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K-SPHERICAL FUNCTIONS ON ABELIAN SEMIGROUPS

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Abstract

We present the form of the solutions $f : S \to \mathbb{C}$ of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K| f(x) f(y) \quad \text{for } x, y \in S,$$

where f satisfies the condition $f(\sum_{\lambda \in K} \lambda x) \neq 0$ for all $x \in S$, $(S, +)$ is an abelian semigroup and K is a subgroup of the automorphism group of S.

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1. Introduction

The functional equation

$$\int_K f(x + \lambda y)\, d\mu(\lambda) = f(x) f(y) \quad \text{for } x, y \in G, \tag{1.1}$$

where $(G, +)$ is a locally compact group, $f : G \to \mathbb{C}$ and K is a compact subgroup of the automorphism group of G with the normalised Haar measure μ, is a generalisation of the cosine equation and it arises in the theory of group representations, being the relation defining K-spherical functions (for the terminology, see [3, page 88]). For accounts of (1.1), see, for example, [4, 12, 15, 16].

D'Alembert's functional equation is a particular case of (1.1), corresponding to the group $K = \mathbb{Z}_2$, namely,

$$f(xy) + f(x\sigma(y)) = 2 f(x) f(y) \quad \text{for } x, y \in S, \tag{1.2}$$

where $(S, +)$ is an abelian group, $\sigma \in \mathrm{Aut}(S)$ is an involution and $f : S \to \mathbb{C}$. Equation (1.2) has been studied in many contexts: groups [9, 11], nilpotent groups [5], metabelian groups [6, 14], abelian semigroups [13], topological groups [7], topological monoids [8] and Banach algebras [1, 2]. For nonabelian groups, the solutions of d'Alembert's functional equation are different from those for the abelian case.

Our work is based on the following results.

THEOREM 1.1 [12, Corollary 3.12], [4, Theorem 1.1]. *Let* $(G,+)$ *be a locally compact abelian Hausdorff topological group and let* K *be a compact Hausdorff topological transformation group of* G *acting by automorphisms on* G. *Let* μ *be the normalised Haar measure on* K. *If* $\varphi \in C(G)$ *is a nonzero solution of*

$$\int_K \varphi(x+\lambda y)\,d\mu(\lambda) = \varphi(x)\varphi(y) \quad \text{for } x,y \in G,$$

then there exists a continuous homomorphism $\chi : G \to \mathbb{C}^*$ *such that*

$$\varphi(x) = \int_K \chi(\lambda x)\,d\mu(\lambda) \quad \text{for } x \in G.$$

If φ *is bounded, then* χ *may be taken as a unitary character.*

If we take the discrete topology on the groups G and K and the normalised counting measure μ in the previous theorem, we obtain the following corollary.

COROLLARY 1.2. *Let* G *be an abelian group and* K *be a finite subgroup of the automorphism group of* G. *Let* $f : G \to \mathbb{C}$, $f \neq 0$, *satisfy*

$$\sum_{\lambda \in K} f(x+\lambda y) = |K| f(x) f(y) \quad \text{for } x,y \in G.$$

Then there exists a homomorphism $m : G \to \mathbb{C}^*$ *such that*

$$f(x) = \frac{1}{|K|}\sum_{\lambda \in K} m(\lambda x) \quad \text{for } x \in G.$$

THEOREM 1.3 [17, Theorem 3.18(d)]. *Suppose that* S *is a topological semigroup,* $n \in \mathbb{N}$, $\chi_1,\dots,\chi_n : S \to \mathbb{C}$ *are different multiplicative functions,* $a_1,\dots,a_n \in \mathbb{C}$ *and* $f = a_1\chi_1 + \cdots + a_n\chi_n : S \to \mathbb{C}$. *If* f *is continuous, then each of the functions* $a_1\chi_1,\dots,a_n\chi_n$ *is also continuous.*

2. Main result

Throughout, $(S,+)$ is an abelian semigroup, K is a subgroup of the automorphism group of S (where we write the action of $\lambda \in K$ on $x \in S$ as λx), $|K| \geq 2$, $\mathbb{C}^*$ is the multiplicative group of complex numbers and the relation $\sim \subseteq S \times S$ is given by

$$\forall_{x,y\in S}\,(x \sim y \Leftrightarrow \exists_{z\in S}\,(x+z = y+z)). \tag{2.1}$$

First, we give an example of a semigroup that is not a group and admits a nontrivial finite group of automorphisms.

