMMA 7. *Let R be an f-ring and consider the following conditions for an l-ideal I of R:*
(1) *I is a P-ideal and* $I^{\perp} \neq (0)$;
(2) *I is an lp-ideal and* $I^{\perp} \neq (0)$;
(3) *I is a maximal (proper) polar in R.*

Then (1) *implies* (2), (2) *implies* (3), *and* (3) *implies* (2). *If in addition R has no non-zero nilpotent elements then* (3) *implies* (1), *and hence in this case the conditions are equivalent.*

PROOF. (1) implies (2), obviously.

(2) $\Rightarrow$ (3). Since $I^{\perp} \neq (0)$ $I^{\perp\perp}$ is a proper l-ideal, and since $I \subseteq I^{\perp\perp}$, $I^{\perp\perp}$ is an lp-ideal. By the previous proposition, $I^{\perp\perp}$ is a minimal lp-ideal, so $I = I^{\perp\perp}$, and again by that proposition, I is a maximal polar of R. (3) implies (2): This, also, follows from the previous proposition.

Now suppose R has no nilpotents. To complete the proof it suffices to show that (2) implies (1). Therefore suppose $I^{\perp} \neq (0)$, I is an lp-ideal, and that $ab \in I$. Then $I = I^{\perp\perp} \supseteq (ab)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}$, and since I is an lp-ideal it follows that $a \in I$ or $b \in I$.

THEOREM 1. *If R is an f-ring the following conditions are equivalent:*
(1) *LP(R) is discrete;*
(2) *each lp-ideal is a direct summand;*
(3) *R is a direct sum of totally ordered rings with no proper l-ideals;*
(4) *each l-ideal of R is a direct summand;*
(5) *each lp-ideal is a polar.*

PROOF. (1) *implies* (2): by lemma 5. (2) implies (3): For each lp-ideal P_{λ}, $R = P_{\lambda} \oplus T_{\lambda}$ where T_{λ} is a totally ordered ring. We show that R is the direct sum of these totally ordered rings. Firstly by proposition 4, each T_{λ} has no has no proper l-ideals. Thus the direct sum ΣT_{λ} of these l-ideals is contained in R. If there were an element $r \in R \backslash (\Sigma T_{\lambda})$ then there would be an lp-ideal $P_{\alpha} \supseteq \Sigma T_{\lambda}$

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such that $r \notin P_\alpha$. By assumption P_α is a direct summand, so $R = P_\alpha \oplus T_\alpha$, and by choice of P_α, $T_\alpha \subseteq P_\alpha$. Hence $R = P_\alpha \oplus T_\alpha \subseteq P_\alpha$, which is a contradiction. Thus $R = \Sigma T_\lambda$.

(3) *implies* (4): If J is any l-ideal of of of R, and $R = \Sigma_\Lambda R_\alpha$, where for each $\alpha \in \wedge$, R_α is a totally order ring with no proper l-ideals, then $J = \Sigma(R_\alpha \cap J)$ and for each $\alpha R_\alpha \cap J = R_\alpha$ or $R_\alpha \cap J = (0)$. Thus $J = \Sigma_{\Lambda'} R$ for some subset $\Lambda' \subseteq \Lambda$.

(4) *implies* (5): Trivial since any direct summand is a polar.

(5) *implies* (1): Let $P \in LP(R)$. Then by the previous lemma $P^\perp \neq (0)$. Since P is a polar, proposition 4 implies P is a minimal lp-ideal; hence $LP(R)$ equals $\mathscr{M}$ — the space of minimal lp-ideals. Also by proposition 4, $P = P^{\perp\perp} = a^\perp$ for all $a \in P^\perp \backslash (0)$. So,

$$P = a^\perp = k((LP)_a) = k(\mathscr{M}_a).$$

Now, it is easy to show that $\mathscr{M}_a$ is open-closed (in $\mathscr{M}$), so $h(P) = hk(\mathscr{M}_a) = \mathscr{M}_a$, which implies P is a direct summand. Hence, by lemma 5, $LP(R)$ is discrete.

There is another characterisation, in terms of the lattice of all l-ideals of R, of those f-rings R for which the structure space $LP(R)$ is discrete, and we note this result now.

Definition 4. The set $\mathscr{L}(R)$ of all l-ideals of an f-ring R is a lattice under the operations $+$ and $\cap$. It is well known that this lattice is distributive. (R) is said to be *complemented* if for each $I \in \mathscr{L}(R)$ there exists an l-ideal J such that $I + J = R$ and $I \cap J = (0)$. Clearly, in this case $I^\perp = J$.

Lemma 8. *(R) is complemented if and only if $LP(R)$ is discrete.*

Proof. Obvious.

Theorem 1 can be strengthened for f-rings with no nilpotent elements which also satisfy another fairly innocuous condition.

Theorem 2. *Suppose that R is an f-ring with no non-zero nilpotent elements, and that each proper l-ideal of R is contained in a P-ideal. Then the following conditions are equivalent:*

(1) *$AP(R)$ is discrete;*

(2) *each P-ideal is a direct summand;*

(3) *each P-ideal is a polar;*

(4) *each P-ideal I is a minimal lp-ideal, and $I^\perp \neq (0)$;*

(5) *each lp-ideal I is a minimal lp-ideal, and $I^\perp \neq (0)$;*

(6) *$LP(R)$ is discrete;*

(7) *R is a direct sum of totally ordered integral domains with no proper l-ideals;*

(8) *each l-ideal of R is a direct summand.*

PROOF. (1) implies (2): Since R has no nilpotents $k(AP(R)) = (0)$, and proposition 3 may be applied.

(2) implies (3): Obviously.

(3) implies (4): Follows from proposition 4.

(4) implies (5): Follows from the hypothesis.

(5) implies (6): Follows from lemma 7.

(6) implies (1): Obviously.

Clearly, by theorem 1, (6), (7), and (8) are equivalent.

REMARKS. (1) Any f-ring with identity satisfies the condition that each proper l-ideal is contained in a P-ideal.

(2) It is possible to have $AP(R) = LP(R)$ even when $LP(R)$ is not discrete. Rings characterised by the property that $AP(R) = LP(R)$ are the subject of another paper.

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