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A NEW MINIMIZATION PRINCIPLE FOR THE POISSON EQUATION LEADING TO A FLEXIBLE FINITE ELEMENT APPROACH

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Abstract

A new minimization principle for the Poisson equation using two variables – the solution and the gradient of the solution – is introduced. This principle allows us to use any conforming finite element spaces for both variables, where the finite element spaces do not need to satisfy the so-called inf–sup condition. A numerical example demonstrates the superiority of this approach.

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1. Introduction

It is often important to get an accurate approximation of the gradient of the solution of a Poisson equation. In that case, a mixed formulation of the Poisson equation is used, where there are two unknowns – the solution and its gradient – in the variational equation. Discretizing a mixed formulation of a partial differential equation is a challenging task as the involved finite element spaces should satisfy a compatibility condition, that is, the so-called inf–sup condition [6]. Although there are many finite element spaces discovered satisfying the compatibility condition for the Poisson equation [2, 6–8, 13, 15], it is not so easy for mixed formulations of other partial differential equations. It is sometimes useful to use a least-squares finite element method to approximate the solution and its gradient simultaneously [3–5]. A least-squares formulation allows the use of any conforming finite element spaces avoiding the compatibility condition.

In this paper, we propose a new minimization principle for the Poisson equation using the solution and the gradient of the solution as two unknowns. This formulation

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is similar to a least-squares formulation in the sense that it allows the use of any conforming finite element spaces avoiding the compatibility condition [3–5, 10]. However, compared to a least-squares finite element method, the source term f can be in the dual of a H^1-space, and the gradient can be discretized using a L^2-conforming finite element space. We also give optimal a priori error estimates for the proposed finite element method.

The structure of the rest of the paper is organized as follows. In the next section, we introduce our formulation and show its well-posedness. We propose finite element methods for the given formulation and prove a priori error estimates in Section 3. A numerical example with discretization errors is presented in Section 4 and a short conclusion is given in Section 5.

2. A new formulation of the Poisson equation

In this section, we introduce a new minimization principle of the Poisson problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded convex domain with polygonal or polyhedral boundary $\partial\Omega$ with the outward pointing normal $\mathbf{n}$ on $\partial\Omega$. We start with the following minimization problem for the Poisson problem.

Problem 2.1. Given $f \in H^{-1}(\Omega)$, we want to find

$$u = \arg\min_{v \in H_0^1(\Omega)} K(v) \tag{2.1}$$

with

$$K(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \ell(v),$$

where

$$\ell(v) = \int_\Omega f v\,dx.$$

We refer to the literature [1, 2, 6–8, 11, 13–15] for different variational formulations of the Poisson equation.

Let $V = H_0^1(\Omega)$ and $\mathbf{Q} = [L^2(\Omega)]^d$. For two vector-valued functions $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^d$ and $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$, let the Sobolev inner product on the Sobolev space $H^k(\Omega)$ ($k \in \mathbb{R}$) be defined as

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{k,\Omega} = \sum_{i=1}^{d} \langle \alpha_i, \beta_i \rangle_{k,\Omega},$$

where $(\boldsymbol{\alpha})_i = \alpha_i$, $(\boldsymbol{\beta})_i = \beta_i$ with $\alpha_i, \beta_i \in H^k(\Omega)$ for $i = 1, \ldots, d$, and the norm $\|\cdot\|_{H^k(\Omega)}$ is induced by this inner product. We use the standard notation $\|\cdot\|_{k,\Omega}$ for the norm in the $H^k(\Omega)$-space. We now introduce a functional

$$J_{\alpha,\gamma}(v, \boldsymbol{\tau}; f) = \tfrac{1}{2}(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\boldsymbol{\tau} - \alpha\nabla v\|_{0,\Omega}^2) - \gamma\ell(v),$$

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where $\alpha > 0$ and γ are two fixed constants, and consider another minimization problem for the two variables $(v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}]$,

$$\operatorname*{arg\,min}_{(v,\boldsymbol{\tau})\in[V\times\mathbf{Q}]} J_{\alpha,\gamma}(v, \boldsymbol{\tau}; f). \tag{2.2}$$

