

LOCAL LINEAR APPROXIMATIONS OF JUMP DIFFUSION PROCESSES

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Abstract

Local linear approximations have been the main component in the construction of a class of effective numerical integrators and inference methods for diffusion processes. In this note, two local linear approximations of jump diffusion processes are introduced as a generalization of the usual ones. Their rate of uniform strong convergence is also studied.

Keywords: Jump diffusion process; stochastic differential equation; numerical integration; local linearization method; pseudo-likelihood method

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1. Introduction

In the last ten years, the jump diffusion processes defined through stochastic differential equations (SDEs) have become an important mathematical tool for describing the dynamics of several phenomena, e.g. the dynamics of asset prices in the market and the firing of neurons. Since exact representation of the trajectories of these processes is only possible in a few cases, approximate representations are required. Different types of such approximation have already been proposed in [5], [6], and [7], and are essentially based on Itô–Taylor expansions of the jump diffusion process. The good convergence properties of the approximations based on these expansions and their numerical instability in many cases of nonlinear SDE [1], [8], [9] are well known.

Our main objective in this paper is to investigate another kind of approximation of the jump diffusion processes, namely local linear approximations. In the framework of ‘pure’ diffusion processes, this kind of approximation has been recently proposed as a stable alternative to the above-mentioned conventional approximations based on Itô–Taylor expansions [1], [3], [9], [12] and they have been the key in the derivation of effective inference methods for SDEs [10], [13], [14], [15]. Therefore, the present study is well motivated.

Specifically, in this note the local linear approximations for diffusion processes are extended to the more general case of jump diffusion processes, and their rate of uniform strong convergence is studied.

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2. Local linear approximations

Let (Ω, \mathcal{F}, P) be the underlying complete probability space we consider, and let $\{\mathcal{F}_t, t \geq t_0\}$ be an increasing, right-continuous family of complete σ -subalgebras of \mathcal{F} . Consider a d -dimensional jump diffusion process \mathbf{x} defined by the following SDE:

$$d\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t)) dt + \sum_{i=1}^m \mathbf{g}_i(t) dw^i(t) + \sum_{i=1}^p \mathbf{h}_i(t, \mathbf{x}(t)) dq^i(t) \quad \text{for } t \geq t_0, \quad (1)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

Here \mathbf{f} , \mathbf{g}_i , and \mathbf{h}_i are differentiable functions, $\mathbf{w} = (w^1, \dots, w^m)$ is an m -dimensional \mathcal{F}_t -adapted standard Wiener process, and each $q^i(t)$ could be either an \mathcal{F}_t -adapted Poisson counting process $n^i(t)$ with intensity μ^i , or an \mathcal{F}_t -adapted compensated Poisson process $n^i(t) - \mu^i t$. It is assumed that $w^i(t)$ and $q^j(t)$ are all independent, with zero probability of there being simultaneous jumps. Linear growth restriction, uniform Lipschitz, and smoothness conditions are assumed of the functions \mathbf{f} , \mathbf{g}_i , and \mathbf{h}_i in order to ensure the existence of a unique strong solution to (1).

Consider the time discretization $(\tau)_\delta = \{\tau_n, n = 0, 1, \dots\}$, with maximum step size $\delta \in (0, 1)$, defined as a sequence of \mathcal{F} -stopping times that satisfy

$$t_0 = \tau_0 < \dots < \tau_n < \dots < \infty \quad \text{and} \quad \sup_n (\tau_{n+1} - \tau_n) \leq \delta$$

with probability 1, where τ_n is \mathcal{F}_{τ_n} -measurable for each $n \in \mathbb{N}$. In addition, let us write

$$n_t = \max\{n = 0, 1, \dots : \tau_n \leq t\} < \infty$$

for all $t \in \mathbb{R}$.

In the next subsection, the basic definitions of the local linear approximations of diffusion processes are briefly presented.