EXAMPLE 2.1. Let $S = \{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq z\}$ and $K = \{O_k : k \in \{0,1,\dots,n\}\}$, where $O_k(r\cos\phi, r\sin\phi, z) = (r\cos(\phi + 2\pi k/n), r\sin(\phi+2\pi k/n), z)$ for $\phi \in [0,2\pi)$, $z \geq r \geq 0$. Then S is a convex cone, so it is a semigroup, S is not a group (for example, $(0,0,-1) \notin S$) and K is a finite subgroup of the automorphism group of S.

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The first lemma is an easy consequence of the definition of equivalence relation $\sim$.

Lemma 2.2. *The relation $\sim$ given by (2.1) is an equivalence relation; $S/_\sim$ with the operation $+ : (S/_\sim)^2 \to S/_\sim$ defined by*

$$[x]_\sim + [y]_\sim := [x+y]_\sim \quad \textit{for } x, y \in S$$

is a cancellative abelian semigroup, and the function $\varkappa : S \to S/_\sim$ given by

$$\varkappa(x) = [x]_\sim \quad \textit{for } x \in S \tag{2.2}$$

is a semigroup epimorphism.

Definition 2.3. For each $x \in S$, we define the element $\widetilde{x} \in S$ by the formula

$$\widetilde{x} = \sum_{\lambda \in K \setminus \{\mathrm{Id}\}} \lambda x.$$

Lemma 2.4. *For all $x, y \in S$ and $\lambda \in K$,*

$$\widetilde{x + \lambda y} = \widetilde{x} + \lambda \widetilde{y},$$
$$\lambda(x + \widetilde{x}) = x + \widetilde{x}.$$

Moreover, if the function $f : S \to \mathbb{C}$ satisfies

$$\sum_{\lambda \in K} f(x + \lambda y) = |K| f(x) f(y) \quad \textit{for } x, y \in S, \tag{2.3}$$

then

$$f(x + (y + \widetilde{y})) = f(x) f(y + \widetilde{y}) \quad \textit{for } x, y \in S.$$

Proof. Let $x, y \in S$, $\lambda \in K$. Then

$$\begin{aligned}\widetilde{x + \lambda y} &= \sum_{\mu \in K \setminus \{\mathrm{Id}\}} \mu(x + \lambda y) = \sum_{\mu \in K \setminus \{\mathrm{Id}\}} \mu x + \sum_{\mu \in K \setminus \{\mathrm{Id}\}} (\mu \circ \lambda) y = \sum_{\mu \in K \setminus \{\mathrm{Id}\}} \mu x + \sum_{\mu \in K \setminus \{\lambda\}} \mu y \\ &= \sum_{\mu \in K \setminus \{\mathrm{Id}\}} \mu x + \lambda \sum_{\mu \in K \setminus \{\mathrm{Id}\}} \mu y = \widetilde{x} + \lambda \widetilde{y}\end{aligned}$$

and

$$\lambda(x + \widetilde{x}) = \lambda \sum_{\mu \in K} \mu x = \sum_{\mu \in K} \mu x = x + \widetilde{x}.$$

Let $f : S \to \mathbb{C}$ satisfy (2.3). We observe that

$$\begin{aligned}|K| f(x) f(y + \widetilde{y}) &= \sum_{\lambda \in K} f(x + \lambda(y + \widetilde{y})) \\ &= \sum_{\lambda \in K} f(x + y + \widetilde{y}) = |K| f(x + y + \widetilde{y}),\end{aligned}$$

for $x, y \in S$, which completes the proof. □

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Lemma 2.5. *Let G be an abelian group such that $S/_\sim \leq G$ and $G = S/_\sim - S/_\sim$. For every automorphism $\lambda \in \mathrm{Aut}(S)$, there exists a unique $\lambda_G \in \mathrm{Aut}(G)$ such that*

$$\lambda_G \circ \varkappa = \varkappa \circ \lambda, \tag{2.4}$$

where $\varkappa$ is defined by (2.2). Moreover,

$$(\lambda \circ \tau)_G = \lambda_G \circ \tau_G \quad \text{for } \lambda, \tau \in K,$$

and the set

$$K_G := \{\lambda_G : \lambda \in K\} \tag{2.5}$$

is a subgroup of the automorphism group of G.

Proof. Let $\lambda \in K$. We define $\lambda_G : G \to G$ by the formula

$$\lambda_G(\varkappa(x) - \varkappa(y)) := \varkappa(\lambda x) - \varkappa(\lambda y) \quad \text{for } x, y \in S.$$

Let $x, y, u, v \in S$ be such that $\varkappa(x) - \varkappa(y) = \varkappa(u) - \varkappa(v)$. Then $\varkappa(x + v) = \varkappa(y + u)$, so there exists $z \in S$ such that $x + v + z = y + u + z$. Hence $\lambda x + \lambda v + \lambda z = \lambda y + \lambda u + \lambda z$, which yields $\varkappa(\lambda x) + \varkappa(\lambda v) = \varkappa(\lambda y) + \varkappa(\lambda u)$. From this, $\lambda\varkappa(x) - \lambda\varkappa(y) = \lambda\varkappa(u) - \lambda\varkappa(v)$, so λ_G is well defined.

