Canad. Math. Bull. Vol. **27** (4), 1984

ON THE MONOTONIC VARIATION OF THE ZEROS OF ULTRASPHERICAL POLYNOMIALS WITH THE PARAMETER[1]

BY

RENATO SPIGLER[2]

ABSTRACT. We show that $f(\lambda)x_{n,k}^{(\lambda)}$ increases with λ, for $0<\lambda<1$, $x_{n,k}^{(\lambda)}$ being the kth zero of the ultraspherical polynomial $P_n^{(\lambda)}(x)$ and $f(\lambda)$ a suitable function of λ. As a consequence, some inequalities for $x_{n,k}^{(\lambda)}$ and an estimate for $\partial x_{n,k}^{(\lambda)}/\partial\lambda$ can be obtained.

1. **Introduction.** In this paper we show that $f(\lambda)x_{n,k}^{(\lambda)}$ increases with λ, for $0<\lambda<1$, where $x_{n,k}^{(\lambda)}$ denotes the kth positive zero of the ultraspherical polynomial $P_n^{(\lambda)}(x)$ and $f(\lambda)$ is a suitable function of λ, which may also depend on n.

The choice $f(\lambda)=\lambda^\alpha$, for some α, $0<\alpha<1$, for example, improves a result obtained in [3]. However, we obtain also bounds for $x_{n,k}^{(\lambda)}/x_{n,k}^{(\lambda+\varepsilon)}$ which do *not* blow up as $\lambda\to 0$. Moreover, we give an estimate for the derivative $\partial x_{n,k}^{(\lambda)}/\partial\lambda$, sharper than that might be obtained from [3]. This approach also provides inequalities for the zeros $x_{n,k}^{(\lambda)}$.

The basic idea is to use a more general scaling than in [3] of the independent variable in the Gegenbauer differential equation and use a version of Sturm's theorem proved in [1].

There is a physical interpretation for the zeros of the classical orthogonal polynomials (cf. [4, pp. 140–141]). Confining ourselves to the ultraspherical case, this can be stated as follows.

Suppose that two electrical charges, whose common value is $q>0$, are located at $x=1$ and $x=-1$. Suppose that there are $n\geq 2$ unit charges at some points of the interval $[-1,+1]$. Then, when the system attains the equilibrium, the positions of the n unit charges coincide with the zeros $x_{n,k}^{(\lambda)}$ of the ultraspherical polynomial $P_n^{(\lambda)}(x)$, with $\lambda=2q-1/2$.

It follows that studying the variations of $x_{n,k}^{(\lambda)}$ with the parameter λ amounts to analyze the displacements of the unit charges from their position of

Received by the editors October 6, 1983.

1980 Mathematics Subject Classification: Primary 33A45; Secondary 34A50.

Key words and phrases: zeros of ultraspherical polynomials, Sturm comparison theorem.

(1) This work was supported, in part, by the Italian C.N.R.

(2) On leave from the University of Padua, Italy.

https://doi.org/10.4153/CMB-1984-074-1 Published online by Cambridge University Press

equilibrium, when the value of the charges at the end-points changes. The classical result $\partial x_{n,k}^{(\lambda)}/\partial\lambda < 0$ (cf. e.g. [4, p. 121]), from this viewpoint, states simply that, when q increases, all the unit charges are pushed towards the origin, by effect of the increased repulsive force.

2. **The main result.** Consider the differential equation

$$y''(t) + p_\lambda(t)y(t) = 0, \qquad t \in (-1, 1), \tag{2.1}$$

where

$$p_\lambda(t) = (n+\lambda)^2/(1-t^2) + (2+4\lambda-4\lambda^2+t^2)/4(1-t^2)^2, \tag{2.2}$$

which is satisfied by $u(t) = (1-t^2)^{\lambda/2+1/4}P_n^{(\lambda)}(t)$, [4, p. 82]. $P_n^{(\lambda)}(x)$ and $u(x)$ have the same zeros in $(-1, 1)$.