2.1. Local linear approximations of diffusion processes

Let us consider the d -dimensional diffusion process \mathbf{z} defined by the SDE

$$d\mathbf{z}(t) = \overline{\mathbf{f}}(t, \mathbf{z}(t)) dt + \mathbf{G}(t) d\mathbf{w}(t) \quad \text{for } t \geq a, \quad (2)$$

$$\mathbf{z}(a) = \mathbf{z}_0,$$

where $\overline{\mathbf{f}}$ is a differentiable function and $\mathbf{G}(t) = [\mathbf{g}_1(t), \dots, \mathbf{g}_i(t)]$ is a $d \times m$ matrix. Here \mathbf{g}_i and \mathbf{w} are defined as before.

Definition 1. ([3].) For a given time discretization $(\tau)_\delta$, the order- γ local linear discretization ($\gamma = 1, 1.5$) of the diffusion process \mathbf{z} is defined by the recurrent relation

$$\mathbf{y}_{\tau_{n+1}} = \mathbf{y}_{\tau_n} + \mathbf{F}_\gamma(\tau_n, \mathbf{y}_{\tau_n}; \tau_{n+1} - \tau_n) + \boldsymbol{\xi}(\tau_n, \mathbf{y}_{\tau_n}; \tau_{n+1} - \tau_n), \quad (3)$$

where \mathbf{y}_{τ_0} is a given initial point and \mathbf{F}_γ and $\boldsymbol{\xi}$ are vector functions defined as

$$\boldsymbol{\xi}(t, \mathbf{y}; h) = \int_t^{t+h} \exp(\mathbf{J}(t, \mathbf{y})(t + h - s)) \mathbf{G}(s) d\mathbf{w}(s), \quad (4)$$

$$\mathbf{F}_\gamma(t, \mathbf{y}; h) = \int_0^h \exp(\mathbf{J}(t, \mathbf{y})s) \boldsymbol{\phi}_\gamma(h - s; t, \mathbf{y}) ds. \quad (5)$$

Here $J(t, y)$ is the Jacobian matrix of \bar{f} evaluated at $(t, y) \in \mathbb{R} \times \mathbb{R}^d$, and

$$\phi_\gamma(s; t, y) = \begin{cases} \bar{f}(t, y) + \left(\frac{\partial \bar{f}(t, y)}{\partial t} + \frac{1}{2} \sum_{k,l=1}^d [G(t)G^\top(t)]^{k,l} \frac{\partial^2 \bar{f}(t, y)}{\partial y^k \partial y^l} \right) s, & \gamma = 1.5, \\ \bar{f}(t, y) + \frac{\partial \bar{f}(t, y)}{\partial t} s, & \gamma = 1. \end{cases}$$

On the basis of this discretization, various local linearization schemes for the numerical integration of SDEs have been proposed [1], [2], [4], [8], that differ in how the integrals (4) and (5) are computed.

Definition 2. ([3].) For a given time discretization $(\tau)_\delta$, the stochastic process

$$y_\gamma^\delta = \{y_\gamma^\delta(t), t \in [a, b]\}$$

is called the order- γ local linear approximation ($\gamma = 1, 1.5$) of the diffusion process z if

$$y_\gamma^\delta(t) = y_{\tau_{n_t}} + F_\gamma(\tau_{n_t}, y_{\tau_{n_t}}; t - \tau_{n_t}) + \xi(\tau_{n_t}, y_{\tau_{n_t}}; t - \tau_{n_t}), \tag{6}$$

where $y_{\tau_{n_t}}$ is the order- γ local linear discretization (3).

It is obvious that the local linear approximation (6) is a continuous-time stochastic process that coincides with the local linear discretization (3) on the time discretization $(\tau)_\delta$. In [3], some properties of such an approximation were studied, including its convergence rate to the underlying diffusion process.

2.2. Local linear approximations of jump diffusion processes

Consider the sequence of jump times $\{\sigma\}_{\mu^i} = \{\sigma_{i,n}, n = 0, 1, 2, \dots\}$, associated to $q^i(t)$, which is defined as an increasing sequence of random variables such that $\sigma_{i,n+1} - \sigma_{i,n}$ is exponentially distributed with parameter μ^i for all n and i . Without loss of generality, it is assumed that $\{\sigma\}_{\mu^i} \subset (\tau)_\delta$ for all $i = 1, \dots, p$. In addition, let us assume that only the first r Poisson processes q^i are compensated, and let $(\tau)_\delta \subset [t_0, T]$, with $T < \infty$.