For $x, y, u, v \in S$,

$$\begin{aligned}\lambda_G(\varkappa(x) - \varkappa(y)) + \lambda_G(\varkappa(u) - \varkappa(v)) &= \varkappa(\lambda x) - \varkappa(\lambda y) + \varkappa(\lambda u) - \varkappa(\lambda v)\\ &= \varkappa(\lambda(x + u)) - \varkappa(\lambda(y + v))\\ &= \lambda_G(\varkappa(x + u) - \varkappa(y + v))\\ &= \lambda_G(\varkappa(x) - \varkappa(y) + \varkappa(u) - \varkappa(v))\end{aligned}$$

and

$$\varkappa(x) - \varkappa(y) = \varkappa(\lambda(\lambda^{-1}x)) - \varkappa(\lambda(\lambda^{-1}y)) = \lambda_G(\varkappa(\lambda^{-1}x) + \varkappa(\lambda^{-1}y)).$$

Now if $\lambda_G(\varkappa(x) - \varkappa(y)) = \lambda_G(\varkappa(u) - \varkappa(v))$, then

$$\begin{aligned}\varkappa(\lambda(x + v)) - \varkappa(\lambda(y + u)) &= \varkappa(\lambda x) - \varkappa(\lambda y) - (\varkappa(\lambda u) - \varkappa(\lambda v))\\ &= \lambda_G(\varkappa(x) - \varkappa(y)) - \lambda_G(\varkappa(u) - \varkappa(v)) = 0.\end{aligned}$$

Hence, there exists $z \in S$ such that $\lambda x + \lambda v + z = \lambda y + \lambda u + z$. Then $x + v + \lambda^{-1}z = y + u + \lambda^{-1}z$, so $\varkappa(x + v) = \varkappa(y + u)$, which means that $\varkappa(x) - \varkappa(y) = \varkappa(u) - \varkappa(v)$. Hence λ_G is an automorphism.

Observe that

$$\lambda_G(\varkappa(x)) = \lambda_G(\varkappa(x + x) - \varkappa(x)) = \varkappa(\lambda(x + x)) - \varkappa(\lambda x) = \varkappa(\lambda x) \quad \text{for } x \in S,$$

which proves (2.4). If the automorphism $\sigma : G \to G$ satisfies $\sigma \circ \varkappa = \varkappa \circ \lambda$ for some $\lambda \in \mathrm{Aut}(S)$, then

$$\sigma(\varkappa(x) - \varkappa(y)) = \sigma(\varkappa(x)) - \sigma(\varkappa(y)) = \varkappa(\lambda x) - \varkappa(\lambda y) = \lambda_G(\varkappa(x) - \varkappa(y)) \quad \text{for } x, y \in S,$$

which shows the uniqueness of λ_G.

Finally,

$$(\lambda \circ \tau)_G \circ \varkappa = \varkappa \circ (\lambda \circ \tau) = \lambda_G \circ (\varkappa \circ \tau) = \lambda_G \circ (\tau_G \circ \varkappa) = (\lambda_G \circ \tau_G) \circ \varkappa \quad \text{for } \lambda, \tau \in K.$$

In view of the above identity, it is easy to check that K_G is group. □

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Theorem 2.6. *If the function $f : S \to \mathbb{C}$ satisfies (2.3) and*

$$f(x + \widetilde{x}) \neq 0 \quad \text{for } x \in S, \tag{2.6}$$

then the function $h : G \to \mathbb{C}$ given by the formula

$$h(\varkappa(x) - \varkappa(y)) := \frac{f(x + \widetilde{y})}{f(y + \widetilde{y})} \quad \text{for } x, y \in S,$$

is well defined,

$$h(\varkappa(x)) = f(x) \quad \text{for } x \in S, \tag{2.7}$$

and h satisfies

$$\sum_{\lambda_G \in K_G} h(x + \lambda_G y) = |K_G| h(x) h(y) \quad \text{for } x, y \in G, \tag{2.8}$$

and $\varkappa$, λ_G, K_G are defined, respectively, by (2.2), (2.4) and (2.5), where G is an abelian group such that $S/_\sim \leq G$ and $G = S/_\sim - S/_\sim$.

