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ASYMPTOTIC EXPANSIONS OF INVARIANT METRICS OF STRICTLY PSEUDOCONVEX DOMAINS

SIQI FU

ABSTRACT. In this paper we obtain the asymptotic expansions of the Carathéodory and Kobayashi metrics of strictly pseudoconvex domains with C^∞ smooth boundaries in $\mathbb{C}^n$. The main result of this paper can be stated as following:

MAIN THEOREM. *Let Ω be a strictly pseudoconvex domain with C^∞ smooth boundary. Let $F_\Omega(z,X)$ be either the Carathéodory or the Kobayashi metric of Ω. Let $\delta(z)$ be the signed distance from z to $\partial\Omega$ with $\delta(z) < 0$ for $z \in \Omega$ and $\delta(z) \geq 0$ for $z \notin \Omega$. Then there exist a neighborhood U of $\partial\Omega$, a constant $C > 0$, and a continuous function $C(z,X)\colon (U \cap \Omega) \times \mathbb{C}^n \longrightarrow \mathbb{R}$ such that*

$$F_\Omega(z,X) = \left\{ \frac{|\langle \partial\delta(z), X\rangle|^2}{\delta^2(z)} + \frac{L_\delta(z,X)}{|\delta(z)|} \right\}^{1/2} + C(z,X)$$

and $|C(z,X)| \leq C|X|$ for $z \in U \cap \Omega$ and $X \in \mathbb{C}^n$.

0. **Introduction.** In this paper we obtain the asymptotic expansions of the Carathéodory and Kobayashi metrics of strictly pseudoconvex domains with C^∞ smooth boundary in $\mathbb{C}^n$.

Let us first recall some definitions. Let Ω be a bounded domain in $\mathbb{C}^n$ and Δ the unit disc. Let $H(\Delta,\Omega)$ be the set of holomorphic mappings from Δ to Ω and $H(\Omega,\Delta)$ the set of holomorphic mappings from Ω to Δ. The Carathéodory and Kobayashi metrics of Ω are defined respectively by

$$F^C_\Omega(z,X) = \sup\left\{ |(Xf)(z)| \;\middle|\; f \in H(\Omega,\Delta), f(z) = 0 \right\}$$

and

$$F^K_\Omega(z,X) = \inf\{1/\lambda \mid f \in H(\Delta,\Omega), f(0) = z, f'(0) = \lambda X, \lambda > 0\}$$

for $z \in \Omega$ and $X = \sum_{i=1}^n X_i \partial/\partial z_i \in T^{1,0}_z(\Omega)$.

Let

$$\delta(z) = \begin{cases} -\operatorname{dist}(z,\partial\Omega) & z \in \Omega \\ \operatorname{dist}(z,\partial\Omega) & z \notin \Omega. \end{cases}$$

It is known from [K-P] that $\delta(z)$ is of the class C^k in a neighborhood of $\partial\Omega$ whenever $\partial\Omega$ is of class $C^k (k \geq 1)$. For a C^2 function $f(z)$ we use $L_f(z,X)$ to denote its Levi form at z in the direction of X. *i.e.*

$$L_f(z,X) = \sum_{i,j=1}^n \frac{\partial^2 f(z)}{\partial z_i \partial \bar{z}_j} X_i \overline{X_j}.$$

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MAIN THEOREM. *Let Ω be a strictly pseudoconvex domain with C^∞ smooth boundary. Let $F_\Omega(z,X)$ be either the Carathéodory or the Kobayashi metric of Ω. Then there exist a neighborhood U of $\partial\Omega$, a constant $C > 0$, and a continuous function $C(z,X)$: $(U\cap\Omega)\times\mathbb{C}^n \longrightarrow \mathbb{R}$ such that*

$$F_\Omega(z,X) = \left\{ \frac{|\langle\partial\delta(z),X\rangle|^2}{\delta^2(z)} + \frac{L_\delta(z,X)}{|\delta(z)|} \right\}^{1/2} + C(z,X)$$

and $|C(z,X)| \le C|X|$ for $z \in U\cap\Omega$ and $X \in \mathbb{C}^n$.