It is well known [11] that the solution to (1) is given by

$$x(t) = x(t-) + \sum_{i=1}^p h_i(t, x(t-)) \Delta n_t^i, \tag{7}$$

where Δn_t^i is the increment of the process n^i at the time instant t and $x(t-)$ denotes the solution to (2) with

$$\bar{f}(t, z(t)) = f(t, z(t)) - \sum_{j=1}^r h_j(t, z(t)) \mu^j t \tag{8}$$

and initial condition $z(\sigma_{i,n}) = x(\sigma_{i,n})$, for all times t between two consecutive jump times $\sigma_{i,n}$ and $\sigma_{j,m}$.

This leads to the following two definitions.

Definition 3. For a given time discretization $(\tau)_\delta$, the order- γ local linear discretization ($\gamma = 1, 1.5$) of the jump diffusion process x is defined by the recurrent relation

$$y_{\tau_{n+1}} = y_{\tau_{n+1}-} + \sum_{i=1}^p h_i(\tau_{n+1}, y_{\tau_{n+1}-}) \Delta n_{\tau_{n+1}}^i, \tag{9}$$

where $y_{\tau_{n+1}-}$ denotes the local linear discretization of diffusion process z at time τ_{n+1} .

Definition 4. For a given time discretization $(\tau)_\delta$, the stochastic process

$$\mathbf{y}_\gamma^\delta = \{\mathbf{y}_\gamma^\delta(t), t \in [t_0, T]\}$$

is called the order- γ local linear approximation ($\gamma = 1, 1.5$) of the jump diffusion process \mathbf{x} if

$$\mathbf{y}_\gamma^\delta(t) = \mathbf{y}_\gamma^\delta(t-) + \sum_{i=1}^p \mathbf{h}_i(t, \mathbf{y}_\gamma^\delta(t-)) \Delta n_t^i, \tag{10}$$

where $\mathbf{y}_\gamma^\delta(t-)$ denotes the local linear approximation of diffusion process \mathbf{z} at time t .

3. Convergence of the local linear approximations

In order to study the rate of convergence of the above local linear approximations, other definitions (from [7]) are needed.

Let \mathcal{M} be the set of all the multi-indices $\alpha = (j_1, \dots, j_{l(\alpha)})$ with $j_i \in \{0, 1, \dots, m\}$ and $i = 1, \dots, l(\alpha)$. By $l(\alpha)$ we denote the length of the multi-index α and by $n(\alpha)$ the number of its zero components. The multi-indices in \mathcal{M} obtained by deleting the first and the last components of α are denoted by $-\alpha$ and $\alpha-$, respectively. The multi-index of zero length is denoted by v .

Denote by \mathcal{A}_γ the hierarchical set defined as

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\},$$

and by $\mathcal{B}(\mathcal{A}_\gamma) = \{\alpha \in \mathcal{M} \setminus \mathcal{A}_\gamma : -\alpha \in \mathcal{A}_\gamma\}$ the remainder set of \mathcal{A}_γ , for $\gamma = 1, 1.5$. Denote by H_α the set of adapted right-continuous processes $h = \{h(t), t \geq t_0\}$ with left limits, on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\{I_{\alpha-}[h(\cdot)]_{\rho,t}, t \geq t_0\} \in H_{(j_{l(\alpha)})}$, where $I_{\alpha-}[h]_{\rho,t}$ denotes the multiple Itô integral of h corresponding to the multi-index $\alpha-$, evaluated at the times ρ and t . Here, $H_{(j)} = H_{(1)}$ for $j = \{2, \dots, m\}$, and $H_{(1)}, H_{(0)}$, and H_v are the sets of processes h such that, respectively, $\int_0^t |h(s, \varpi)|^2 ds < \infty$, $\int_0^t |h(s, \varpi)| ds < \infty$, and $|h(t, \varpi)| < \infty$ hold with probability 1 (with $\varpi \in \Omega$).