This minimization problem is equivalent to finding $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = \gamma\ell(v), \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}],$$

where the bilinear form $a(\cdot, \cdot)$ is defined as

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - \alpha\nabla u, \boldsymbol{\tau} - \alpha\nabla v)_{0,\Omega}.$$

Standard arguments can be used to show the continuity of the bilinear form $a(\cdot, \cdot)$ on the space $V \times \mathbf{Q}$. Now we show that the bilinear form $a(\cdot, \cdot)$ is coercive on $V \times \mathbf{Q}$.

Lemma 2.2. *Let $\alpha > 0$. For $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$, the bilinear form $a(\cdot, \cdot)$ satisfies*

$$a((u, \boldsymbol{\sigma}), (u, \boldsymbol{\sigma})) \geq \frac{\alpha^2}{\alpha^2 + 2C_1}(\|u\|^2_{1,\Omega} + \|\boldsymbol{\sigma}\|^2_{0,\Omega}),$$

where C_1 is the constant in the Poincaré inequality

$$\|u\|^2_{1,\Omega} \leq C_1\|\nabla u\|^2_{0,\Omega}.$$

Proof. The proof follows from a triangle inequality and Poincaré inequality:

$$\begin{aligned}
\|u\|^2_{1,\Omega} + \|\boldsymbol{\sigma}\|^2_{0,\Omega} &\leq \frac{C_1}{\alpha^2}\|\alpha\nabla u\|^2_{0,\Omega} + \|\boldsymbol{\sigma}\|^2_{0,\Omega} \\
&\leq \frac{2C_1}{\alpha^2}[\|\boldsymbol{\sigma} - \alpha\nabla u\|^2_{0,\Omega} + \|\boldsymbol{\sigma}\|^2_{0,\Omega}] + \|\boldsymbol{\sigma}\|^2_{0,\Omega} \\
&\leq \frac{2C_1 + \alpha^2}{\alpha^2}(\|\boldsymbol{\sigma}\|^2_{0,\Omega} + \|\boldsymbol{\sigma} - \alpha\nabla u\|^2_{0,\Omega}) \\
&= \frac{2C_1 + \alpha^2}{\alpha^2}a((u, \boldsymbol{\sigma}), (u, \boldsymbol{\sigma})).
\end{aligned}$$

□

Corollary 2.3. *Since the bilinear form $a(\cdot, \cdot)$ is continuous and coercive on $V \times \mathbf{Q}$, and the linear form $\ell(v)$ is also continuous on V for $f \in H^{-1}(\Omega)$, the problem of finding $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that*

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = \gamma\ell(v), \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}],$$

has a unique solution from the Lax–Milgram lemma [7].

Remark 2.4. In contrast to the standard least-squares method, where we need $f \in L^2(\Omega)$, we have $f \in H^{-1}(\Omega)$. Thus, the standard least-squares method cannot handle the situation if the source function is not in L^2, whereas the new approach requires exactly the same regularity for f as the standard Galerkin approach. Moreover, the gradient can be discretized just by using L^2-conforming finite elements.

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Let $(u_e, \boldsymbol{\sigma}_e) \in V \times \mathbf{Q}$ be the solution of the minimization problem (2.2). We now choose α and γ in such a way that the solution $(u, \boldsymbol{\sigma})$ of the minimization problem (2.2) satisfies $u = u_e$ and $\boldsymbol{\sigma}_e = \nabla u$. Here the natural norm for an element $(v, \boldsymbol{\tau}) \in V \times \mathbf{Q}$ of the product space $V \times \mathbf{Q}$ is $\sqrt{\|v\|_{1,\Omega}^2 + \|\boldsymbol{\tau}\|_{0,\Omega}^2}$. Thus, (2.2) leads to the problem of finding $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - \alpha\nabla u, \boldsymbol{\tau} - \alpha\nabla v)_{0,\Omega} - \gamma\ell(v) = 0, \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}].$$