A similar asymptotic expansion for the Bergman metric was given by Fefferman [Fe] with more information about $C(z,X)$. The boundary behavior of the Carathéodory and Kobayashi metrics was first studied by Graham [G] in 1975. Since then there have been extensive studies of the boundary behavior of these two invariant metrics and their applications. A detailed account of the development in this direction can be found, to name only a few, in [D-F] for the case of pseudoconvex domains with analytic boundaries, [Ca] for the case of pseudoconvex domains of finite type in $\mathbb{C}^2$, [M1] for the case of strictly pseudoconvex domains in $\mathbb{C}^n$, [H] for the case of pseudoconvex domains of homogeneous finite diagonal type, [Ch] for the case of pseudoconvex domains of finite type in C^n, and the references cited in these papers.

This paper is organized as follows: In the first section, we study the first and the second derivatives of distance functions for general real hypersurfaces. This enables us to express the Levi forms of the distance functions by Levi forms of other defining functions with simpler forms. In the second section, we obtain the asymptotic expansion for the Kobayashi metric by a scaling procedure similar to those in [M2]. There are two major differences in our method. First, we use the biholomorphic image of a ball which oscillates the boundary of the strictly pseudoconvex domain to the 4th order instead of the 3rd order to approximate the domain. Secondly, the results in the first section allow us to treat the holomorphic tangent vectors as the vectors of the interior points rather than projecting them to the boundary, decomposing into complex normal and tangential components. These eventually avoid the loss of accuracy. In the third section, we deduce the asymptotic expansion of the Carathéodory metric from a sharp comparison to the Kobayashi metric. This is done by using the embedding theorem of Fornaess [Fo], Lempert's theorem on invariant metrics of convex domains [L], and the localization theorem of the Kobayashi metric of Forsterneric–Rosay[F-R].

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1. **Distance functions of hypersurfaces.** Let $\Omega \subset\subset \mathbb{R}^{N+1}$ be a domain with C^∞ smooth boundary (actually C^3 is enough). Let

$$\delta(p) = \begin{cases} -\operatorname{dist}(p,\partial\Omega) & p\in\Omega \\ \operatorname{dist}(p,\partial\Omega) & p\notin\Omega \end{cases}$$

where $p = (x,y) = (x_1,x_2,\dots,x_N,y)$. Thus $\delta(z)$ is smooth in a neighborhood of $\partial\Omega$. We then extend $\delta(x)$ to be in $C^\infty(\mathbb{R})$.

Let $p \in \Omega$ near the boundary and $p_0 = \pi(p)$ the nearest point to p on the boundary. After translations and rotations, we may assume that p_0 is the origin O and the negative y-axis is the normal direction at p_0. Therefore, in the new coordinate system $p = (0,0,\dots,0,d)$. There exits a neighborhood U of the origin such that

$$\partial\Omega \cap U = \{(x,y) \in U \mid y = f(x_1,x_2,\dots,x_N)\}$$

where $f(x_1,x_2,\dots,x_N) = \sum_{i,j=1}^N a_{ij}x_ix_j + O(|x|^3)$.

We have the following theorem.

THEOREM 1.1. *If $d > 0$ is sufficiently small, then*

$$\frac{\partial\delta(p)}{\partial x_i} = 0, \quad \frac{\partial\delta(p)}{\partial y} = -1 \tag{1.1}$$

and

$$\frac{\partial^2\delta(p)}{\partial x_i\partial x_j} = a_{ij} + O(d), \quad \frac{\partial^2\delta(p)}{\partial x_i\partial y} = 0 \tag{1.2}$$

PROOF. For $(x_1,x_2,\dots,x_N,y) \in \Omega$ near p_0, set $\pi(x,y) = \big(t_1,t_2,\dots,t_N, f(t_1,t_2,\dots,t_N)\big)$, where $t_i = t_i(x,y)$ $(1 \le i \le N)$ are smooth functions. For $t' = (t_1',t_2',\dots,t_N')$ near the origin,

$$d = \sum_{j=1}^N (t_j' - x_j)^2 + \big(f(t_1',t_2',\dots,t_N') - y\big)^2$$

obtains a minimum at $t' = t$. Thus

$$2(t_i - x_i) + 2\big(f(t) - y\big)\frac{\partial f}{\partial t_i} = 0. \tag{1.3}$$