Furthermore, let

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^d \bar{f}^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^d \sum_{j=1}^m G^{k,j} G^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}$$

be the diffusion operator for the SDE (2), let

$$L^j = \sum_{k=1}^d G^{k,j} \frac{\partial}{\partial x^k} \quad \text{for } j = 1, \dots, m,$$

and let the vector \mathbf{G}^j be the j th column of \mathbf{G} . Denote by $C^{1,2}$ the space of functions from $\mathbb{R} \times \mathbb{R}^d$ to \mathbb{R}^d that are once and twice continuously differentiable in their first and second arguments, respectively.

The following lemma presents a known result on the boundedness and convergence rate of the local linear approximation (6) for SDEs. It shall be useful to demonstrate an analogous result for the local linear approximation (10) in the case of SDEs with jumps.

Lemma 1. ([3, Lemma 8].) *Suppose that the Itô coefficient functions λ_α , defined for each multi-index $\alpha = (j_1, \dots, j_l)$ by*

$$\lambda_\alpha = \begin{cases} L^{j_1} \dots L^{j_{l-1}} \bar{f}, & j_l = 0, \\ L^{j_1} \dots L^{j_{l-1}} \mathbf{G}^{j_l}, & j_l \neq 0, \end{cases}$$

satisfy $|\lambda_\alpha(t, \mathbf{u}) - \lambda_\alpha(t, \mathbf{v})| \leq K_1 |\mathbf{u} - \mathbf{v}|$ for all $\alpha \in \mathcal{A}_\gamma$, $t \in [a, b]$, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$; $\lambda_{-\alpha} \in C^{1,2}$ and $\lambda_\alpha \in H_\alpha$ for all $\alpha \in \mathcal{A}_\gamma \cup \mathcal{B}(\mathcal{A}_\gamma)$; and $|\lambda_\alpha(t, \mathbf{u})| \leq K_1(1 + |\mathbf{u}|)$ for all $\alpha \in \mathcal{A}_\gamma \cup \mathcal{B}(\mathcal{A}_\gamma)$, $t \in [a, b]$, and $\mathbf{u} \in \mathbb{R}^d$. In addition, suppose that the inequality $|\phi_\gamma(\zeta; s, \mathbf{u})| \leq K_2$ holds for all $\zeta \in [0, 1]$, $s \in [a, b]$, and $\mathbf{u} \in \mathbb{R}^d$. Then the order- γ local linear approximation (6) of the solution to (2) satisfies

$$\mathbb{E} \left(\sup_{a \leq t \leq b} |y_\gamma^\delta(t)|^2 \mid \mathcal{F}_a \right) \leq K_3(1 + |y_\gamma^\delta(a)|^2) \tag{11}$$

and

$$\mathbb{E} \left(\sup_{a \leq t \leq b} |z(t) - y_\gamma^\delta(t)|^2 \mid \mathcal{F}_a \right) \leq K_4 |z_0 - y_\gamma^\delta(a)|^2 + (K_5(1 + |z_0|^2) + K_3(1 + |y_\gamma^\delta(a)|^2)) \delta^{2\gamma}, \tag{12}$$

for $b < \infty$ and $\gamma = 1, 1.5$. Here K_1, K_2, K_3, K_4 , and K_5 are positive constants.

The next lemma gives an upper bound for the second moment of the local linear approximation of the jump diffusion process. A key step in the proof is the successive application of (11) in each time interval between two consecutive jumps.

Lemma 2. *Let \bar{f} be defined as in (8), and assume that the functions \bar{f} and \mathbf{G}^{j_l} satisfy the conditions of Lemma 1 for $t \in [t_0, T]$. In addition, suppose that the inequality*

$$|\mathbf{h}_i(t, \mathbf{u})| \leq K_6(1 + |\mathbf{u}|) \tag{13}$$

holds for $t \in [t_0, T]$ and $\mathbf{u} \in \mathbb{R}^d$. Then the order- γ local linear approximation (10) of the solution to (1) satisfies

$$\mathbb{E} \left(\sup_{t_0 \leq s \leq T} |y_\gamma^\delta(s)|^2 \mid \mathcal{F}_{t_0} \right) \leq D(1 + |y_\gamma^\delta(t_0)|^2),$$

where D is a positive constant.

