

POSITIVE PERIODIC SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Abstract

In this work, we consider the periodic boundary value problem

$$\begin{cases} x'' + a(t)x = f(t, x) + c(t), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where $a, c \in L^1(0, T)$ and f is a Carathéodory function. An existence theorem for positive periodic solutions is proved in the case where the associated Green function is nonnegative. Our result is valid for systems with strong singularities, and answers partially the open problem raised in Torres [*Weak singularities may help periodic solutions to exist*, *J. Differential Equations* **232** (2007), 277–284].

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1. Introduction and preliminaries

Consider the periodic boundary value problem

$$x'' + a(t)x = f(t, x) + c(t), \tag{1}$$

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{2}$$

where $a, c \in L^1(0, T)$ and $f : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ is a function of L^1 -Carathéodory type, that is, it is continuous in the second variable and for every $0 < r < s$ there exists $h_{r,s} \in L^1(0, T)$ such that $|f(t, x)| \leq h_{r,s}(t)$ for all $x \in [r, s]$ and almost every $t \in [0, T]$. We are interested in equation (1) with a singularity in $x = 0$, which means

$$\lim_{x \rightarrow 0^+} f(t, x) = +\infty \quad \text{uniformly almost every } t \in [0, T].$$

The singularities appear when electrostatic or gravitational forces are considered.

In recent years, the periodic boundary value problem (1)–(2) has attracted attention from many specialists in differential equations. Lazer and Solimini [6] studied this

problem in the special case $f(t, x) = 1/x^\lambda$. They showed that the strong force assumption $\lambda \geq 1$ is necessary in some sense for the existence of positive periodic solutions. Since then, the strong force condition has become standard in related work; see, for instance, [1, 3–5, 7–9, 13, 14] and the references therein. Compared with the case of strong singularities, the study of the existence of periodic solutions under the presence of weak singularities is much more recent. Here we refer the reader to [2, 9–11].

In [12], the boundary value problem (1)–(2) is studied by using Schauder's fixed point theorem, and some existence results prove that in some situations weak singularities may help create periodic solutions. However, 'the validity of such results under a strong force assumption remains as an open problem' (see [12]). The purpose of this paper is to give a partial answer to this open problem, namely, we prove that [12, Theorem 2] is valid under a strong force assumption if $a > 0$ and $\|a\|_1 T < 1$ (see Theorem 1 below).

Let us fix some notation. Given $a \in L^1(0, T)$, we write $a > 0$ if $a \geq 0$ for almost every $t \in [0, T]$ and it is positive in a set of positive measure. We denote by a_* and a^* the essential infimum and supremum of a , respectively. The usual supremum norm and L^1 -norm are denoted by $\|\cdot\|$ and $\|\cdot\|_1$, respectively.

2. Main results

Throughout this paper, we assume the following hypothesis:

(H1) Green's function $G(t, s)$ of the linear equation $x'' + a(t)x = 0$ with periodic condition (2) is nonnegative for every $(t, s) \in [0, T] \times [0, T]$.

Under condition (H1), the unique T -periodic solution of $x''(t) + a(t)x = h(t)$ is given by

$$x(t) = \int_0^T G(t, s)h(s) ds.$$

We remark that, when $a(t) = k^2 > 0$, it is not hard to see that condition (H1) is equivalent to $0 < k^2 \leq (\pi/T)^2$. For a nonconstant $a(t)$, we refer to an L^p -criterion given as [12, Lemma 1], where the condition $a > 0$ is needed. So the assumption $a > 0$ in this paper is natural.

The following theorem is [12, Theorem 1], and will be used in the proof of our main result.

THEOREM A. *Assume (H1) and that there exist $b > 0$ and $\lambda > 0$ such that*

$$0 \leq f(t, x) \leq \frac{b(t)}{x^\lambda} \quad \text{for all } x > 0, \text{ for almost every } t.$$

If $\gamma_ > 0$, then there exists a positive T -periodic solution of (1).*

LEMMA 1. Assume that $a > 0$. There exists $0 < \eta < 1$ such that, for any $t_0, t_1 \in [0, T]$,

$$0 \leq \int_{t_0}^{t_1} \int_{t_0}^r a(s) \, ds \, dr \leq \eta T \|a\|_1. \tag{3}$$

PROOF. The first part of inequality (3) is clear. So it suffices to prove the second part. We can easily see that there exist $T_1, T_2, T_3 \in (0, T)$, $T_1 < T_2 < T_3$, such that

$$\int_0^{T_i} a(s) \, ds = \frac{i \|a\|_1}{4}, \quad i = 1, 2, 3.$$

Let $l = \min\{T_1, T_2 - T_1, T_3 - T_2, T - T_3\}$. For any $t \in [0, T]$, set

$$A(t) = \left\{ \delta(t) : \left| \int_t^{t+\delta(t)} a(s) \, ds \right| = \frac{\|a\|_1}{2} \right\}.$$

Then $|\delta(t)| \geq l$ for any $\delta(t) \in A(t)$, $t \in [0, T]$.