Letting the test functions $\boldsymbol{\tau} = \mathbf{0}$ and $v = 0$ successively in the above equation leads to

$$\begin{aligned} -(\boldsymbol{\sigma} - \alpha\nabla u, \alpha\nabla v)_{0,\Omega} - \gamma\ell(v) &= 0, \quad v \in V, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - \alpha\nabla u, \boldsymbol{\tau})_{0,\Omega} &= 0, \quad \boldsymbol{\tau} \in \mathbf{Q}. \end{aligned} \tag{2.3}$$

The second equation immediately yields

$$(2\boldsymbol{\sigma} - \alpha\nabla u, \boldsymbol{\tau})_{0,\Omega} = 0, \quad \boldsymbol{\tau} \in \mathbf{Q},$$

and hence $\alpha = 2$ ensures that $\boldsymbol{\sigma} = \nabla u$. Using $\boldsymbol{\sigma} = \nabla u$ in the first equation of (2.3),

$$-\alpha(1 - \alpha)(\nabla u, \nabla v)_{0,\Omega} - \gamma\ell(v) = 0.$$

We have the standard variational problem for the Poisson equation if $\gamma = \alpha(\alpha - 1)$ and, thus, setting $\alpha = 2$, we get $\gamma = 2$. Now we have the following problem.

Problem 2.5. Given $f \in H^{-1}(\Omega)$, the variational equation for the minimization problem is to find $(u, \boldsymbol{\sigma}) \in [V \times \mathbf{Q}]$ such that

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = 2\ell(v), \quad (v, \boldsymbol{\tau}) \in [V \times \mathbf{Q}],$$

where the bilinear form $a(\cdot, \cdot)$ is defined as

$$a((u, \boldsymbol{\sigma}), (v, \boldsymbol{\tau})) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - 2\nabla u, \boldsymbol{\tau} - 2\nabla v)_{0,\Omega}.$$

From the above discussion we have the following theorem.

Theorem 2.6. *Let u and $(\tilde{u}, \tilde{\boldsymbol{\sigma}})$ be the solutions of Problems 2.1 and 2.5, respectively. Then we have $\tilde{u} = u$ and $\tilde{\boldsymbol{\sigma}} = \nabla u$.*

Remark 2.7. The idea can be easily generalized to a general differential equation, which can be put in a minimization framework. For example, consider the solution of the linear elastic problem of finding the displacement field $\mathbf{u} \in \mathbf{V} = [H_0^1(\Omega)]^d$ such that [6]

$$\mathbf{u} = \arg\min_{\mathbf{v} \in \mathbf{V}} \frac{1}{2} \int_\Omega \varepsilon(\mathbf{v}) : C\varepsilon(\mathbf{v})\, dx - \ell(\mathbf{v}),$$

where $\varepsilon(\mathbf{v}) = (\nabla\mathbf{v} + [\nabla\mathbf{v}]^T)/2$ is the symmetric part of the gradient, C is Hooke's tensor [12], and $\ell(\cdot)$ is a linear form

$$\ell(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx.$$

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By defining a pseudo-stress $\boldsymbol{\sigma} = \sqrt{C}\varepsilon(\mathbf{v})$, with $\sqrt{C}$ denoting the square root of the tensor C, we can put this in the above framework with

$$a(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{v}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - 2\sqrt{C}\varepsilon(\mathbf{u}), \boldsymbol{\tau} - 2\sqrt{C}\varepsilon(\mathbf{v}))_{0,\Omega}.$$

We note that since C is a symmetric positive definite tensor, its square root is well defined. Similarly, if the functional $K(\cdot)$ in the minimization problem (2.1) is given as

$$K(v) = \frac{1}{2}\int_{\Omega} |\kappa \nabla v|^2 \, dx - \ell(v),$$

where $\kappa : \mathbb{R}^d \to\in \mathbb{R}^{d\times d}$ is a positive definite matrix function, the same formulation is obtained using $\boldsymbol{\sigma} = \sqrt{\kappa}\nabla u$ and the bilinear form

$$a(u, \boldsymbol{\sigma}, v, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{0,\Omega} + (\boldsymbol{\sigma} - 2\sqrt{\kappa}\nabla u, \boldsymbol{\tau} - 2\sqrt{\kappa}\nabla v)_{0,\Omega},$$

where $\sqrt{\kappa}$ is the square root of the matrix function κ.