Proof. Let $N_T = \sum_{i=1}^p n^i(T)$ be the total number of jumps up to time T , and let

$$\{t\}_{N_T} = \{t_j, j = 0, \dots, N_T\}$$

be a sequence of times such that $\{t\}_{N_T} \subset (\tau)_\delta$, $t_j \in \{\tau_0 \cup \{\sigma\}_{\mu^1} \cup \dots \cup \{\sigma\}_{\mu^p}\}$, and $t_j < t_{j+1}$, for all $j = 0, \dots, N_T - 1$. Furthermore, let

$$Z_s = \{n^i(t_j) : t_j \leq s, t_j \in \{t\}_{N_T}, i = 1, \dots, p\}$$

for $s \geq t_0$, and define

$$e_j = \mathbb{E} \left(\sup_{t_0 \leq s \leq t_j} |y_\gamma^\delta(s)|^2 \mid \mathcal{F}_{t_0}, Z_{t_j} \right)$$

with $t_j \in \{t\}_{N_T}$; here, $E(\cdot \mid \mathcal{F}_{t_0}, Z_{t_j})$ denotes the conditional expectation with respect to \mathcal{F}_{t_0} and Z_{t_j} . Then

$$e_{j+1} \leq e_j + \Delta e_{j+1},$$

where $\Delta e_{j+1} = E(\sup_{t_j < s \leq t_{j+1}} |y_\gamma^\delta(s)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}})$. From (10) and (13) we find that

$$\begin{aligned} \Delta e_{j+1} &\leq 2 E\left(\sup_{t_j < s \leq t_{j+1}} |y_\gamma^\delta(s-)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\quad + 2p \sum_{i=1}^p E\left(\sup_{t_j < s \leq t_{j+1}} |h_i(s, y_\gamma^\delta(s-)) \Delta n_s^i|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\leq 2(1 + 2pK_6^2) E\left(\sup_{t_j \leq s \leq t_{j+1}} |y_\gamma^\delta(s-)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) + 4pK_6^2. \end{aligned}$$

By definition, for all $s \in [t_j, t_{j+1}]$, $y_\gamma^\delta(s-)$ is the local linear approximation of the solution to an SDE (with no jumps). Therefore, by using Lemma 1 in this time interval, we find that

$$\begin{aligned} E\left(\sup_{t_j \leq s \leq t_{j+1}} |y_\gamma^\delta(s-)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) &= E\left(E\left(\sup_{t_j \leq s \leq t_{j+1}} |y_\gamma^\delta(s-)|^2 \mid \mathcal{F}_{t_j}\right) \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\leq K_3(1 + E(|y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}})) \\ &\leq K_3\left(1 + E\left(\sup_{t_0 \leq s \leq t_j} |y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_j}\right)\right). \end{aligned}$$

Thus,

$$\Delta e_{j+1} \leq C_1 e_j + C_2,$$

where $C_1 = 2(1 + 2pK_6^2)K_3$ and $C_2 = C_1 + 4pK_6^2$, and it follows that

$$e_{j+1} \leq (1 + C_1)e_j + C_2, \tag{14}$$

which implies that

$$\begin{aligned} e_{j+1} &\leq (1 + C_1)^{j+1} e_0 + \frac{C_2}{C_1} ((1 + C_1)^j - 1) \\ &\leq (1 + C_1)^{j+1} \left(e_0 + \frac{C_2}{C_1}\right) \\ &\leq \frac{C_2}{C_1} (1 + C_1)^{j+1} (1 + e_0). \end{aligned}$$