Let us introduce the scaling $t = x/f(\lambda)$, $f(\lambda)$ being a suitable function of λ (to be chosen), for $0<\lambda<1$, with $f(\lambda)>0$, $f'(\lambda)>0$ for $0<\lambda<1$, $f\in C^1(0, 1)$.

The functions of x, $u(x/f(\lambda))$, $u(x/f(\lambda+\varepsilon))$ have, on the interval $(0, f(\lambda))$ and $(0, f(\lambda+\varepsilon))$, the zeros $f(\lambda)x_{n,k}^{(\lambda)}$ and $f(\lambda+\varepsilon)x_{n,k}^{(\lambda+\varepsilon)}$, $k = 1, 2, \ldots [n/2]$, being $\varepsilon > 0$ and $x_{n,k}^{(\lambda)}$. The kth positive zero of the ultraspherical polynomial $P_n^{(\lambda)}(x)$. They satisfy the differential equations

$$z''(x) + \psi_\lambda(x)z(x) = 0, \qquad w''(x) + \psi_{\lambda+\varepsilon}(x)w(x) = 0,$$

respectively, where

$$\psi_\nu(x) \equiv [f(\nu)]^{-2}p_\nu(x/f(\nu)).$$

We shall prove that $\psi_\lambda(x)$ is a *decreasing* function of λ, for $0<\lambda<1$, $0<x<f(\lambda)$ and suitable choice of $f(\lambda)$. In fact

$$\begin{aligned}\psi_\lambda(x) = p_\lambda(x/f(\lambda))/f^2(\lambda) &= (n+\lambda)^2/(f^2(\lambda)-x^2)\\ &\quad + [2f^2(\lambda)(1+2\lambda-2\lambda^2)+x^2]/4(f^2(\lambda)-x^2)^2,\end{aligned} \tag{2.3}$$

and $d\psi_\lambda(x)/d\lambda \le 0$ provided that

$$\begin{aligned}&[2(n+\lambda)(f^2-x^2) + 2ff'(n+\lambda)^2 + ff'(1+2\lambda-2\lambda^2) + f^2(1-2\lambda)](f^2-x^2)\\ &\qquad - ff'[4(n+\lambda)^2(f^2-x^2) + 2f^2(1+2\lambda-2\lambda^2)+x^2] \le 0.\end{aligned} \tag{2.4}$$

After some straightforward algebra, (2.4) becomes:

$$\begin{aligned}&-(f^2-x^2)[2(n+\lambda)^2f' - (2n+1)f]f - 2x^2(n+\lambda)(f^2-x^2)\\ &\qquad - ff'(1+2\lambda-2\lambda^2)(f^2+x^2) - x^2ff' \le 0.\end{aligned} \tag{2.4'}$$

Now, this is certainly satisfied for $f(\lambda)>0$, $f'(\lambda)>0$, $0<x<f(\lambda)$, $0<\lambda<1$ (actually for $0<\lambda<(1+\sqrt{3})/2$), *and* $2(n+\lambda)^2f' - (2n+1)f > 0$, i.e.:

$$f'(\lambda)/f(\lambda) \ge (2n+1)/2(n+\lambda)^2. \tag{2.5}$$

By integrating this differential inequality, we get

$$(2.5')\qquad f(\lambda)\geq f(\lambda_0)\exp\{(2n+1)(\lambda-\lambda_0)/2(n+\lambda)(n+\lambda_0)\},$$

where $\lambda_0\geq 0$ and $f(\lambda_0)>0$ are arbitrary.

Note that (2.5) gives only a *sufficient* condition.

