ON THE EXISTENCE OF THE BURKILL INTEGRAL

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1. Introduction. The main problem of the present paper is the existence of the Burkill integral of an interval function $f(I)$ which is not supposed to be continuous. Little is known about this case, though otherwise the theory of the integral can be considered as complete: we may refer to Ringenberg's comprehensive paper **(2)** in which further references are given.

We shall deal with the problem by introducing the notion of *infinitesimal additivity* and will show that the indefinite integral can be continuous even when $f(I)$ is not. Finally we apply the main result to the generalised arc length of a curve; the result appears not to be known even with respect to the familiar notion of arc length.

Let $R(0 \leqslant x_j \leqslant A_j; j = 1,2, \ldots, n)$ be a fixed interval in the Euclidean space $Eu_n(n \geqslant 1)$, and let an interval function $f(I)$ be defined for any closed interval

$$I(a_{1j} \leqslant x_j \leqslant a_{2j}; 0 \leqslant a_{1j} < a_{2j} \leqslant A_j) \subset R.$$

Any $f(I)$ is supposed to be finite for every $I \subset R$. The following result is known **(2; 3**, p. 168**)**.

THEOREM 1. *If* (i) $f(I)$ *increases by subdivision* (*abbreviation* $f(I) \subset SA$), (ii) *the upper Burkill integral of* $|f(I)|$ *over* R *is finite, and* (iii) $f(I)$ *is continuous on* R, *then its Burkill integral over* R *and, therefore, over any* $I \subset R$, *exists.*

We replace the condition of continuity by a much weaker one.

THEOREM 1$'$. (a) *Theorem* 1 *holds when the condition* (iii) *is replaced by* (iii$'$): $f(I)$ *is infinitesimally additive on* R (see 2.1).

(b) *When* R *is the linear interval* $\langle 0,A \rangle$ *and* $f(I)$ *is subadditive, then* $f(I)$ *is Burkill integrable if and only if* (i) *its lower Burkill integral is bounded above and* (ii) $f(I)$ *is infinitesimally additive on* R.

Again for non-continuous $f(I)$, $\int_I f$ can be continuous (§ 5).

2. Some additional definitions and notations. A representation of an interval I in the form $I = I_1 + \ldots + I_m = \sum I_k$ is said to be a subdivision $\mathfrak{S}$ of I, or $\mathfrak{S}(I)$. The I_k's are always required to be *finite in number and not to overlap*. We write $f(\mathfrak{S})$ for $\sum f(I_k)$; $||I_k||$ for the diameter of I_k;$||\mathfrak{S}||$ for max $||I_k||$(k $= 1, \ldots, m$). When the I_k's are arranged in rows and columns $(n \geqslant 2)\mathfrak{S}$ is said to be a mesh-division. Any finite number of non-overlapping

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intervals form a figure, denoted by F or $\sum I_k$. While I is closed, I^0 is open; $|I|$ is its Lebesgue measure, in Eu_2, for example, the area of the interval.

The upper, lower Burkill integral; the Burkill integral, respectively, of $f(I)$ over $I \subset R$ are denoted by $U_I f$, $L_I f$; $\int_I f$. The existence of the latter integral is meant to imply its finiteness.

If, for any $I \subset R$,

$$f(I) \leqslant f(I_1) + f(I_2) \qquad (I = I_1 + I_2),$$

then $f(I)$ is said to be subadditive ($f(I) \in st$); if, for any $\mathfrak{S}$, $f(I) \leqslant f(\mathfrak{S})$, then $f(I) \in SA$. Clearly $SA \subset st$; and $SA \equiv st$ in Eu_1.

By i we denote any oriented interval of $n - 1$ dimensions such that $i^0 \subset R^0 (n \geqslant 2)$, while in Eu_1, i is any point of $R^0 = (0, A)$. Thus in Eu_2, i is any line segment parallel to one of the axes which does not form part of the perimeter of R and has no point outside R. When $n = 2$ and both end-points of i, or $n = 3$ and all the sides of i lie on $R - R^0$, etc., we use sometimes the notation i^*. A function $f(I)$ is said to be infinitesimally additive if for any fixed i

2.1 $$f(I_1) + f(I_2) - f(I) \to 0 \qquad (I = I_1 + I_2 \subset R)$$

whenever $I_1 I_2 = i$ and $|I| \to 0$. An interval i^* is said to be irregular if there is at least one sub-interval $i \subset i^*$ for which 2.1 does not hold. We remark that, if $f(I)$ is of bounded variation over R (abbreviation: $f \in V$; $V_I f$ = total variation of f over I), then the limits of

$$f(I_1), f(I_2), f(I) \ (i \text{ fixed}, I_1 | I_2 = i, |I_1 + I_2| = |I| \to 0)$$

exist, and the *irregular i^* are countable.*

3. Some lemmas

LEMMA 1. *If $\int_R f$ exists then, given $\epsilon > 0$, there is a $\delta > 0$ such that whenever a figure*

$$\sum I_k \subset R \quad \textit{and} \quad \max ||I_k|| < \delta \ (k = 1,2, \ldots),$$