By [K-P], we have

$$\frac{\partial\delta}{\partial x_i}(x,y) = -\frac{t_i - x_i}{\delta(x,y)} \tag{1.4}$$

and

$$\frac{\partial\delta}{\partial y}(x,y) = -\frac{f(t) - y}{\delta(x,y)}. \tag{1.5}$$

Hence

$$\frac{\partial\delta}{\partial x_i}(p) = 0, \quad \frac{\partial\delta}{\partial y}(p) = -1.$$

Taking $\partial/\partial x_j$ to both sides of (1.3), we obtain

$$\frac{\partial t_i}{\partial x_j} - \varepsilon_{ij} + \left(\sum_{k=1}^N \frac{\partial f(t)}{\partial t_k}\frac{\partial t_k}{\partial x_j}\right)\frac{\partial f(t)}{\partial t_i} + \big(f(t) - y\big)\sum_{k=1}^N \frac{\partial^2 f(t)}{\partial t_i\partial t_k}\frac{\partial t_k}{\partial x_j} = 0. \tag{1.6}$$

Taking $\partial/\partial y$ to both sides of (1.3), we obtain

$$(1.7)\qquad \frac{\partial t_j}{\partial y}+\Big(\sum_{k=1}^{N}\frac{\partial f(t)}{\partial t_k}\frac{\partial t_k}{\partial y}-1\Big)\frac{\partial f(t)}{\partial t_i}+\big(f(t)-y\big)\sum_{k=1}^{N}\frac{\partial^2 f(t)}{\partial t_k\partial t_i}\frac{\partial t_k}{\partial y}=0.$$

Evaluating (1.6) and (1.7) at $p=(0,\dots,0,d)$, we have

$$(1.6')\qquad \frac{\partial t_i}{\partial x_j}(p)-\varepsilon_{ij}-d\sum_{k=1}^{N}a_{ik}\frac{\partial t_k}{\partial x_j}(p)=0\quad \text{for } i=1,2,\dots,N$$

and

$$(1.7')\qquad \frac{\partial t_i}{\partial y}(p)-d\sum_{k=1}^{N}a_{ik}\frac{\partial t_k}{\partial y}(p)=0\quad \text{for } i=1,2,\dots,N$$

Using Cramer's rule to solve the system of linear equations (1.6′) and (1.7′), we obtain that when $d>0$ is sufficiently small

$$(1.8)\qquad \frac{\partial t_i}{\partial x_j}(p)=\begin{cases} a_{ij}d+O(d^2) & i\neq j\\ 1+a_{ij}d+O(d^2) & i=j\end{cases}$$

and

$$(1.9)\qquad \frac{\partial t_i}{\partial y}(p)=0.$$

On the other hand, it follows from (1.4) that

$$\frac{\partial^2\delta}{\partial x_i\partial x_j}(x,y)=-\frac{(\frac{\partial t_i}{\partial x_j}-\varepsilon_{ij})\delta(x,y)-(t_i-x_i)\frac{\partial\delta}{\partial x_j}(x,y)}{\delta^2(x,y)}$$

and

$$\frac{\partial^2\delta}{\partial x_i\partial y}(x,y)=-\frac{\frac{\partial t_i}{\partial y}\delta(x,y)-(t_i-x_i)\frac{\partial\delta}{\partial y}(x,y)}{\delta^2(x,y)}.$$

Therefore by (1.8) and (1.9), we have

$$\frac{\partial^2\delta}{\partial x_i\partial x_j}(p)=a_{ij}+O(d),\qquad \frac{\partial^2\delta}{\partial x_i\partial y}(p)=0.$$ ■

We state a special case of Theorem 1.1 as the following.

COROLLARY 1.2. *Let $\Omega\subset\subset\mathbb{C}^n$ have a smooth boundary and assume that the origin $O\in\partial\Omega$ and the negative x_1-axis is the normal direction at O. Let $z^0=(\tau,0,\dots,0)$. Then when $\tau>0$ is sufficiently small, we have*

$$\partial\delta(z^0)=(\partial\delta/\partial z_1,\partial\delta/\partial z_2,\dots,\partial\delta/\partial z_n)=(-1/2,0,\dots,0)$$

and

$$L_\delta(z^0,X)=L_f(z^0,X)+O(\tau)|X|^2$$

where f is a smooth function near the origin O such that

$$\partial\Omega\cap U=\{\operatorname{Re} z_1=f(\operatorname{Im} z_1,z_2,\dots,z_n)\}.$$

2. **Asymptotic expansion of the Kobayashi metric.** Let Ω be a bounded domain with C^∞ smooth boundary near a strictly pseudoconvex boundary point p_0. Let U be a tubular neighborhood of p_0 such that $p = \pi(z) \in U \cap \partial\Omega$ for each $z \in U$. In this section, we will use C to denote constants which may be different in different appearances.