Proof. First, we observe that, for all $x, y \in S$,

$$\varkappa(x) = \varkappa(y) \Rightarrow f(x) = f(y). \tag{2.9}$$

Indeed, let $x, y \in S$ be such that $\varkappa(x) = \varkappa(y)$. Then there exists $z \in S$ such that $x + z = y + z$. Hence

$$\begin{aligned}
|K| f(x) f(z + \widetilde{z}) &= \sum_{\lambda \in K} f(x + \lambda(z + \widetilde{z})) = \sum_{\lambda \in K} f(x + (z + \widetilde{z})) = |K| f(x + z + \widetilde{z}) \\
&= |K| f(y + z + \widetilde{z}) = \sum_{\lambda \in K} f(y + z + \widetilde{z}) \\
&= \sum_{\lambda \in K} f(y + \lambda(z + \widetilde{z})) = |K| f(y) f(z + \widetilde{z}),
\end{aligned}$$

which means that $f(x) = f(y)$.

Let $x, y, u, v \in S$ be such that $\varkappa(x) - \varkappa(y) = \varkappa(u) - \varkappa(v)$. Then $\varkappa(x + v) = \varkappa(y + u)$, so we obtain $\varkappa(x + \widetilde{y} + v + \widetilde{v}) = \varkappa(u + \widetilde{v} + y + \widetilde{y})$. In view of Lemma 2.4 and (2.9),

$$f(x + \widetilde{y}) f(v + \widetilde{v}) = f(x + \widetilde{y} + v + \widetilde{v}) = f(u + \widetilde{v} + y + \widetilde{y}) = f(u + \widetilde{v}) f(y + \widetilde{y}),$$

so

$$\frac{f(x + \widetilde{y})}{f(y + \widetilde{y})} = \frac{f(u + \widetilde{v})}{f(v + \widetilde{v})},$$

which shows that h is well defined.

Observe that

$$h(\varkappa(x)) = h(\varkappa(x + x) - \varkappa(x)) = \frac{f(x + x + \widetilde{x})}{f(x + \widetilde{x})} = \frac{f(x) f(x + \widetilde{x})}{f(x + \widetilde{x})} = f(x) \quad \text{for } x \in S.$$

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Let $x, y, u, v \in S$. In view of Lemma 2.4,

$$\begin{aligned}
\frac{1}{|K_G|} &\sum_{\lambda_G \in K_G} h(\varkappa(x) - \varkappa(y) + \lambda_G(\varkappa(u) - \varkappa(v))) \\
&= \frac{1}{|K|} \sum_{\lambda \in K} h(\varkappa(x + \lambda u) - \varkappa(y + \lambda v))) \\
&= \frac{1}{|K|} \sum_{\lambda \in K} \frac{f(x + \lambda u + \widetilde{y + \lambda v})}{f(y + \lambda v + \widetilde{y + \lambda v})} = \frac{1}{|K|} \sum_{\lambda \in K} \frac{f(x + \widetilde{y} + \lambda(u + \widetilde{v}))}{f(y + \widetilde{y} + \lambda(v + \widetilde{v}))} \\
&= \frac{1}{|K|} \sum_{\lambda \in K} \frac{f(x + \widetilde{y} + \lambda(u + \widetilde{v}))}{f(y + \widetilde{y} + v + \widetilde{v})} = \frac{f(x + \widetilde{y}) f(u + \widetilde{v})}{f(y + \widetilde{y}) f(v + \widetilde{v})} \\
&= h(\varkappa(x) - \varkappa(y)) h(\varkappa(u) - \varkappa(v)),
\end{aligned}$$

which completes the proof. □

Remark 2.7. If S is a group and $f : S \to \mathbb{C}$ is a nonzero function which satisfies (2.3), then using Lemma 2.4 we easily see that f satisfies (2.6).

Theorem 2.8. *Let $f : S \to \mathbb{C}$. Then f satisfies (2.3) and (2.6) if and only if there exists a homomorphism $m : S \to \mathbb{C}^*$ such that*

$$f(x) = \frac{1}{|K|} \sum_{\lambda \in K} m(\lambda x) \quad \textit{for } x \in S. \tag{2.10}$$

Proof. ($\Rightarrow$) In view of Theorem 2.6, there exists a function $h : G \to \mathbb{C}$ such that h satisfies (2.8) and (2.7), where G is an abelian group such that $S/_\sim \le G$ and $G = S/_\sim - S/_\sim$. Hence, in view of Corollary 1.2, there exists a homomorphism $m_G : G \to \mathbb{C}^*$ such that

$$h(x) = \frac{1}{|K_G|} \sum_{\lambda_G \in K_G} m_G(\lambda_G x) \quad \text{for } x \in G.$$