3.11 $$\left| \sum \left\{ f(I_k) - \int_{I_k} f \right\} \right| < \tfrac{1}{2}\epsilon$$

3.12 $$\sum \left| f(I_k) - \int_{I_k} f \right| < \epsilon.$$

While 3.11 is known (**2**; **3**, p. 167), 3.12 is deduced from it by considering separately the intervals I_k for which the corresponding differences occurring in the sum in 3.11 are $\geqslant 0$ or < 0, respectively.

We proceed to some elementary existence theorems.

LEMMA 2. *The integral $\int_R f$ exists if, and only if, given $\epsilon > 0$, there is a $\delta > 0$ such that for subdivisions $\mathfrak{S}_1, \mathfrak{S}_2$ of R*

3.21 $$|f(\mathfrak{S}_1) - f(\mathfrak{S}_2)| < \epsilon, \qquad (||\mathfrak{S}_j(R)|| < \delta, j = 1, 2);$$

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or if

3.221 $$\left| \sum_k \left\{ f(I_k) - \sum_l f(I_{kl}) \right\} \right| < \epsilon;$$

or

3.222 $$\sum_k \left| f(I_k) - \sum_l f(I_{kl}) \right| < \epsilon$$

whenever $R = \sum I_k$, $\max||I_k|| < \delta$ and $I_k = \sum_l I_{kl}$.

Part 3.21 is trivial. The necessity of the condition of 3.221 follows from it; so does the sufficiency, when $\mathfrak{S}_1$ is taken as $\sum I_k$ and $\sum_{k,l} I_{kl}$ as a subdivision both of $\mathfrak{S}_1(R)$ and of $\mathfrak{S}_2(R)$. Hence the condition 3.222 is sufficient also. Its necessity is deduced from 3.12 by the additivity of the Burkill integral.

LEMMA 3a. *When $f(I) \in SA$ the integral exists if, and only if,* (i) $L_R f < \infty$, *and* (ii) *given $\epsilon > 0$, there is a $\delta > 0$ such that for any figure $F = \sum I_k \subset R$, with $||F|| < \delta$ and $I_k = I_{k1} + I_{k2}$,*

(3.3) $$\sum_k \{f(I_{k1}) + f(I_{k2}) - f(I_k)\} < \epsilon.$$

LEMMA 3b. *In Eu_1,* (ii) *can be replaced by the weaker condition* (ii′) *$f(I)$ is infinitesimally additive* (see 2.1).

Proof af Lemma 3a. The necessity of the condition follows immediately from 3.221. To prove the converse we may consider Eu_n for $n = 2$ only. Clearly $L_R f \geqslant f(R) > -\infty$. We show that, for any $\mathfrak{S}(R)$, $f(\mathfrak{S}) \leqslant L_R f$; which implies that

$$U_R f \leqslant L_R f, \quad U_R = L_R = \int_R f.$$

Since $f(I) \in SA$ we may for convenience suppose that $\mathfrak{S}(R)$ $(R = \sum J_j)$ be a mesh-division. Fixing ϵ and δ according to (ii), we find an $\mathfrak{S}^*(\mathrm{R})$ $(\mathrm{R} = \sum I_k)$ such that $f(\mathfrak{S}^*) < L_R f + \epsilon$, and that $||\mathfrak{S}^*||$ is smaller than δ and the sides of each J_j. Denote the I_k's that lie in one, two or four of the J_j by I_p, I_q or I_r, respectively, and set

$$I_q = I_{q1} + I_{q2},\ I_r = I_{r1} + \ldots + I_{r4},$$

where each I_{ql}, I_{rl} belongs to *one* J_j only. As $f(I) \in SA$,

$$\begin{aligned} f(S) &\leqslant \sum_p f(I_p) + \sum_q \sum_{l=1}^{2} f(I_{ql}) + \sum_r \sum_{l=1}^{4} f(I_{rl}) \\ &= \sum_k f(I_k) + \sum_q \left\{ \sum_{l=1}^{2} f(I_{ql}) - f(I_q) \right\} + \sum_r \left\{ \sum_{l=1}^{4} f(I_{rl}) \right. \\ &\quad \left. - f(I_{r1} + I_{r2}) - f(I_{r3} + I_{r4}) \right\} + \sum_r \{f(I_{r1} + I_{r2}) \\ &\quad + f(I_{r3} + I_{r4}) - f(I_r)\} \\ &< f(S^*) + 3\epsilon < L_R f + 4\epsilon \end{aligned}$$

by (ii). Taking $\epsilon \to 0$ we complete the proof.

