A NOTE ON NON-NEGATIVE MATRICES

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1. Introduction. This note can be regarded as an addendum to the paper **(4)**. On the complex Hilbert space of vectors $x = (x_1, x_2, \ldots,)$ a matrix A is said to be bounded if there exists a constant M such that $||Ax|| \leqslant M||x||$ whenever $||x||^2 = \sum|x_k|^2 < \infty$; the least such M is denoted by $||A||$. Only bounded matrices A and vectors x satisfying $||x|| < \infty$ will be considered in the sequel. The spectrum of A, denoted by $sp(A)$, is the set of values for which the resolvent $R(\lambda) = (A - \lambda I)^{-1}$ fails to be bounded. The notation $A \geqslant 0$ or $A > 0$, where $A = (a_{ij})$, means that, for all i and j, $a_{ij} \geqslant 0$ or $a_{ij} > 0$ respectively. There was stated in **(4)** the following theorem (also contained in some results of Bonsall, cf. the references cited in **(4)**) generalizing results of Perron and Frobenius for finite matrices:

(I) *If* $A \geqslant 0$, *then* $\mu = \sup|\lambda|$, *where* λ *is in* $\mathrm{sp}(A)$, *also belongs to* $\mathrm{sp}(A)$.

The proof in **(4)** of this theorem is not correct for arbitrary bounded $A \geqslant 0$, although it is valid for any such matrix with a spectrum identical with the set of (function theoretical) singularities of its resolvent, that is, with the set of singularities of at least one element of the resolvent. However, although the spectrum always contains this latter set, there exist bounded matrices, even satisfying $A \geqslant 0$, for which a number can belong to the spectrum and, at the same time, be an analytic point of each element of the resolvent. Such a matrix is given by $B = (b_{ij})$ where $b_{ij} = 1$ or 0 according as j is, or is not, $i + 1$. In fact, $B \geqslant 0$, $\mathrm{sp}(B)$ is the unit disk $|\lambda| \leqslant 1$, and 0 is the only singularity of $R(\lambda)$ (see **(6**, p. 145**)**). In view of this circumstance, an alternate simple proof of (I) will be given in § 2 below.

A few remarks relating to **(4)** will be made in § 3. In § 4, generalizations of certain theorems stated in a recent paper of Birkhoff and Varga **(1**, pp. 356-357**)**, will be given.

2. In order to prove (I), it is sufficient to show that $R(\lambda)$ is bounded whenever $R(|\lambda|)$ is bounded. Now $R(\lambda)$ is given by $R(\lambda) = -\sum A^n/\lambda^{n+1}$ whenever $|\lambda|$ is sufficiently large, in fact, whenever $|\lambda|$ exceeds the spectral radius,

$$\mu\left(= \lim_{n\to\infty} ||A^n||^{1/n}\right),$$

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of A (cf., for example, **(5**, p. 421**)**). Since $A \geqslant 0$, also $A^n \geqslant 0$, and so if $R(\lambda) = (R_{ij}(\lambda))$, it is clear that, for $|\lambda| > \mu$, $|R_{ij}(\lambda)| \leqslant -R_{ij}(|\lambda|)$. If $x = (x_1, x_2, \ldots,)$ and $X = (|x_1|, |x_2|, \ldots,)$, then $||x|| = ||X||$ $(< \infty)$. Consequently, $||R(\lambda)x||^2 = \sum_i|\sum_j R_{ij}(\lambda)x_j|^2 \leqslant \sum_i(\sum_j R_{ij}(|\lambda|)|x_j|)^2 = ||R(|\lambda|)X||^2 \leqslant ||R(|\lambda|)||^2||x||^2$, and so $||\mathrm{R}(\lambda)|| \leqslant ||R(|\lambda|)||$ whenever $|\lambda| > \mu$. But if μ is not in sp(A), then $||R(\mu)|| < \infty$, and it follows from the continuity of $||R(\lambda)||$ on the complement of the spectrum, and from the fact that $||R(\lambda)|| \to \infty$ whenever λ is not in sp(A) and tends to a point of sp(A), that $||R(\lambda)|| \leqslant ||R(\mu)|| < \infty$ for $|\lambda| = \mu$. Since sp(A) is closed, this last inequality implies that the spectral radius is less than μ, a contradiction, and the proof of (I) is now complete.

3. The third theorem in **(4)** can be stated as

(III) *If* $A \geqslant 0$ *and if* μ *of* (I) *is a pole of the resolvent* $R(\lambda) = (A - \lambda I)^{-1}$ *then there exists a characteristic vector* $x \geqslant 0$ *of* A *belonging to* μ, *thus* $Ax = \mu x$ $(x \neq 0)$.