Now we apply the version of Sturm's theorem proved in [1], as in [3]. We have only to prove the validity of the limit-condition:

$$(2.6)\qquad l\equiv \lim_{x\to 0+}\{u'(x/f(\lambda))u(x/f(\lambda+\varepsilon))/f(\lambda) \\ -u(x/f(\lambda))u'(x/f(\lambda+\varepsilon))/f(\lambda+\varepsilon)\}=0.$$

Setting $l\equiv\lim_{x\to 0_+}F(x)$, we have:

$$(2.7)\qquad \begin{aligned} F(x)&=[u'(o)+(x/f(\lambda))u''(o)+\cdots][u(o)+(x/f(\lambda+\varepsilon))u'(o)+\cdots]/f(\lambda)\\ &\quad-[u(o)+(x/f(\lambda))u'(o)+\cdots][u'(o)+(x/f(\lambda+\varepsilon))u''(o)+\cdots]/f(\lambda+\varepsilon)\\ &=[1/f(\lambda)-1/f(\lambda+\varepsilon)]u(o)u'(o)+x[1/f^2(\lambda)-1/f^2(\lambda+\varepsilon)]\\ &\quad\times u(o)u''(o)+0(x^2).\end{aligned}$$

Therefore $l=0$, because the ultraspherical polynomials enjoy the property that $u(o)=0$ or $u'(o)=0$.

Thus, for every $\varepsilon>0$

$$(2.8)\qquad f(\lambda)x_{n,k}^{(\lambda)}<f(\lambda+\varepsilon)x_{n,k}^{(\lambda+\varepsilon)},$$

for n, k fixed.

Let us introduce, for short, the

DEFINITION 2.1. We call *acceptable* a function $f(\lambda)$, possibly depending on n, such that $f(\lambda)>0$, $f'(\lambda)>0$ for $0<\lambda<1$, $f\in C^1(0,1)$ and satisfying (2.4′) for all $x\in(0,f(\lambda))$.

In particular, we get an *acceptable* function when (2.4′) is replaced by (2.5), in the Definition 2.1.

Then we proved the following:

THEOREM 2.2. *If $x_{n,k}^{(\lambda)}$ is the k-th positive zero of the ultraspherical polynomial $P_n^{(\lambda)}(x)$, $k=1,2,\ldots,[n/2]$, with $0<\lambda<1$, and $f(\lambda)$ is an acceptable function, then $f(\lambda)x_{n,k}^{(\lambda)}$ increases with λ, for $0<\lambda<1$.*

3. **Some consequences.** Together with $x_{n,k}^{(\lambda)}>x_{n,k}^{(\lambda+\varepsilon)}$, which follows from (6.21.3) of [4, p. 121], (2.8) yields:

$$(3.1)\qquad 1<x_{n,k}^{(\lambda)}/x_{n,k}^{(\lambda+\varepsilon)}<f(\lambda+\varepsilon)/f(\lambda),\qquad k=1,2,\ldots,[n/2].$$

This relation permits us to estimate the *Lipschitz constant* of $x_{n,k}^{(\lambda)}$ as a

https://doi.org/10.4153/CMB-1984-074-1 Published online by Cambridge University Press

function of λ. In fact we obtain

(3.2) $$|x_{n,k}^{(\lambda+\varepsilon)}-x_{n,k}^{(\lambda)}|<[f(\lambda+\varepsilon)-f(\lambda)]x_{n,k}^{(\lambda+\varepsilon)}/f(\lambda).$$

As $x_{n,k}^{(\lambda)}$ is differentiable with respect to λ, we get the estimate for the derivative

(3.3) $$|\partial x_{n,k}^{(\lambda)}/\partial\lambda|\le(f'(\lambda)/f(\lambda))x_{n,k}^{(\lambda)}<f'(\lambda)/f(\lambda),$$

or better

COROLLARY 3.1. *Under the hypotheses of Theorem* 2.1, *we have*

(3.3′) $$|\partial(\log x_{n,k}^{(\lambda)})/\partial\lambda|\le f'(\lambda)/f(\lambda).$$

Considering for $f(\lambda)$ the r.h.s. of (2.5′), with $\lambda_0=0$, $f(\lambda_0)=1$, i.e.