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Proof of Lemma 3b. We proceed as before, but observe that in Eu_1 the I_k's are either of the type I_p or I_q and that the number of the I_q's is less than N, the number of the J_j's. Taking a suitable $\mathfrak{S}^*$ we show that, given $\epsilon > 0$,

$$f(\mathfrak{S}) < L_R f + \epsilon + N\epsilon;$$

therefore, $U_R = L_R$, so that the integral exists. Conversely, to obtain 2.1 we take $F = I \supset i$ and observe that $||F|| = |I|$ in this case.

Finally we state two results which are not difficult to prove.

If $f(I) \in SA$ and $U_R f$ is finite and $L_I f$ additive, then $\int_R f$ exists.

In Eu_1, if $U_R|f| < \infty$ and $f(I)$ is infinitesimally additive, then both $U_I f$ and $L_I f$ are additive.

4. Proof of Theorem 1′. Again we show that (iii′) is not a necessary condition.

Part (b) follows from Lemma 3b. To deal with (a) we take $n = 2$. We have to show that, for any

$$\mathfrak{S}(R)\ \ (R = \sum I_k),\ f(\mathfrak{S}) \leqslant L_R f;$$

we may suppose that $\mathfrak{S}$ be a mesh-division since $f(I) \in SA$. Let $i_1^*, \ldots, i_n^*$ and $i_{N+1}^*, \ldots, i_{N+T}^*$ be the lines generating $\mathfrak{S}$, parallel to the x_1 and x_2 axis, respectively. If the variation of $f(I)$ on R is zero at each of these lines then, given $\epsilon > 0$, we deduce by a known argument **(3**, pp. 166, 168**)** that $f(\mathfrak{S}) < L_R f + 3\epsilon$ and take $\epsilon \to 0$. Suppose now that the variation of $f(I)$ on R is not zero at i_1^*, say. Let $I_1, I_3, \ldots, I_{2T+1}$ be the intervals lying between $x_2 = 0$ and i_1^*; $I_2, I_4, \ldots, I_{2T+2}$ between i_1^* and i_2^*. We can draw i' between $x_2 = 0$ and i_1^*, i'' between i_1^* and i_2^*, both lines parallel to i_1^* and arbitrarily near it, and such that the variation of $f(I)$ on R vanishes at i' and i''; these lines divide I_{2k-1} or I_{2k} $(k = 1,2, \ldots, T+1)$ into $I_{2k-1,1}$ and $I_{2k-1,2}$, or into $I_{2k,1}$ and $I_{2k,2}$, respectively, where $I_{2k-1,2}$ and $I_{2k,1}$ are adjacent. Set $I_{2k-1,2} + I_{2k,1} = I'_{2k}$. As $f(I) \in SA$,

$$f(I_{2k-1}) + f(I_{2k}) \leqslant f(I_{2k-1,1}) + f(I'_{2k}) + f(I_{2k}) + \Lambda;$$
$$\Lambda = f(I_{2k-1,2}) + f(I_{2k,1}) - f(I'_{2k}) \to 0\ \ (i' \to i_1^*,\ i'' \to i_1^*),$$

since $f(I)$ is infinitesimally additive. Proceeding in this way we replace $\mathfrak{S}(R)$ by a $\mathfrak{S}'(R)$ such that the variation at each of the lines producing $\mathfrak{S}'(R)$ is zero and that

$$f(\mathfrak{S}) \leqslant f(\mathfrak{S}') + \epsilon;$$

which completes the proof.

Clearly (iii′) is weaker than the condition that $f(I)$ be continuous; in Eu_n $(n \geqslant 2)$ however, (iii′) is not a necessary condition either. Consider the square $0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1$ and take $f(I) = 0$ except in the following cases.

(a) One side of I is formed by a segment, of length $l(0 < l \leqslant 1)$ say, of the line $x_1 = \frac{1}{2}$. Then take $f(I) = l$.

(b) Part of the line $x_1 = \frac{1}{2}$ is contained in the interior of I, and the total length l of the closed segment concerned is < 1. Then $f(I) = 2l$.

Plainly $f(I) \in SA$, $f(I)$ is integrable, and $\int_R f = 2$. Yet $f(I)$ is not infinitesimally additive. Take i^* as the segment $0 \leqslant x_2 \leqslant 1$ of $x_1 = \frac{1}{2}$. Then

$$l = 1, f(I_1) = f(I_2) = 1, \qquad (I_1 I_2 = i^*, I = I_1 + I_2)$$

while $f(I) = 0$. Thus the term

$$f(I_1) + f(I_2) - f(I) = 2$$

and does not tend to zero.