Let $z^0 \in U$ and $p = \pi(z^0) \in U \cap \partial\Omega$. After a translation and unitary transformation, we may assume that p is the origin O and the negative $\operatorname{Re} z_1$-axis is the outward normal direction. Then after a unitary transformation, $\partial\Omega$ is defined by the following equation near p:

$$\operatorname{Re} z_1 = \operatorname{Re}\Big\{a(i \operatorname{Im} z_1)^2 + \sum_{j=2}^{n} b_j \cdot (i \operatorname{Im} z_1) z_j + \sum_{j,k=2}^{n} c_{jk} z_j z_k\Big\} + \sum_{i=2}^{n} a_i |z_i|^2 + O(|z'|^3 + |\operatorname{Im} z_1|^3)$$

also $z^0 = (\tau, 0, \ldots, 0)$, $\tau = \operatorname{dist}(z^0, \partial\Omega)$, where $a, a_i > 0$; $b_j, c_{jk} \in \mathbb{C}$.

The following lemma is slightly different from Lemma 2 in [Fe]. We provide the proof because we will need the lemma in the exact form as stated below.

LEMMA 2.1. *After possible shrinking of U, there exists a biholomorphic mapping $\Phi: U \longrightarrow \Phi(U), (z, z') \longrightarrow (\xi, \xi')$ such that:*

1) $\Phi(z^0) = (\nu, 0, \ldots, 0)$, where $\nu = \tau + O(\tau^2)$;

2) When τ is sufficiently small,

$$\Phi_{*_{z^0}} = \begin{pmatrix} 1 + O(\tau) & O(\tau) & O(\tau) & \cdots & O(\tau) \\ O(\tau) & a_2 + O(\tau) & O(\tau) & \cdots & O(\tau) \\ O(\tau) & O(\tau) & a_3 + O(\tau) & \cdots & O(\tau) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O(\tau) & O(\tau) & O(\tau) & \cdots & a_n + O(\tau) \end{pmatrix} \tag{2.1}$$

where the O's are functions which depend smoothly on z_0 for $z_0 \in U$;

3) In the new coordinate system (ξ_1, ξ'), $\partial\Omega$ takes the following form

$$\operatorname{Re} \xi_1 = |\xi'|^2 + P_4(\operatorname{Im} \xi_1, \xi') + O(|\xi'|^5 + |\operatorname{Im} \xi_1|^5) \tag{2.2}$$

where $P_4(\operatorname{Im} \xi_1, \xi')$ is a real valued 4th order polynomial in $\operatorname{Im} \xi_1$, ξ', $\bar{\xi}'$ satisfying

$$P_4(\operatorname{Im} \xi_1, \xi') \geq C(|\operatorname{Im} \xi_1|^4 + |\xi'|^4) \tag{2.3}$$

for some constant $C > 0$.

PROOF OF THE LEMMA. Let $\Phi_1: \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $(z_1, z') \longrightarrow (\zeta_1, \zeta')$, where

$$\zeta_1 = z_1 - a z_1^2 - \sum_{j=2}^{n} b_j z_1 z_j - \sum_{j,k=2}^{n} c_{jk} z_j z_k,$$
$$\zeta_r = a_r z_r, \quad 2 \leq r \leq n.$$

Then $z^1 = \Phi_1(z^0) = (\nu_1, 0, \ldots, 0)$ with $\nu_1 = \tau - a\tau^2 = \tau + O(\tau^2)$, and

$$\Phi_{1*_{z^0}} = \begin{pmatrix} 1 + O(\tau) & O(\tau) & O(\tau) & \cdots & O(\tau) \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}. \tag{2.4}$$