(3.4) $$g(\lambda)\equiv\exp\{(2n+1)\lambda/2n(n+\lambda)\},$$

formulae (3.1), (3.3′) can be rewritten for $g(\lambda)$ as

(3.5) $$1<x_{n,k}^{(\lambda)}/x_{n,k}^{(\lambda+\varepsilon)}<\exp\{(2n+1)\varepsilon/2(n+\lambda)(n+\lambda+\varepsilon)\},$$

(3.6) $$|\partial(\log x_{n,k}^{(\lambda)})/\partial\lambda|\le(2n+1)/2(n+\lambda)^2.$$

Several remarks are now in order.

REMARK 3.1. Formulae (3.5), (3.6) do *not* blow up as λ approaches 0, other than in [3].

REMARK 3.2. Inequality (3.5) holds for *negative* zeros of $P_n^{(\lambda)}(x)$, as well. In fact, $\psi_\lambda(x)$ is an *even* function of x. On the other hand, $P_n^{(\lambda)}(-x)=(-1)^nP_n^{(\lambda)}(x)$, (see e.g. [4, p. 80]).

REMARK 3.3. The result (3.1) can be used to obtain some inequalities for $x_{n,k}^{(\lambda)}$. From the monotonic character of $f(\lambda)x_{n,k}^{(\lambda)}$, in fact, we get

(3.7) $$(f(\lambda_1)/f(\lambda))x_{n,k}^{(\lambda_1)}\le x_{n,k}^{(\lambda)}\le(f(\lambda_2)/f(\lambda)x_{n,k}^{(\lambda_2)},$$

for $0\le\lambda_1\le\lambda\le\lambda_2\le1$. For a given *acceptable* $f(\lambda)$, knowing the zeros of two *particular* ultraspherical polynomials, $P_n^{(\lambda_1)}(x)$, $P_n^{(\lambda_2)}(x)$, (e.g. Čebyšev, for $\lambda=0$, $\lambda=1$), we can derive bounds for $x_{n,k}^{(\lambda)}$, for every $\lambda\in(\lambda_1,\lambda_2)$.

We observe that the differential inequality (2.5) is also satisfied by $f(\lambda)\equiv\lambda$, which yields the result of [3]. On the other hand, looking for solutions of the form $f(\lambda)\equiv\lambda^\alpha$, $0<\alpha<1$, we obtain from it

$$f(\lambda)/f'(\lambda)\equiv\lambda/\alpha\le2(\lambda^2+2n\lambda+n^2)/(2n+1),$$

i.e., setting $a\equiv1/(2\alpha)$:

$$P_a(\lambda)\equiv\lambda^2+[2n-a(2n+1)]\lambda+n^2\ge0.$$

As the discriminant of $P_a(\lambda)$ is $\Delta=[2n-a(2n+1)]^2-4n^2=$

https://doi.org/10.4153/CMB-1984-074-1 Published online by Cambridge University Press

$a(2n+1)[a(2n+1)-4n]$, we obtain $\Delta \le 0$ for $a \le 4n/(2n+1)$, i.e. $P_a(\lambda) \ge 0$ for

$$\alpha \ge (2n+1)/8n. \tag{3.8}$$

We conclude that, if $\alpha \ge \max_{n\ge 1}(2n+1)/8n = \frac{3}{8}$, (3.8) holds *uniformly* (in n) for *all* $n \ge 1$, and therefore $\psi_\lambda(x)$ is a monotonic decreasing function of λ, for all $n \ge 1$. If $\alpha \ge (2n_0+1)/8n_0$ for some $n_0 \ge 1$, then $P_a(\lambda) \ge 0$ for all $n \ge n_0$ and therefore $\psi_\lambda(x)$ decreases with λ only for $n \ge n_0$.

Inequalities (3.1), (3.3′) become, in this case

$$1 < x_{n,k}^{(\lambda)}/x_{n,k}^{(\lambda+\varepsilon)} < (1+\varepsilon/\lambda)^\alpha, \qquad k=1,2,\ldots,[n/2], \qquad \forall \varepsilon > 0, \tag{3.9}$$

$$|\partial(\log x_{n,k}^{(\lambda)})/\partial\lambda| \le \alpha/\lambda. \tag{3.10}$$

If the parameter α is chosen greater than or equal to 3/8, these hold uniformly in n, for $n \ge 1$; if $\alpha \ge (2n_0+1)/8n_0$ for some positive integer n_0, they hold only for $n \ge n_0$. As $0 < \alpha < 1$, these estimates are sharper than the corresponding ones with $\alpha = 1$; (3.9) with $\alpha = 1$ was proved in [3]: they share the property of blowing up as $\lambda \to 0$.

