

GENERALIZED MAILLET DETERMINANT

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§ 1. Introduction

In this paper, we shall study a generalization of the Maillet determinant. Let p be an odd prime, and $G = (\mathbf{Z}/p\mathbf{Z})^*$. We shall identify any integer and its image in G if there is no fear of confusion. For any integer a , let $R(a)$ denote an integer satisfying

$$R(a) \equiv a \pmod{p}, \quad 0 \leq R(a) < p.$$

Maillet studied the following determinant

$$D_p = \det (R(ab^{-1}))_{1 \leq a, b \leq (p-1)/2}$$

which is called the Maillet determinant and he raised the question whether $D_p \neq 0$ for all p . Carlitz-Olson [1] proved that the Maillet determinant is not zero by showing the following formula:

$$D_p = \pm p^{(p-3)/2} h^-$$

where h^- is the first factor of the class number of $\mathbf{Q}(\zeta_p)$, ζ_p the primitive p -th root of unity.

Carlitz considered a generalization of D_p in [2]. We consider another generalization of the determinant D_p . Let S be a subset of G . S is called a CM-type if

$$S \cup (-S) = G, \quad S \cap (-S) = \phi.$$

Clearly $\{1, 2, \dots, (p-1)/2\}$ is a CM-type. For any CM-type S , we define a determinant D_S by

$$D_S = \det (R(ab^{-1}))_{a, b \in S}.$$

We call D_S the generalized Maillet determinant for S . Since $D_S = D_{-S}$, we may only consider CM-types which contain 1.

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Let χ be a Dirichlet character mod p . For any CM-type S , we define $c_\chi = c_\chi(S)$ by

$$c_\chi = \sum_{a \in S} \chi(a).$$

Let $B_{1,\chi}$ denote the first generalized Bernoulli number. If χ is odd, $B_{1,\chi} \neq 0$. Therefore, we can define a rational number A_S by

$$A_S = \frac{2}{p-1} \sum_{\chi:\text{odd}} c_\chi c_{\bar{\chi}} B_{1,\chi}^{-1}.$$

Then we have

THEOREM. *For any CM-type S which contains 1, we have*

$$D_S = -\frac{1}{2}(-p)^{(p-3)/2}(1 + A_S)h^-.$$

We shall prove this theorem and see the connection between our theorem and Carlitz-Olson’s formula.

§ 2. Proof of the theorem

We need the following lemma, which is well-known as the Dedekind determinant [3]:

LEMMA. *Let S be a CM-type, and f be an odd function on G . Then the determinant $D(f) = \det (f(ab^{-1}))_{a,b \in S}$ is independent of S , and*

$$D(f) = \sum_{\chi:\text{odd}} \frac{1}{2} \sum_{a \in G} \chi(a)f(a).$$

We define the determinant $D_S(x)$ as follows:

$$D_S(x) = \det (R(ab^{-1}) + x)_{a,b \in S}.$$

Since $R(a) - (p/2)$ is an odd function, by Lemma

$$\begin{aligned} D_S\left(-\frac{p}{2}\right) &= \sum_{\chi:\text{odd}} \frac{1}{2} \sum_{a \in G} \chi(a)\left(R(a) - \frac{p}{2}\right) \\ &= \sum_{\chi:\text{odd}} \frac{p}{2} B_{1,\chi} = -\frac{1}{2}(-p)^{(p-3)/2}h^-. \end{aligned}$$

And so, it suffices to show that

$$\left[A_S\left(x + \frac{p}{2}\right) + \frac{p}{2}\right]D_S = \frac{p}{2}(1 + A_S)D_S(x).$$