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In the new coordinate system (ζ_1, ζ'), $\partial\Omega$ takes the following form near p:

$$\operatorname{Re}(\zeta_1) = |\zeta'|^2 + \operatorname{Re}\Big\{ \sum_{j,k,l=2}^{n} a_{jkl}\zeta_j\zeta_k\zeta_l + \sum_{j,k,l=2}^{n} b_{jkl}\zeta_j\zeta_k\bar{\zeta}_l + \sum_{j,k=2}^{n} c_{jk}(i\operatorname{Im}\zeta_1)\zeta_j\zeta_k$$
$$+ \sum_{j,k=2}^{n} d_{jk}(i\operatorname{Im}\zeta_1)\zeta_j\bar{\zeta}_k + \sum_{j=2}^{n} e_j(i\operatorname{Im}\zeta_1)^2\zeta_j + f\cdot(i\operatorname{Im}\zeta_1)^3\Big\} + O(|\zeta'|^4 + |\operatorname{Im}\zeta_1|^4)$$

where a's, b's, c's, d's, e's and f are complex numbers.

Let $\Phi_2\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $(\zeta_1, \zeta') \longrightarrow (w_1, w')$, where

$$w_1 = \zeta_1 - f\zeta_1^3 - \sum_{j=2}^{n} e_j\zeta_1^2\zeta_j - \sum_{j,k=2}^{n} c_{jk}\zeta_1\zeta_j\zeta_k - \sum_{j,k,l=2}^{n} a_{jkl}\zeta_j\zeta_k\zeta_l,$$
$$w_r = \zeta_r + \frac{1}{2}\sum_{j,k=2}^{n} b_{jkr}\zeta_j\zeta_k + \frac{1}{2}\sum_{j=2}^{n} d_{jr}\zeta_1\zeta_j, \quad 2 \le r \le n.$$

Then $z^2 = \Phi_2(z^1) = (\nu_2, 0, \ldots, 0)$ with $\nu_2 = \tau + O(\tau^2)$, and

$$(2.5) \qquad \Phi_{2*_{z^1}} = \begin{pmatrix} 1+O(\tau) & 0 & 0 & \cdots & 0 \\ 0 & 1+O(\tau) & O(\tau) & \cdots & O(\tau) \\ 0 & O(\tau) & 1+O(\tau) & \cdots & O(\tau) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & O(\tau) & O(\tau) & \cdots & 1+O(\tau) \end{pmatrix}$$

In the new coordinate system (w_1, w'), $\partial\Omega$ takes the following form near p:

$$\operatorname{Re} w_1 = |w'|^2 + O(|w'|^4 + |\operatorname{Im} w_1|^4).$$

Let $\Phi_3\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $(w_1, w') \longrightarrow (\xi_1, \xi')$, where

$$\xi_1 = w_1 + Cw_1^4,$$
$$\xi' = w' - Cw_1w'.$$

Then when $C > 0$ is sufficiently large, $\partial\Omega$ takes the following form near p in the new coordinate system (ξ_1, ξ'):

$$\operatorname{Re}\xi_1 = |\xi'|^2 + P_4(\operatorname{Im}\xi_1, \xi') + O(|\xi'|^5 + |\operatorname{Im}\xi_1|^5)$$

where

$$(2.6) \qquad P_4(\operatorname{Im}\xi_1, \xi') \ge C(|\operatorname{Im}\xi_1|^4 + |\xi'|^4).$$

In this case, $z^3 = \Phi_3(z^2) = (\nu, 0, \ldots, 0)$ with $\nu = \nu_2 + C\nu_2^4 = \tau + O(\tau^2)$, and

$$(2.7) \qquad \Phi_{3*_{z^2}} = \begin{pmatrix} 1+O(\tau) & 0 & 0 & \cdots & 0 \\ 0 & 1+O(\tau) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+O(\tau) \end{pmatrix}.$$

Set $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$, then (2.1) follows from (2.4), (2.5) and (2.7). Therefore we conclude the proof of the lemma. ∎

Now, we can prove the following theorem which does not require the global strict pseudoconvexity of the domain.