3. Finite element approximation and a priori error estimate

Let $\mathcal{T}_h$ be a quasi-uniform partition of the domain Ω in simplices, convex quadrilaterals, or hexahedra having the mesh size h. Let $\hat{T}$ be a reference simplex, square, or cube, where the reference simplex is defined as

$$\hat{T} = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; x_i > 0, i = 1, \ldots, d, \text{ and } \sum_{i=1}^{d} x_i < 1 \right\}$$

and the reference square or cube $\hat{T} = (0, 1)^d$.

The finite element space is defined by affine maps F_T from a reference element $\hat{T}$ to a physical element $T \in \mathcal{T}_h$. For $k \in \mathbb{N}$, let $Q_k(\hat{T})$ be the space of polynomials of degree less than or equal to k in $\hat{T}$ in the variables $x_1, \ldots, x_d$ if $\hat{T}$ is the reference simplex; the space of polynomials in $\hat{T}$ of degree less than or equal to k with respect to each variable $x_1, \ldots, x_d$ if $\hat{T}$ is the reference square or cube.

Then the finite element space based on the mesh $\mathcal{T}_h$ is defined as the space of continuous functions whose restrictions to an element T are obtained by maps of given polynomial functions of the reference element [6, 7, 9], that is,

$$S_h = \{v_h \in H^1(\Omega) \mid v_h|_T = \hat{v}_h \circ F_T^{-1}, \hat{v}_h \in Q_k(\hat{T}), T \in \mathcal{T}_h\}.$$

We now define $V_h = S_h \cap H_0^1(\Omega)$ and two other finite element spaces

$$L_h = \{v_h \in L^2(\Omega) v_h|_T \in Q_r(T), T \in \mathcal{T}_h\}, \quad \mathbf{Q}_h = [L_h]^d,$$

where $r \in \mathbb{N} \cup \{0\}$. Now a discrete formulation of our problem is to find $(u_h, \boldsymbol{\sigma}_h) \in [V_h \times \mathbf{Q}_h]$ such that

$$a((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) = 2\ell(v_h), \quad (v_h, \boldsymbol{\tau}_h) \in [V_h \times \mathbf{Q}_h].$$

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Since $[S_h]^d \subset \mathbf{Q}$, we can also use $\mathbb{S}_h = [S_h]^d$ to discretize the gradient of the continuous problem. This leads to a problem of finding $(u_h, \boldsymbol{\sigma}_h) \in [V_h \times \mathbb{S}_h]$ such that

$$a((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) = 2\ell(v_h), \quad (v_h, \boldsymbol{\tau}_h) \in [V_h \times \mathbb{S}_h], \tag{3.1}$$

which utilizes equal order interpolation. However, different order interpolations can be used for the solution and gradient as before. Since the discrete formulation is conforming, the bilinear form $a(\cdot, \cdot)$ and the linear form $\ell(\cdot)$ are both continuous on the corresponding spaces. The coercivity also follows from the continuous setting.

THEOREM 3.1. *The discrete problem of finding* $(u_h, \boldsymbol{\sigma}_h) \in [V_h \times \mathbf{Q}_h]$ *or* $(u_h, \boldsymbol{\sigma}_h) \in [V_h \times \mathbb{S}_h]$ *such that*

$$a((u_h, \boldsymbol{\sigma}_h), (v_h, \boldsymbol{\tau}_h)) = 2\ell(v_h), \quad (v_h, \boldsymbol{\tau}_h) \in [V_h \times \mathbf{Q}_h], \quad \text{or} \quad (v_h, \boldsymbol{\tau}_h) \in [V_h \times \mathbb{S}_h]$$

has a unique solution, and the solution satisfies

$$\|u - u_h\|_{1,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le c\Big(\inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} + \inf_{\boldsymbol{\tau}_h \in \mathbb{S}_h \text{ or } \boldsymbol{\tau}_h \in \mathbf{Q}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{0,\Omega}\Big),$$

where u is the exact solution of the problem (2.1) *and* $\boldsymbol{\sigma} = \nabla u$.