Actually it was assumed in **(4)** that μ should be positive; the proof given there, § 5, makes it clear, however, that this need not be assumed. If λ is real and satisfies $\lambda > \mu$, then $R(\lambda) = -\sum A^n/\lambda^{n+1} \leqslant 0$, and the matrix inequality $c_{-N} \leqslant 0$ $(N \geqslant 1,\ c_{-N} \neq 0)$ needed in the representation

$$R(\lambda) = \sum_{n=-N}^{\infty} c_n(\lambda - \mu)^n$$

of **(4**, § 5**)**, is still assured.

Whether the assumption in (III) that μ be a pole of $R(\lambda)$ can be weakened to the (implied) condition that μ be an isolated point of sp(A) and belong to the point spectrum will remain undecided. It is even conceivable that only the assumption that μ be in the point spectrum is needed in the hypothesis of (III).

Incidentally, the statement of **(3)**, and mentioned in **(4)**, that *if* $A \geqslant 0$ *and is completely continuous, and if the diagonal elements of every power* A^n *are zero, then zero is the only point of* sp(A), surely cannot be true if the assumption of complete continuity is omitted, as the matrix B cited earlier in this paper shows.

4. Generalizations of certain theorems for finite matrices stated in a recent paper of Birkhoff and Varga **(1**, pp. 256–257**)**, will be given in this section. Corresponding to the terminology of **(1)**, a matrix A will be called non-negative or positive according as $A \geqslant 0$ or $A > 0$, essentially non-negative if $a_{ij} \geqslant 0$ for $i \neq j$, and irreducible (also indecomposable, cf. the references cited in **(1)**) if, for any i and j, there exists a finite sequence $i = k(0), k(1), \ldots, k(N) = j$ for which $a_{k(n-1),k(n)} \neq 0$ for $k = 1, 2, \ldots, N$. A vector $x = (x_1, x_2, \ldots,)$ will be called non-negative or positive according as $x_i \geqslant 0$ or $x_i > 0$ for all i.

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(i) *If A is essentially non-negative then* $\nu = \max Re(\text{sp}(A))$ *is in* $\text{sp}(A)$; *moreover,* $\nu > Re(\lambda)$ *if* $\lambda \neq \nu$ *and* λ *is in* $\text{sp}(A)$. *In case ν is a pole of the resolvent* $R(\lambda) = (A - \lambda I)^{-1}$, *then A has a non-negative characteristic vector x belonging to ν.*

In fact, (i) follows readily from (I) and (III) if these latter theorems are applied to the matrix $C = A + \alpha I$ which is non-negative if α is positive and sufficiently large. It is to be noted that the resolvent of C is given by $R(\lambda - \alpha)$.

Furthermore,

(ii) *If A is essentially non-negative and irreducible and if ν of* (i) *is a pole of* $R(\lambda)$, *then* (a) *ν is a simple pole of* $R(\lambda)$, (b) *ν is a simple characteristic number, and* (c) *there exists a positive characteristic vector x of A belonging to ν.*

Assertion (ii) follows from (IV) of **(4)**, namely,

(IV) *If* $C \geqslant 0$, *if for every pair, i, j there exists an integer* $M = M(i,j)$ *such that* $(C^M)_{ij} > 0$, *and if μ of* (I) *is a pole of* $R(\lambda) = (C - \lambda I)^{-1}$, *then* (a) *$\mu$ is a simple pole of* $R(\lambda)$, (b) *μ is a simple characteristic number, and* (c) *there exists a characteristic vector* $x > 0$ *belonging to μ.*

In order to see this, let (IV) be applied to $C = A + \alpha I$, which is non-negative for α positive and sufficiently large, and note that the condition $(C^M)_{ij} > 0$ for some positive integer $M = M(i,j)$ is a consequence of the present assumption of irreducibility of A, provided that α is sufficiently large. (In this connection for finite matrices, see **(2**, p. 20**)**). For, let $\alpha > 0$ be chosen so large that the diagonal elements c_{ii} of C are positive. Since $c_{ij} = a_{ij}$ if $i \neq j$, it is then clear that the irreducibility of A implies that of C. Consequently, since $C \geqslant 0$, there exists for any pair i, j a positive integer $M = M(i,j)$ and a finite sequence $i = k(0), k(1), \ldots, k(M) = j$ such that

$$d = \prod_{n=1}^{M} c_{k(n-1),k(n)} > 0.$$

But $(C^M)_{ij}$ is given by a sum of non-negative terms one of which is d and so $(C^M)_{ij} > 0$. Thus, as remarked above, (ii) follows from (IV) of **(4)**. Incidentally, the above argument makes clear that a non-negative matrix, here C, satisfies $(C^M)_{ij} > 0$ for every pair i, j and some positive integer $M = M(i,j)$ if and, in fact, only if, it is irreducible.

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