THEOREM 2.2. *Let Ω be a bounded domain with C^∞ smooth boundary near a strictly pseudoconvex point $p_0 \in \partial\Omega$. Then there exist a neighborhood U of p_0, a constant $C > 0$, and a function $C(z,X)\colon (U\cap\Omega)\times\mathbb{C}^n \longrightarrow \mathbb{R}$ such that*

$$F^K_\Omega(z,X) = \left\{\frac{|\langle\partial\delta(z),X\rangle|^2}{\delta^2(z)} + \frac{L_\delta(z,X)}{|\delta(z)|}\right\}^{1/2} + C(z,X)$$

and $|C(z,X)| \le C|X|$ for $z \in U\cap\Omega$ and $X \in \mathbb{C}^n$.

PROOF. Let U be the tubular neighborhood of p_0. Let $z_0 \in U\cap\Omega$ and let $\tau = d(z_0,\partial\Omega)$. Suppose that Φ is the biholomorphic mapping obtained by Lemma 2.1. It follows from (2.1) that

$$\Phi_{*_O} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} \tag{2.8}$$

By (2.2), (2.3) and (2.8) there exist $C_1 > C_2 > 0$, $C_3 > 0$ and $r > 0$, all independent on z^0, such that

i) Φ is biholomorphic on $B(O,3r)$;

ii) $B(O,C_2r) \subset \Phi\big(B(O,2r)\big) \subset B(O,C_1r)$;

iii) $\operatorname{Re}\xi_1 \ge C_3|\xi|^2$ on $\overline{\Phi\big(\Omega\cap B(O,2r)\big)}$.

Now it follows from Theorem 2.1 and Lemma 2.2 in [F-R] that

$$F^K_{\Omega\cap B(O,2r)}(z^0,X) \ge F^K_\Omega(z^0,X) \ge (1-C\tau)F^K_{\Omega\cap B(O,2r)}(z^0,X). \tag{2.9}$$

Let $D = \Phi\big(\Omega\cap B(O,2r)\big)$ and let $z^3 = \Phi(z_0) = (\nu,0,\ldots,0)$. When τ is sufficiently small we have

$$\operatorname{Re}\nu = \tau + O(\tau^2) > \frac{\tau}{2}, \quad \operatorname{Im}\nu = O(\tau^2). \tag{2.10}$$

Define $\Psi\colon D \longrightarrow \mathbb{C}^n$ with $\Psi(z^3) = 0$ by

$$\eta_1 = \frac{\xi_1 - \nu}{\xi_1 + \nu}$$

$$\eta_r = \frac{2\sqrt{\nu}\xi_r}{\xi_1 + \nu} \quad \text{for } 2 \le r \le n$$

where $\arg(\sqrt{\nu}) < \frac{\pi}{4}$. Thus Ψ is biholomorphic on a neighborhood of $\bar{D}$.

CLAIM. *For $\xi \in \partial D$,*

$$\Big|\|\Psi(\xi)\|^2 - 1\Big| < C\tau. \tag{2.11}$$

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PROOF OF THE CLAIM. Denote

$$L(\xi) = |\xi_1 + \nu|^2\big(\|\Phi(\xi)\|^2 - 1\big) = 4|\nu|M(\xi) + 4\big(|\nu| \operatorname{Re} \xi_1 - \operatorname{Re}(\bar{\nu}\xi_1)\big)$$

where $M(\xi) = -\operatorname{Re} \xi_1 + |\xi'|^2$. Since $\operatorname{Re} \xi_1 \geq 0$ on $\bar{D}$, by (2.10) we have

(2. 12) $$|\xi_1 + \nu|^2 \geq C(|\xi_1|^2 + \tau^2), \quad \big||\nu| \operatorname{Re} \xi_1 - \operatorname{Re}(\bar{\nu}\xi_1)\big| \leq C\tau^2|\xi_1|^2.$$

Let $\partial D = \Phi\Big(\bar{\Omega} \cap \big(\partial B(O, 2r)\big)\Big) \cup \Phi\big(\partial\Omega \cap B(O, 2r)\big) = V_1 \cup V_2$. If $\xi \in V_1$, then

(2. 13) $$|\xi_1| \geq \operatorname{Re} \xi_1 \geq C|\xi|^2.$$

It follows from (2.12) that $|\xi_1 + \nu|^2 \geq Cr^4$. Thus

$$\big|\|\Psi(\xi)\|^2 - 1\big| = \frac{|L(\xi)|}{|\xi_1 + \nu|^2} \leq \frac{C\tau + C\tau^2}{Cr^4} \leq C\tau.$$