Without loss of generality, we may assume that $t_0 < t_1$. If there does not exist any $\delta(t_0) \in A(t_0)$ such that $t_0 + \delta(t_0) \in [t_0, t_1]$, we can get easily that, for any $r \in [t_0, t_1]$, $\int_{t_0}^r a(s) \, ds \leq \|a\|_1/2$. So

$$\int_{t_0}^{t_1} \int_{t_0}^r a(s) \, ds \, dr \leq \frac{1}{2} T \|a\|_1.$$

If there exists $\delta(t_0) \in A(t_0)$ such that $t_0 + \delta(t_0) \in [t_0, t_1]$, then $\delta(t_0) > l$, and

$$\begin{aligned} \int_{t_0}^{t_1} \int_{t_0}^r a(s) \, ds \, dr &= \left(\int_{t_0}^{t_0+\delta(t_0)} + \int_{t_0+\delta(t_0)}^{t_1} \right) \int_{t_0}^r a(s) \, ds \, dr \\ &\leq \delta(t_0) \|a\|_1/2 + |t_1 - t_0 - \delta(t_0)| \|a\|_1 \\ &\leq \delta(t_0) \|a\|_1/2 + (T - \delta(t_0)) \|a\|_1 \\ &= (T - \delta(t_0)/2) \|a\|_1 \\ &\leq (T - l/2) \|a\|_1. \end{aligned}$$

Let $\eta = \max\{(1/2), 1 - (l/2T)\}$. Then (3) is true. □

Now we are in a position to give our main result.

THEOREM 1. Let us assume (H1) and that there exist $b, \hat{b} > 0$ and $\lambda > 0$ such that

$$(H2) \quad 0 \leq \frac{\hat{b}(t)}{x^\lambda} \leq f(t, x) \leq \frac{b(t)}{x^\lambda} \quad \text{for all } x > 0, \text{ for almost every } t.$$

If $\gamma_* = 0$, $a > 0$ and $T \|a\|_1 < 1$, then there exists a positive T -periodic solution of (1).

PROOF. We consider the following family of systems,

$$x'' + a(t)x(t) = f(t, x) + c(t) + \frac{a(t)}{n}, \tag{4}$$

with n any natural number. Noticing that $\int_0^T G(t, s)a(s) ds = 1$ and replacing $c(t)$ by $c(t) + a(t)/n$, it is easy to get from Theorem A that there exists a positive T -periodic solution x_n of (4) for each n .

For the sake of convenience, we assume that

$$\begin{aligned}\gamma(t) &= \int_0^T G(t, s)c(s) ds, & \beta(t) &= \int_0^T G(t, s)b(s) ds, \\ \hat{\beta}(t) &= \int_0^T G(t, s)\hat{b}(s) ds.\end{aligned}$$

Now we complete the proof in the following two steps.

STEP 1. Suppose that $x_n(t) \geq (1 - \|a\|_1 T)\|x_n\|$ for all $t \in [0, T]$. Clearly, the T -periodic solution x_n of (4) satisfies

$$x_n(t) = \int_0^T G(t, s)[f(s, x_n(s)) + c(s) + a(s)/n] ds.$$

So by (H2),

$$\begin{aligned}x_n(t) &\leq \int_0^T G(t, s)f(s, x_n(s)) ds + \gamma(t) + 1/n \\ &\leq \int_0^T \frac{G(t, s)b(s)}{x_n^\lambda(s)} ds + \gamma(t) + 1/n \\ &\leq \frac{\beta^*}{((1 - \|a\|_1 T)\|x_n\|)^\lambda} + \gamma^* + 1.\end{aligned}$$

This implies that the sequence $\{x_n\}$ is uniformly bounded, that is, there exists $R < \infty$ such that $\|x_n\| \leq R$ for all n . As a result,

$$\begin{aligned}x_n(t) &= \int_0^T G(t, s)[f(s, x_n(s)) + c(s) + a(s)/n] ds \\ &\geq \int_0^T G(t, s)\frac{\hat{b}(s)}{R^\lambda} ds \\ &\geq \frac{\hat{\beta}_*}{R^\lambda} =: r.\end{aligned}$$

We note that $\hat{\beta}_* > 0$ as a consequence of (H1) and $\hat{b} > 0$. Thus $0 < r \leq x_n(t) \leq R$ for all n and $t \in [0, T]$.