Now, it is clear that

$$(p - 1)R(a) = \sum_{\chi \in \hat{G}} \bar{\chi}(a) \sum_{b \in G} \chi(b)R(b),$$

where \hat{G} denotes the character group of G . If χ is not trivial, then

$$\sum_{b \in G} \chi(b)R(b) = pB_{1,\chi}.$$

Therefore,

$$(p - 1)R(a) = \frac{p(p - 1)}{2} + p \sum_{\chi:\text{odd}} \bar{\chi}(a)B_{1,\chi}$$

because $B_{1,\chi} = 0$ for any non-trivial even character χ . We define the rational number $A(a)$ as follows:

$$A(a) = \sum_{\chi:\text{odd}} c_{\bar{\chi}}\chi(a)B_{1,\chi}^{-1}.$$

Then it is clear that $A_s = \sum_{a \in S} A(a)$. For $b \in S$,

$$\begin{aligned} &\sum_{a \in S} A(a)(R(ab^{-1}) + x) \\ &= \sum_{a \in S} A(a) \left(x + \frac{p}{2} + \frac{p}{p - 1} \sum_{\chi:\text{odd}} \bar{\chi}(ab^{-1})B_{1,\chi} \right) \\ &= \frac{p - 1}{2} A_s \left(x + \frac{p}{2} \right) + \frac{p}{p - 1} \sum_{\chi:\text{odd}} \sum_{a \in S} A(a)\bar{\chi}(ab^{-1})B_{1,\chi}. \end{aligned}$$

And then

$$\sum_{\chi:\text{odd}} \sum_{a \in S} A(a)\bar{\chi}(ab^{-1})B_{1,\chi} = \sum_{\chi:\text{odd}} \sum_{\psi:\text{odd}} c_{\bar{\psi}}B_{1,\psi}^{-1}\chi(b)B_{1,\chi} \sum_{a \in S} \psi\bar{\chi}(a).$$

We have

$$\sum_{a \in S} \psi\bar{\chi}(a) = \begin{cases} \frac{p - 1}{2} & \text{if } \chi = \psi \\ 0 & \text{if } \chi \neq \psi \end{cases}$$

because $\psi\bar{\chi}$ is even and S is a complete system of representatives of $G/(\pm 1)$. Hence

$$\begin{aligned} \sum_{\chi:\text{odd}} \sum_{a \in S} A(a)\bar{\chi}(ab^{-1})B_{1,\chi} &= \frac{p - 1}{2} \sum_{\chi:\text{odd}} c_{\bar{\chi}}\chi(b) \\ &= \frac{p - 1}{2} \sum_{a \in S} \sum_{\chi:\text{odd}} \chi(a^{-1}b) = \left(\frac{p - 1}{2} \right)^2. \end{aligned}$$

Consequently,

$$\sum_{a \in S} A(a)(R(ab^{-1}) + x) = \frac{p-1}{2} \left[A_s \left(x + \frac{p}{2} \right) + \frac{p}{2} \right].$$

Therefore, there exists an integer $a_i \in S$ such that $A(a_i) \neq 0$.

Now we put the matrix $M(x) = (m_{a,b})_{a,b \in S}$ as follows:

$$m_{a,b} = \begin{cases} R(ab^{-1}) + x & \text{if } a \neq a_i \\ 1 & \text{if } a = a_i. \end{cases}$$

Then, by some properties of matrices, we have

$$A(a_i)D_s(x) = \frac{p-1}{2} \left[A_s \left(x + \frac{p}{2} \right) + \frac{p}{2} \right] \det M(x),$$

and

$$A(a_i)D_s = \frac{p(p-1)}{4} (1 + A_s) \det M(0).$$

Since $\det M(x) = \det M(0) \neq 0$ and $A(a_i) \neq 0$,

$$\left[A_s \left(x + \frac{p}{2} \right) + \frac{p}{2} \right] = \frac{p}{2} (1 + A_s) D_s(x).$$

This completes the proof of the theorem.