By using the above inequality and taking $j = N_T$, we find that

$$E\left(\sup_{t_0 \leq s \leq T} |y_\gamma^\delta(s)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{N_T}}\right) \leq \frac{C_2}{C_1} (1 + C_1)^{1+N_T} (1 + |y_\gamma^\delta(t_0)|^2). \tag{15}$$

Finally, by taking into account the fact that

$$E(\kappa^{N_T}) = E(e^{N_T \log(\kappa)}) = \exp\left((\kappa - 1)(T - t_0) \sum_{i=1}^p \mu^i\right) \tag{16}$$

for all constant $\kappa > 1$, we obtain

$$\begin{aligned} \mathbb{E}\left(\sup_{t_0 \leq s \leq T} |y_\gamma^\delta(s)|^2 \mid \mathcal{F}_{t_0}\right) &= \mathbb{E}\left(\mathbb{E}\left(\sup_{t_0 \leq s \leq T} |y_\gamma^\delta(s)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{N_T}}\right) \mid \mathcal{F}_{t_0}\right) \\ &\leq \frac{C_2}{C_1}(1 + C_1) \exp\left(C_1(T - t_0) \sum_{i=1}^p \mu^i\right) (1 + |y_\gamma^\delta(t_0)|^2), \end{aligned}$$

which completes the proof.

The following theorem states the convergence rate of the local linear approximation of the jump diffusion process. A key step in the proof is the successive application of (12) in each time interval between two consecutive jumps.

Theorem 1. *Let \bar{f} be defined as in (8), and suppose that the functions \bar{f} and G^j satisfy the conditions of Lemma 1 for $t \in [t_0, T]$. In addition, suppose that the inequality*

$$|\mathbf{h}_i(t, \mathbf{u}) - \mathbf{h}_i(t, \mathbf{v})| \leq K_6|\mathbf{u} - \mathbf{v}| \tag{17}$$

holds for $t \in [t_0, T]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Then the order- γ local linear approximation (10) of the solution to (1) satisfies

$$\begin{aligned} \mathbb{E}\left(\sup_{t_0 \leq t \leq T} |\mathbf{x}(t) - y_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}\right) \\ \leq D_4|\mathbf{x}_0 - y_\gamma^\delta(t_0)|^2 + (D_5(1 + |\mathbf{x}_0|^2) + D_3(1 + |y_\gamma^\delta(t_0)|^2))\delta^{2\gamma} \end{aligned}$$

for $T < \infty$ and $\gamma = 1, 1.5$. Here D_3, D_4 , and D_5 are positive constants.

Proof. As in the proof of Lemma 2, let $N_T = \sum_{i=1}^p n^i(T)$ be the total number of jumps up to time T , and let $\{t\}_{N_T} = \{t_j, j = 0, \dots, N_T\}$ be a sequence of times such that $\{t\}_{N_T} \subset (\tau)_\delta$, $t_j \in \{\tau_0 \cup \{\sigma\}_{\mu^1} \cup \dots \cup \{\sigma\}_{\mu^p}\}$, and $t_j < t_{j+1}$, for all $j = 0, \dots, N_T - 1$. Furthermore, let

$$Z_s = \{n^i(t_j) : t_j \leq s, t_j \in \{t\}_{N_T}, i = 1, \dots, p\}$$

for $s \geq t_0$, and define

$$e_j = \mathbb{E}\left(\sup_{t_0 \leq t \leq t_j} |\mathbf{x}(t) - y_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_j}\right),$$

with $t_j \in \{t\}_{N_T}$. In this case, we have $e_{j+1} \leq e_j + \Delta e_{j+1}$, where

$$\Delta e_{j+1} = \mathbb{E}\left(\sup_{t_j < t \leq t_{j+1}} |\mathbf{x}(t) - y_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right).$$