If $\xi \in V_2$, then (2.3) and (2.13) imply that

$$|M(\xi)| = |P_4(\operatorname{Im} \xi_1, \xi') + O(|\xi'|^5 + |\operatorname{Im} \xi|^5)| \leq C|\xi|^4 \leq C|\xi_1|^2.$$

Hence

$$\big|\|\Psi(\xi)\|^2 - 1\big| \leq \frac{C\tau|\xi_1|^2 + C\tau^2|\xi_1|}{|\xi_1|^2 + \tau^2} \leq C\tau.$$

This concludes the proof of the Claim. ∎

Now, by the Claim

$$B(O, 1 - C\tau) \subset \Psi(D) \subset B(O, 1 + C\tau).$$

By the length decreasing property of the Kobayashi metric, we obtain

(2. 14) $$(1 - C\tau)|Y| \leq F^K_{\Psi(D)}(O, Y) \leq (1 + C\tau)|Y|.$$

Combining (2.9) and (2.14), we have

(2. 15) $$(1 - C\tau)|(\Psi \circ \Phi)_{*_{z^0}}(X)| \leq F^K_\Omega(z^0, X) \leq (1 + C\tau)|(\Psi \circ \Phi)_{*_{z^0}}(X)|.$$

Since

(2. 16) $$\Psi_{*_{z^3}} = \begin{pmatrix} \frac{1}{2\nu} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\nu}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\sqrt{\nu}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{\nu}} \end{pmatrix},$$

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by (2.1) and the fact that $\nu = \tau + O(\tau^2)$, we obtain

$$(\Psi \circ \Phi)_{*_{z^0}} = \begin{pmatrix} \frac{1}{2\tau} + O(1) & O(1) & O(1) & \cdots & O(1) \\ O(1) & \frac{a_2}{\sqrt{\tau}} + O(1) & O(1) & \cdots & O(1) \\ O(1) & O(1) & \frac{a_3}{\sqrt{\tau}} + O(1) & \cdots & O(1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O(1) & O(1) & O(1) & \cdots & \frac{a_n}{\sqrt{\tau}} + O(1) \end{pmatrix}.$$

Hence

$$|(\Psi \circ \Phi)_{*_{z^0}}(X)| = \Big\{ \frac{1}{4\tau^2}|X_1|^2 + \frac{1}{\tau}\sum_{i=2}^{n} a_i |X_i|^2 + O(1)|X|^2 \Big\}^{1/2}.$$

Therefore, by Corollary 1.2

$$\begin{aligned} |(\Psi \circ \Phi)_{*_{z^0}}(X)| &= \Big\{ \frac{|\langle \partial\delta(z^0), X\rangle|^2}{\tau^2} + \frac{L_\delta(z^0, X)}{\tau} + O(1)|X|^2 \Big\}^{1/2} \\ &= \Big\{ \frac{|\langle \partial\delta(z^0), X\rangle|^2}{\tau^2} + \frac{L_\delta(z^0, X)}{\tau} \Big\}^{1/2} + O(1)|X|. \end{aligned} \tag{2.17}$$

From (2.15) and (2.17), we obtain

$$\begin{aligned} F^K_\Omega(z^0, X) &= |(\Psi \circ \Phi)_{*_{z^0}}(X)| + O(1)|X| \\ &= \Big\{ \frac{|\langle \partial\delta(z^0), X\rangle|^2}{\delta^2(z^0)} + \frac{L_\delta(z^0, X)}{|\delta(z^0)|} \Big\}^{1/2} + O(1)|X|. \end{aligned}$$

Thus we proved Theorem 2.1. ∎

3. **Comparison of the Carathéodory and Kobayashi metrics.** We first prove the following lemma.

LEMMA 3.1. *Let Ω be a strictly pseudoconvex domain with C^∞ smooth boundary in $\mathbb{C}^n$. Let $p \in \partial\Omega$ and V a neighborhood of p. Then there exist a neighborhood W of p and a constant $C_1 > 0$ such that*

$$\big(1 - C_1 d(z, \partial\Omega)\big) F^C_{V\cap\Omega}(z, X) \le F^C_\Omega(z, X) \tag{3.1}$$

for $z \in W \cap \Omega$ and $X \in \mathbb{C}^n$.