We define the function $m : S \to \mathbb{C}^*$ by the formula

$$m(x) = m_G(\varkappa(x)) \quad \text{for } x \in S.$$

Then m is a homomorphism and

$$\begin{aligned}
f(x) = h(\varkappa(x)) &= \frac{1}{|K_G|} \sum_{\lambda_G \in K_G} m_G(\lambda_G \varkappa(x)) = \frac{1}{|K|} \sum_{\lambda \in K} m_G(\varkappa(\lambda x)) \\
&= \frac{1}{|K|} \sum_{\lambda \in K} m(\lambda x) \quad \text{for } x \in S.
\end{aligned}$$

($\Leftarrow$) Assume that f has the form (2.8). Then

$$\begin{aligned}
\sum_{\lambda \in K} f(x + \lambda y) &= \frac{1}{|K|} \sum_{\lambda \in K} \sum_{\mu \in K} m(\mu(x + \lambda y)) = \frac{1}{|K|} \sum_{\lambda \in K} \sum_{\mu \in K} m(\mu x) m(\mu \lambda y) \\
&= \frac{1}{|K|} \sum_{\mu \in K} \Big(m(\mu x) \sum_{\lambda \in K} m(\mu \lambda y) \Big) = \frac{1}{|K|} \sum_{\mu \in K} m(\mu x) \sum_{\lambda \in K} m(\lambda y) \\
&= |K| f(x) f(y) \quad \text{for } x, y \in S.
\end{aligned}$$

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Further,

$$f(x+\widetilde{x}) = \frac{1}{|K|}\sum_{\lambda\in K} m\Big(\lambda\sum_{\mu\in K}\mu x\Big) = m\Big(\sum_{\mu\in K}\mu x\Big) \neq 0 \quad \text{for } x\in S.$$

□

Corollary 2.9. *Assume that S is a topological abelian semigroup and that automorphisms from K are continuous. Let $f : S \to \mathbb{C}$ be such that $f(x+\widetilde{x}) \neq 0$ for all $x\in S$. Then f is continuous and satisfies (2.3) if and only if there exists a continuous homomorphism $m : S \to \mathbb{C}^*$ such that (2.10) holds.*

Proof. In view of Theorem 2.8, f has the form (2.10) so we only have to show the continuity. It is easy to see that if m is continuous and f is given by (2.10), then f is continuous.

Assume now that f is continuous. Combine the terms of $|K|f = \sum_{\lambda\in K} m\circ\lambda \in C(S)$ having the same multiplicative functions $m\circ\lambda$ and write

$$|K|f = c_0 m + \sum_{i=1}^{N} c_i m\circ\lambda_i, \tag{2.11}$$

where $\lambda_1,\lambda_2,\ldots,\lambda_N\in K$ are such that $m, m\circ\lambda_1,\ldots,m\circ\lambda_N$ are different and where the $c_0, c_1,\ldots,c_N$ are positive integers. The first term of (2.11) corresponds to $\lambda_0 = \mathrm{Id}\in K$. Since $|K|f$ is continuous, so are the individual terms in the sum (2.11) (by Theorem 1.3). In particular, $c_0 m\in C(S)$, which implies that $m\in C(S)$. □

The paper [10] contains many theorems which are based on solutions of (2.3) on groups. Using Theorem 2.8, we can obtain analogous results for semigroups (without changing proofs) provided that K is abelian. We give one example.

Theorem 2.10 (see [10, Theorem 4]). *Let X be a complex linear space. Assume that K is abelian. The functions $f : S\to X$, $f\neq 0$, $\varphi : S\to\mathbb{C}$, $\varphi(\sum_{\lambda\in K}\lambda x)\neq 0$ for all $x\in S$ satisfy*

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|\varphi(y)f(x) \quad \textit{for } x,y\in S,$$

if and only if there exist a homomorphism $m : S\to\mathbb{C}^*$, $A_0^\lambda\in X$ and k-additive symmetric maps $A_k^\lambda : S^k\to X$, $1\le i\le |K_0|-1$, $\lambda\in K_1$, such that

$$\varphi(x) = \frac{1}{|K|}\sum_{\lambda\in K} m(\lambda x) \quad \textit{for } x\in S,$$

$$f(x) = \sum_{\lambda\in K_1} m(\lambda x)\Big[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\ldots,x)\Big] \quad \textit{for } x\in S,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0 \quad \textit{for } x,y\in S, \lambda\in K_1, 1\le i\le k\le |K_0|-1,$$

where $K_0 := \{\lambda\in K : m\circ\lambda = m\}$ and K_1 is the set of representatives of cosets of the quotient group K/K_0.

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