By using (7), (10), and (17) we find that

$$\begin{aligned} \Delta e_{j+1} &\leq 2\mathbb{E}\left(\sup_{t_j < t \leq t_{j+1}} |\mathbf{x}(t-) - y_\gamma^\delta(t-)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\quad + 2p \sum_{i=1}^p \mathbb{E}\left(\sup_{t_j < t \leq t_{j+1}} |\mathbf{h}_i(t, \mathbf{x}(t-))\Delta n_t^i - \mathbf{h}_i(t, y_\gamma^\delta(t-))\Delta n_t^i|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\leq D_1 \mathbb{E}\left(\sup_{t_j < t \leq t_{j+1}} |\mathbf{x}(t-) - y_\gamma^\delta(t-)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right), \end{aligned}$$

where $D_1 = 2(1 + pK_6^2)$. By definition, for all $t \in [t_j, t_{j+1}]$, $y_\gamma^\delta(t-)$ is the local linear approximation of the solution to an SDE (with no jumps). Therefore, by using Lemma 1 in this time interval, we find that

$$\begin{aligned} R &:= E\left(\sup_{t_j < t \leq t_{j+1}} |\mathbf{x}(t-) - y_\gamma^\delta(t-)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\leq E\left(E\left(\sup_{t_j \leq t \leq t_{j+1}} |\mathbf{x}(t-) - y_\gamma^\delta(t-)|^2 \mid \mathcal{F}_{t_j}\right) \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right) \\ &\leq K_4 E(|\mathbf{x}(t_j) - y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}) \\ &\quad + \left(K_5 \left(1 + E\left(|\mathbf{x}(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right)\right) + K_3 \left(1 + E\left(|y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}\right)\right)\right) \delta^{2\gamma}. \end{aligned}$$

From inequalities (14) and (15) it follows that

$$\begin{aligned} E(|y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}) &\leq E\left(\sup_{t_0 \leq t \leq t_j} |y_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_j}\right) \\ &\leq E\left(\sup_{t_0 \leq t \leq T} |y_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{N_T}}\right) \\ &\leq \frac{C_2}{C_1} (1 + C_1)^{1+N_T} (1 + |y_\gamma^\delta(t_0)|^2). \end{aligned} \tag{18}$$

The inequality

$$\begin{aligned} E(|\mathbf{x}(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}) &\leq E\left(\sup_{t_0 \leq t \leq T} |\mathbf{x}(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{N_T}}\right) \\ &\leq \frac{C_2^*}{C_1^*} (1 + C_1^*)^{1+N_T} (1 + |\mathbf{x}(t_0)|^2) \end{aligned} \tag{19}$$

can be analogously derived for the jump diffusion process \mathbf{x} by following the proof of Lemma 2, using the fact that the well-known inequality

$$E\left(\sup_{t_0 \leq t \leq T} |z(t)|^2 \mid \mathcal{F}_{t_0}\right) \leq K_5(1 + |z(t_0)|^2),$$

rather than (11), holds for the solution to (2). In (19), $C_1^* = 2(1 + 2pK_6^2)K_5$ and $C_2^* = C_1^* + 4pK_6^2$ are positive constants. However, we also have

$$E(|\mathbf{x}(t_j) - y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{j+1}}) \leq E\left(\sup_{t_0 \leq t \leq t_j} |\mathbf{x}(t_j) - y_\gamma^\delta(t_j)|^2 \mid \mathcal{F}_{t_0}, Z_{t_j}\right). \tag{20}$$

From (18), (19), and (20) it follows that

$$R \leq K_4 e_j + D_2 \delta^{2\gamma},$$

where

$$\begin{aligned} D_2 &= 2\alpha\beta^{N_T+1}(K_5(1 + |\mathbf{x}(t_0)|^2) + K_3(1 + |y_\gamma^\delta(t_0)|^2)), \\ \alpha &= \max\left\{\frac{C_2}{C_1}, \frac{C_2^*}{C_1^*}\right\}, \quad \beta = \max\{(1 + C_1), (1 + C_1^*)\}. \end{aligned} \tag{21}$$