PROOF. By Property 1 in [Fo], there exist a strictly convex domain Ω' with smooth boundary, a Stein neighborhood Ω_1 of Ω, a holomorphic mapping $\Phi: \Omega_1 \longrightarrow \mathbb{C}^n$, and a neighborhood U of p with $p \in U \subset V \cap \Omega$ such that

i) $\Phi(\Omega) \subset \Omega'$;
ii) $\Phi(\bar\Omega) \subset \bar\Omega'$;
iii) $\Phi(U \setminus \bar\Omega) \subset \mathbb{C}^n \setminus \bar\Omega'$;
iv) $\Phi\big|_U : U \longrightarrow \Phi(U)$ is a biholomorphism.

Let W be a neighborhood of p with $W \subset\subset U$. Then

$$\begin{aligned} F^C_\Omega(z,X) &\geq F^C_{\Omega'}\big(\Phi(z),\Phi_*(X)\big) \\ &= F^K_{\Omega'}\big(\Phi(z),\Phi_*(X)\big) \quad \text{(by Lempert's Theorem)} \\ &\geq \Big(1 - C_2 d\big(\Phi(z),\partial\Omega'\big)\Big) F^K_{\Omega'\cap\Phi(U)}\big(\Phi(z),\Phi_*(X)\big) \quad \text{(by Theorem 2.1 in [F-R])} \\ &= \Big(1 - C_2 d\big(\Phi(z),\partial\Omega'\big)\Big) F^K_{\Omega\cap U}(z,X) \end{aligned}$$

where $C_2 > 0$ is a constant. On the other hand, it follows from the Hopf Lemma that there exist a constant $C_3 > 0$ such that

$$\frac{1}{C_3} d(z,\partial\Omega) \leq d\big(\Phi(z),\partial\Omega'\big) \leq C_3 d(z,\partial\Omega) \quad \text{for } z \in W \cap \Omega$$

after possible shrinking of W. Thus, (3.1) is valid with $C_1 = C_2C_3$. ∎

THEOREM 3.2. *Let Ω be a strictly pseudoconvex domain with C^∞ smooth boundary. Then there exist a constant $C > 0$ and a continuous function $C(z,X)\colon \Omega \times \mathbb{C}^n \longrightarrow \mathbb{R}$ such that*

$$F^K_\Omega(z,X) = F^C_\Omega(z,X) + C(z,X)$$

and $0 \leq C(z,X) \leq C|X|$ for $z \in \Omega$ and $X \in \mathbb{C}^n$.

PROOF. Let $p \in \partial\Omega$ and V a neighborhood of p such that $V \cap \Omega$ is biholomorphic to a strictly convex domain. Then Lempert's Theorem implies that

$$F^C_{\Omega\cap V}(z,X) = F^K_{\Omega\cap V}(z,X). \tag{3.2}$$

On the other hand, it follows from Lemma 3.1 that there exist a neighborhood W of p and a constant $C_1 > 0$ such that

$$F^C_\Omega(z,X) \geq \big(1 - C_1 d(z,\partial\Omega)\big) F^C_{\Omega\cap V}(z,X) \tag{3.3}$$

for $z \in W \cap \Omega$ and $X \in \mathbb{C}^n$. Thus (3.2) and (3.3) imply that

$$\begin{aligned} F^C_\Omega(z,X) &\geq \big(1 - C_1 d(z,\partial\Omega)\big) F^K_{\Omega\cap V}(z,X) \\ &\geq \big(1 - C_1 d(z,\partial\Omega)\big) F^K_\Omega(z,X) \\ &= F^K_\Omega(z,X) - C_1 d(z,\partial\Omega) F^K_\Omega(z,X) \\ &\geq F^K_\Omega(z,X) - C|X| \end{aligned}$$

for $z \in W \cap \Omega$ and $X \in C^n$, where $C > 0$ is a constant. Theorem 3.2 then follows by a finite covering of $\partial\Omega$. ∎

The main theorem follows easily from Theorem 2.2 and Theorem 3.2.

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Department of Mathematics
Washington University
St. Louis, Missouri 63130
U.S.A.
e-mail: sfu@pear.wustl.edu