PROOF. The proof follows from Galerkin orthogonality [7] and standard arguments. □

We now consider the algebraic formulation of the finite element problem, using the same notation for the vector representation of the solution and the solution as an element in V_h, $\mathbb{S}_h$, and $\mathbf{Q}_h$. Let $\mathtt{A}$, $\mathtt{B}$, and $\mathtt{M}$ be the matrices corresponding to the bilinear forms $\int_\Omega \nabla u_h \cdot \nabla v_h \, dx$, $\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla v_h \, dx$, and $\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h \, dx$, respectively. Let $\vec{f}$ be the vector arising form the discretization of the linear functional $\ell(\cdot)$. Then the linear system associated with the problem (3.1) is written as

$$\begin{bmatrix} -2\mathtt{B} & 4\mathtt{A} \\ 2\mathtt{M} & -2\mathtt{B}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma}_h \\ u_h \end{bmatrix} = \begin{bmatrix} 2\vec{f} \\ \mathbf{0} \end{bmatrix}.$$

REMARK 3.2. Note that if we use an H^1-conforming Lagrange finite element space of order k to discretize the solution u, and a piecewise polynomial space of order $k-1$ to discretize the gradient $\boldsymbol{\sigma}$ (just L^2-conforming space for the gradient), we arrive at the standard Galerkin formulation [7] in the discrete form. Therefore, the real power of the new approach lies in the fact that we can use equal order interpolation leading to a smooth approximation of the gradient.

REMARK 3.3. The solution u is assumed to be in $H_0^1(\Omega)$ only for the purpose of simplicity. In fact, any nonzero Dirichlet condition or mixture of Dirichlet and Neumann boundary conditions are all fine as in the case of the standard Galerkin finite element method.

TABLE 1. Discretization errors for the solution and gradient.

l	$\|u - u_h\|_{1,\Omega}$		$\|u - u_h\|_{0,\Omega}$		$\|\sigma^1 - \sigma_h^1\|_{0,\Omega}$		$\|\sigma^2 - \sigma_h^2\|_{0,\Omega}$	
1	3.41208e-01		3.71839e-02		1.72948e-01		1.72948e-01	
2	1.70261e-01	1.00	1.15857e-02	1.68	5.39510e-02	1.68	5.39510e-02	1.68
3	8.40661e-02	1.02	3.09011e-03	1.91	1.45061e-02	1.89	1.45061e-02	1.89
4	4.18485e-02	1.01	7.86293e-04	1.97	3.74056e-03	1.96	3.74056e-03	1.96
5	2.08998e-02	1.00	1.97503e-04	1.99	9.62306e-04	1.96	9.62306e-04	1.96
6	1.04469e-02	1.00	4.94393e-05	2.00	2.51753e-04	1.93	2.51753e-04	1.93

4. Numerical example

In this section, we consider a numerical example to demonstrate the performance of this new minimization scheme. In fact, we show the discretization errors for the solution u in the L^2- and H^1-norms, and discretization errors for the gradient in the L^2-norm. For this example, we consider the domain of the square $\Omega = [-1, 1]^2$ with the exact solution

$$u(x, y) = (x - y)\exp\{-5.0(x - 0.5)(x - 0.5) - 5.0(y - 0.5)(y - 0.5)\},$$

where the right-hand-side function f of Problem 2.1 and the Dirichlet boundary condition on $\partial\Omega$ are obtained by using this exact solution. The two components of the gradient are denoted by σ^1 and σ^2, and their numerical approximations are denoted by σ_h^1 and σ_h^2, respectively. We start with the initial uniform triangulation of 32 triangles in the first level and then refine uniformly in each level. We have tabulated the discretization errors in Table 1 using the C^0-linear finite element space for the solution and each component of the gradient. Here l denotes the level of refinement. We see that the numerical results are the same as those predicted by the theory. Moreover, the discretization errors show the superiority of the scheme as the discretization errors for the gradient of the solution converge quadratically to the exact solution. This is not normally achieved in any mixed finite element method.