Thus,

$$\Delta e_{j+1} \leq D_1(K_4 e_j + D_2 \delta^{2\gamma})$$

and it follows that

$$e_{j+1} \leq (1 + D_1 K_4) e_j + D_1 D_2 \delta^{2\gamma},$$

which implies that

$$\begin{aligned} e_{j+1} &\leq (1 + D_1 K_4)^{j+1} e_0 + \frac{D_2 \delta^{2\gamma}}{K_4} ((1 + D_1 K_4)^j - 1) \\ &\leq (1 + D_1 K_4)^{j+1} \left(e_0 + \frac{D_2 \delta^{2\gamma}}{K_4} \right) \end{aligned}$$

and, therefore,

$$e_{N_T+1} \leq (1 + D_1 K_4)^{N_T+1} \left(e_0 + \frac{D_2 \delta^{2\gamma}}{K_4} \right).$$

By taking into account expression (21) for D_2 , we have

$$e_{N_T+1} \leq (1 + D_1 K_4)^{N_T+1} e_0 + \frac{2\alpha\rho^{N_T+1}(K_5(1 + |\mathbf{x}(t_0)|^2) + K_3(1 + |\mathbf{y}_\gamma^\delta(t_0)|^2))}{K_4} \delta^{2\gamma},$$

where $\rho = \beta(1 + D_1 K_4)$. That is,

$$\begin{aligned} &\mathbb{E} \left(\sup_{0 \leq t \leq T} |\mathbf{x}(t) - \mathbf{y}_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{N_T}} \right) \\ &\leq C_4 |\mathbf{x}(t_0) - \mathbf{y}_\gamma^\delta(t_0)|^2 + (C_5(1 + |\mathbf{x}(t_0)|^2) + C_3(1 + |\mathbf{y}_\gamma^\delta(t_0)|^2)) \delta^{2\gamma}, \end{aligned}$$

where $C_4 = (1 + D_1 K_4)^{N_T+1}$, $C_5 = 2\alpha\rho^{N_T+1} K_5/K_4$, and $C_3 = 2\alpha\rho^{N_T+1} K_3/K_4$.

Finally, by using (16) we obtain

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\mathbf{x}(t) - \mathbf{y}_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0} \right) &= \mathbb{E} \left(\mathbb{E} \left(\sup_{0 \leq t \leq T} |\mathbf{x}(t) - \mathbf{y}_\gamma^\delta(t)|^2 \mid \mathcal{F}_{t_0}, Z_{t_{N_T}} \right) \mid \mathcal{F}_{t_0} \right) \\ &\leq D_4 |\mathbf{x}(t_0) - \mathbf{y}_\gamma^\delta(t_0)|^2 + \{D_5(1 + |\mathbf{x}(t_0)|^2) \\ &\quad + D_3(1 + |\mathbf{y}_\gamma^\delta(t_0)|^2)\} \delta^{2\gamma}, \end{aligned}$$

where $D_4 = (1 + D_1 K_4)e^{\varepsilon D_1 K_4}$, $D_5 = 2\alpha\rho^{\varepsilon(\rho-1)} K_5/K_4$, $D_3 = 2\alpha\rho^{\varepsilon(\rho-1)} K_3/K_4$, and $\varepsilon = (T - t_0) \sum_{i=1}^p \mu^i$ are positive constants. This completes the proof.

Finally, by using the Lyapunov inequality the following result is straightforward to derive.

Corollary 1. *In addition to the assumptions of Theorem 1, suppose that the initial conditions satisfy*

$$\mathbb{E}(|\mathbf{x}_0|^2) < \infty, \quad \mathbb{E}(|\mathbf{y}_\gamma^\delta(t_0)|^2) < \infty, \quad \mathbb{E}(|\mathbf{x}_0 - \mathbf{y}_\gamma^\delta(t_0)|^2) \leq K_7 \delta^{2\gamma},$$

where K_7 is a positive constant. Then the order- γ local linear approximation (10) of the solution to (1) satisfies

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\mathbf{x}(t) - \mathbf{y}_\gamma^\delta(t)| \right) = O(\delta^\gamma).$$

From a practical point of view, the results above state the uniform order of strong convergence of the local linearization schemes that could be obtained by approximating the integrals (4) and (5) involved in the definition of $\mathbf{y}_{\tau_{n+1}}$ in (9), provided that these approximations have the same order of convergence.

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