Following a suggestion of R. Askey, S. Ahmed [2] used the scaling function $f(\lambda) = \sqrt{(\lambda + \frac{1}{2})}$ and showed that

$$x_{n,k}^{(\lambda)}/x_{n,k}^{(\lambda+\varepsilon)} < (1+\varepsilon/(\lambda+\tfrac{1}{2}))^{1/2}, \tag{3.11}$$

with the usual meaning for n, k, λ, ε. The relation (3.3′) becomes, in this case

$$|\partial(\log x_{n,k}^{(\lambda)})/\partial\lambda| \le 1/(2\lambda+1). \tag{3.12}$$

Final Remark. It is natural, at this point, to compare the various results.

The best estimate for $\partial(\log x_{n,k}^{(\lambda)})/\partial\lambda$ is obviously provided by *the smallest* value of $f'(\lambda)/f(\lambda)$. It is easy to check that this is given by (3.6), correspondingly to $f(\lambda) = g(\lambda)$, defined in (3.4), *when* $n \ge 2$. Moreover, the smallest value of $[f(\lambda+\varepsilon)-f(\lambda)]/f(\lambda)$ is also obtained when $f(\lambda) = g(\lambda)$, *at least* for ε sufficiently small. In fact, setting $(\Delta f)(\varepsilon) \equiv f(\lambda+\varepsilon) - f(\lambda)$, if $f_1(\lambda)$, $f_2(\lambda)$ are *two acceptable* functions and $f_1'(\lambda)/f_1(\lambda) \le f_2'(\lambda)/f_2(\lambda)$, then $(\Delta f_1)(\varepsilon)/f_1(\lambda) \le (\Delta f_2)(\varepsilon)/f_2(\lambda)$, at least for ε sufficiently small. In fact, from $f_1'/f_1 \le f_2'/f_2$, i.e. $\phi_1' \equiv (\log f_1)' \le (\log f_2)' \equiv \phi_2'$, follows $\phi_1(\lambda+\varepsilon) - \phi_1(\lambda) \le \phi_2(\lambda+\varepsilon) - \phi_2(\lambda)$, at least for $\varepsilon > 0$ sufficiently small. Thus $\log(f_1(\lambda+\varepsilon)/f_1(\lambda)) \le \log(f_2(\lambda+\varepsilon)/f_2(\lambda))$, i.e. $f_1(\lambda+\varepsilon)/f_1(\lambda) \le f_2(\lambda+\varepsilon)/f_2(\lambda)$ and therefore $\Delta f_1/f_1 \le \Delta f_2/f_2$.

Therefore $f(\lambda) = g(\lambda)$ yields *the best estimate* available here, also in (3.2), which means that (3.5) is the best obtained.

Added in proof. When the limit-condition (2.6) is being checked, in (2.7), care should be used, as the function $u(\cdot)$ actually depends on λ. The conclusion still holds true.

References

1. S. Ahmed, A. Laforgia and M. E. Muldoon, *On the spacing of the zeros of some classical orthogonal polynomials*, J. London Math. Soc., (2) **25,** 1982, 246–252.

2. S. Ahmed, *On the zeros of orthogonal polynomials*, Abstract Amer. Math. Soc., **3** (1982), 339.

3. A. Laforgia, *A monotonic property for the zeros of ultraspherical polynomials*, Proc. Amer. Math. Soc., **83** (1981), 757–758.

4. G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1975.

Courant Institute of Mathematical Sciences
New York University
New York, New York 10012

https://doi.org/10.4153/CMB-1984-074-1 Published online by Cambridge University Press