5. Conclusion

We have proposed a new minimization principle for the Poisson equation based on the solution and its gradient. One major advantage of this formulation is that a finite element approximation can be performed as in a least-squares finite element method without fulfilling the compatibility condition between two finite element spaces. However, the finite element approach is much easier than a least-squares approach. An optimal a priori error estimate is given for the proposed formulation. Also, a numerical example is presented to demonstrate the optimality of the scheme. An interesting future work will be to develop an iterative solution method for the linear system of equations of the above finite element method.

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References

[1] Y. Achdou, C. Bernardi and F. Coquel, "A priori and a posteriori analysis of finite volume discretizations of Darcy's equations", *Numer. Math.* **96** (2003) 17–42; doi:10.1007/s00211-002-0436-7.

[2] P. B. Bochev and C. R. Dohrmann, "A computational study of stabilized, low-order C^0 finite element approximations of Darcy equations", *Comput. Mech.* **38** (2006) 323–333; doi:10.1007/s00466-006-0036-y.

[3] P. B. Bochev and M. Gunzburger, "Finite element methods of least-squares type", *SIAM Rev.* **40** (1998) 789–837; doi:10.1137/S0036144597321156.

[4] P. B. Bochev and M. Gunzburger, "On least-squares finite element methods for the Poisson equation and their connection to the Dirichlet and Kelvin principles", *SIAM J. Numer. Anal.* **43** (2005) 340–362; doi:10.1137/S003614290443353X.

[5] P. B. Bochev and M. Gunzburger, *Least-squares finite element methods* (Springer, New York, 2009).

[6] D. Braess, *Finite elements. Theory, fast solvers, and applications in solid mechanics*, 2nd edn (Cambridge University Press, Cambridge, 2001).

[7] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods* (Springer, New York, 1994).

[8] F. Brezzi, T. J. R. Hughes, L. D. Marini and A. Masud, "Mixed discontinuous Galerkin methods for Darcy flow", *J. Sci. Comput.* **22–23** (2005) 119–145; doi:10.1007/s10915-004-4150-8.

[9] P. G. Ciarlet, *The finite element method for elliptic problems* (North-Holland, Amsterdam, 1978).

[10] B. Jiang, *The least-squares finite element method: theory and applications in computational fluid dynamics and electromagnetics* (Springer, Berlin, 1998).

[11] B. P. Lamichhane, "Mixed finite element methods for the Poisson equation using biorthogonal and quasi-biorthogonal systems", *Adv. Numer. Anal.* **2013** (2013); Article 189045 1–9 (electronic); doi:10.1155/2013/189045.

[12] B. P. Lamichhane, B. D. Reddy and B. I. Wohlmuth, "Convergence in the incompressible limit of finite element approximations based on the Hu–Washizu formulation", *Numer. Math.* **104** (2006) 151–175; doi:10.1007/s00211-006-0014-5.

[13] K. A. Mardal, X.-C. Tai and R. Winther, "A robust finite element method for Darcy–Stokes flow", *SIAM J. Numer. Anal.* **40** (2002) 1605–1631; doi:10.1137/S0036142901383910.

[14] R. Stenberg, "A nonstandard mixed finite element family", *Numer. Math.* **115** (2010) 131–139; doi:10.1007/s00211-009-0272-0.

[15] J. M. Urquiza, D. N'Dri, A. Garon and M. C. Delfour, "A numerical study of primal mixed finite element approximations of Darcy equations", *Comm. Numer. Methods Engrg.* **22** (2006) 901–915; doi:10.1002/cnm.859.