On the other hand, since $x'_n(0) = x'_n(T)$, by (4),

$$\int_0^T a(s)x_n(s) ds = \int_0^T [f(s, x_n(s)) + c(s) + a(s)/n] ds.$$

Meanwhile, there exists $t_{n0} \in [0, T]$ such that $x'_n(t_{n0}) = 0$ since $x_n(0) = x_n(T)$. Then, by (4) again,

$$x'_n(t) = \int_{t_{n0}}^t [f(s, x_n(s)) + c(s) + a(s)/n - a(s)x_n(s)] ds.$$

Hence,

$$\begin{aligned} \|x'_n\| &= \max_{t \in [0, T]} \left| \int_{t_{n0}}^t [f(s, x_n(s)) + c(s) + a(s)/n - a(s)x_n(s)] ds \right| \\ &\leq \int_0^T [f(s, x_n(s)) + c(s) + a(s)/n] ds + \int_0^T a(s)x_n(s) ds \\ &= 2 \int_0^T a(s)x_n(s) ds \\ &\leq 2R\|a\|_1. \end{aligned}$$

Therefore, the sequence $\{x_n\}$ is uniformly bounded and equicontinuous. By the Arzelà–Ascoli theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges uniformly to a continuous function x , and $0 < r \leq x(t) \leq R, t \in [0, T]$. Moreover,

$$\begin{aligned} x_{n_k}(t) &= \int_0^T G(t, s)[f(s, x_{n_k}(s)) + c(s) + a(s)/n] ds \\ &\rightarrow \int_0^T G(t, s)[f(s, x(s)) + c(s)] ds \end{aligned}$$

as $k \rightarrow \infty$. So $x(t)$ is a positive T -periodic solution of (1).

STEP 2. Suppose that $x_n(t) < (1 - \|a\|_1 T)\|x_n\|$ for some $t \in [0, T]$. We first prove that there exists $t_{n0} \in [0, T]$ such that

$$x_n(t_{n0}) = \|x_n\|, \quad x'_n(t_{n0}) = 0. \tag{5}$$

If $x_n(0) \neq \|x_n\|$, it is clear that there exists $t_{n0} \in (0, T)$ such that (5) holds. If $x_n(0) = \|x_n\|$, it is sufficient to prove that $x'_n(0) = 0$. In fact, we can see easily that $x'_n(0) > 0$ contradicts the fact that 0 is the maximum point of x_n . If $x'_n(0) < 0$, then $x'_n(T) = x'_n(0) < 0$, which is also in contradiction to $x_n(0) = x_n(T) = \|x_n\| = \max\{|x_n(t)| : t \in [0, T]\}$. Therefore, $x'_n(0) = 0$, and (5) is true.

Meanwhile, it follows from the assumption of this step that there exists $t_{n1} \in [0, T]$ such that $x_n(t_{n1}) = (1 - \|a\|_1 T)\|x_n\|$ and $x_n(t) \geq (1 - \|a\|_1 T)\|x_n\|$ if t is between t_{n0} and t_{n1} . Then by (4) and (5),

$$x'_n(t) = \int_{t_{n0}}^t [f(s, x_n(s)) + c(s) + a(s)/n - a(s)x_n(s)] ds$$

and

$$x_n(t) - \|x_n\| = \int_{t_{n0}}^t \int_{t_{n0}}^r [f(s, x_n(s)) + c(s) + a(s)/n - a(s)x_n(s)] ds dr.$$

Therefore, by Lemma 1 we can get

$$\begin{aligned} \|a\|_1 T \|x_n\| &= |x_n(t_{n1}) - \|x_n\|| \\ &= \left| \int_{t_{n0}}^{t_{n1}} \int_{t_{n0}}^r [f(s, x_n(s)) + c(s) + a(s)/n - a(s)x_n(s)] ds dr \right| \\ &\leq \int_{t_{n0}}^{t_{n1}} \int_{t_{n0}}^r \left[\frac{b(s)}{x_n^\lambda(s)} + c(s) + \frac{a(s)}{n} \right] ds dr + \int_{t_{n0}}^{t_{n1}} \int_{t_{n0}}^r a(s)x_n(s) ds dr \\ &\leq \int_{t_{n0}}^{t_{n1}} \left(\frac{\|b\|_1}{((1 - \|a\|_1 T)\|x_n\|)^\lambda} + \|c\|_1 + \|a\|_1 \right) dr + \eta T \|a\|_1 \|x_n\| \\ &\leq \left(\frac{\|b\|_1}{((1 - \|a\|_1 T)\|x_n\|)^\lambda} + \|c\|_1 + \|a\|_1 \right) T + \eta T \|a\|_1 \|x_n\|, \end{aligned}$$

that is,

$$(1 - \eta) \|a\|_1 \|x_n\| \leq \frac{\|b\|_1}{((1 - \|a\|_1 T)\|x_n\|)^\lambda} + \|c\|_1 + \|a\|_1.$$

This also implies the uniform boundedness of $\{x_n\}$ since $\lambda > 0$, and by the same argument at the end of step 1 we can get a positive T -periodic solution of (1). The proof is complete. \square

REMARK. Theorem 1 gives a partial answer to the open problem given in [12]: our result is valid under the strong force assumption $\lambda \geq 1$ if $a > 0$ and $\|a\|_1 T < 1$. However, the case $\|a\|_1 T \geq 1$ under the strong force assumption still remains an open problem.

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