In the rest of this paper, we shall calculate A_s in more convenient form, and show that Carlitz-Olson's formula follows easily from our theorem. Let $Z[G]$ be the group ring of G , and $Z[G]^- = \{\alpha \in Z[G] \mid \sigma_{-1}\alpha = -\alpha\}$ where σ_a is the image in G of an integer a . Let S be a subset of G . We define the element $s(S)$ in $Z[G]$ by $s(S) = \sum_{\sigma \in S} \sigma$. We put the element

$$\theta' = \sum_{\sigma \in G} \frac{R(\sigma)}{p} \sigma^{-1} \quad \text{in } Q[G],$$

and the ideal of $Z[G]$

$$\varphi' = \theta' Z[G] \cap Z[G].$$

Then the Stickelberger element is defined by

$$\theta = \sum_{\sigma \in G} \left(\frac{R(\sigma)}{p} - \frac{1}{2} \right) \sigma^{-1} = \varepsilon^- \theta'$$

where $\varepsilon^- = \frac{1}{2}(1 - \sigma_{-1})$. And the Stickelberger ideal is defined by

$$\varphi = \theta Z[G] \cap Z[G] = \{\alpha \in \varphi' \mid \sigma_{-1}\alpha = -\alpha\}.$$

Moreover, by [4] we have the formula

$$(\mathbf{Z}[G]^- : \varphi) = h^- .$$

For the CM-type $S_0 = \{1, 2, \dots, (p - 1)/2\}$

$$s(S_0) = (\sigma_{-1} + \sigma_2 - 1)\theta' ,$$

and so $s(S_0) \in \varphi'$. Therefore, for any CM-type S , $h^- \{s(S) - s(S_0)\} \in \varphi$ because $s(S) - s(S_0) \in \mathbf{Z}[G]^-$, and then $h^- s(S) \in \varphi'$. Therefore,

$$ks(S) = \theta' \alpha$$

for some $\alpha = \alpha_S \in \mathbf{Z}[G]$ and some integer $k | h^-$. Then we have

PROPOSITION. For any CM-type S ,

$$A_S = \frac{1}{k} \left(\sum_{\sigma \in S} n_\sigma - \sum_{\tau \notin S} n_\tau \right)$$

where $ks(S) = \theta' \alpha$, $\alpha = \sum_{\sigma \in G} n_\sigma \sigma$, $n_\sigma \in \mathbf{Z}$.

Proof. We extend a character χ to a function on $\mathbf{Q}[G]$ by

$$\chi(\alpha) = \sum_{\sigma \in G} n_\sigma \chi(\sigma) \quad \text{for } \alpha = \sum_{\sigma \in G} n_\sigma \sigma \in \mathbf{Q}[G] .$$

Then $c_\chi = \chi(s(S))$, and $B_{1,\chi} = \chi(\theta')$ for any non trivial character χ . Hence

$$kc_\chi = B_{1,\chi} \chi(\sigma)$$

and so,

$$A_S = \frac{2}{(p - 1)k} \sum_{\chi: \text{odd}} c_\chi \chi(\alpha) .$$

For $\sigma \in G$,

$$\sum_{\chi: \text{odd}} \chi(\sigma) = \begin{cases} \frac{p - 1}{2} & \text{for } \sigma = 1 , \\ -\frac{p - 1}{2} & \text{for } \sigma = \sigma_{-1} , \\ 0 & \text{otherwise .} \end{cases}$$

Therefore,

$$\sum_{\chi: \text{odd}} c_\chi \chi(\sigma) = \sum_{\chi: \text{odd}} \sum_{\tau \in S} \chi(\sigma \tau^{-1}) = \begin{cases} \frac{p - 1}{2} & \text{for } \sigma \in S \\ -\frac{p - 1}{2} & \text{for } \sigma \notin S \end{cases}$$

Consequently,

$$A_S = \frac{1}{k} \left(\sum_{\sigma \in S} n_\sigma - \sum_{\tau \notin S} n_\tau \right).$$

This completes the proof.

By the proposition and the theorem, for S_0 we have Carlitz-Olson's formula.

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