5. Continuity of the indefinite integral.

We deduce

THEOREM 2. *Suppose that $f(I)$ is Burkill integrable.*

(a) *Then $F(I) = \int_I f$ is continuous on R if and only if, given $\epsilon > 0$, there are numbers $\delta,\eta > 0$ such that $|\sum f(I_k)| < \epsilon$ whenever $|I| < \delta$, $I = \sum I_k$ and $||I_k|| < \eta$.*

(b) *The continuity of $f(I)$ is necessary and sufficient for that of $F(I)$* (i) *in Eu_1,* (ii) *when*

$$|f(I)| \leqslant |\sum f(I_k)| \qquad (I = \sum I_k),$$

for instance when $f(I)$ increases by subdivision and is non-negative.

The statement (a) is deduced from the inequality

$$\left| \left| \sum f(I_k) \right| - \left| F(I) \right| \right| < \tfrac{1}{2}\epsilon \qquad \left(\max_k \left| \left| I_k \right| \right| < \eta\right)$$

which follows from Lemma 1. Since continuity is well-known to be a sufficient condition (**3**, p. 167), (b ii) is now evident. In Eu_1, we have $|I| = ||I||$. Hence it is necessary that

$$|f(I)| < \epsilon \text{ for } |I| < \min(\delta,\eta).$$

Thus $f(I)$ is continuous.

Note that in Eu_n, $n > 1$, the condition in (a) does not imply continuity of $f(I)$. Take $n = 2$; R as the square

$$0 \leqslant x_j \leqslant 1; f(I) = |I| + l^2$$

when I touches the line $x_1 = 1$ along a segment of length l, $f(I) = |I|$ otherwise. Clearly $f(I)$ *is not* continuous, while $F = \int_R f$ exists; $F(I) = |I|$, which *is* continuous.

6. A rectifiable curve.

The curve $C\{x_1(t), x_2(t), \ldots, x_m(t)\}$ in Eu_m is defined by functions $x_j(t)$ of bounded variation over

$$R = \langle 0,a \rangle, \qquad j = 1,2,\ldots,m, x_j(0-) = x_j(0), x_j(a+) = x_j(a).$$

Its arc length $A_{0,a}$ is

$$A_{0,a} = \text{l.u.bd.} \sum_k F(I_k); \; F(I) = \left\{ \sum_{j=1}^{m} (x_j(I))^2 \right\}^{\frac{1}{2}}, \; \sum I_k = R, \max|I_k| \to 0,$$

where $x_j(I) = x_j(t_2) - x_j(t_1)$ for $I = \langle t_1, t_2 \rangle \subset R$. If all the $x_j(t)$ are continuous then not only the upper bound, but also the proper limit of $\sum F(I_k)$, that is $\int_R F(I)$, is known to exist. We deduce

THEOREM 3. *Given a curve $C\{x_1(t), \ldots, x_n(t)\}$, the Burkill integral $\int_R F(I)$ exists if, and only if, C is normal. This holds for the generalised form of $F(I)$ as defined below.*

Definition 1. A curve $C(\{x_1(t), \ldots, x_m(t)\}$ is *normal* if all $x_j(t) \in V$ and if, for any $t \in R^0$, there is a $\rho_t (0 \leqslant \rho_t \leqslant 1)$ such that

6.1 $$x_j(t) = \rho_t x_j(t-) + (1 - \rho_t) x_j(t+), \qquad j = 1, 2, \ldots, m,$$

that is, if any point associated with t lies on the line segment joining the two points $x_j(t-)$, $x_j(t+)$ $(j = 1, \ldots, m)$; which clearly coincide when all the $x_j(t)$ are continuous at t.

Definition 2 (Generalisation of the arc length). Let the function

$$f(y_1, y_2, \ldots, y_m) (0 \leqslant y_j < \infty)$$

be (i) non-negative, (ii) continuous, (iii) strictly increasing concerning each y_j, (iv) homogeneous of degree one and (v) subadditive, i.e.

$$f(y_1 + z_1, \ldots, y_m + z_m) \leqslant f(y_1, \ldots, y_m) + f(z_1, \ldots, z_m) \qquad \begin{pmatrix} 0 \leqslant y_j < \infty \\ 0 \leqslant z_j < \infty \end{pmatrix}$$

and such that there is equality only if the z_j and y_j are effectively proportional (that is, for some finite $\sigma > 0$, $y_j = \sigma z_j$ or $z_j = \sigma y_j$). Clearly

$$F(I) = f(|x_1(I)|, \ldots, |x_m(I)|) \in sA$$

as the $x_j(I)$ are additive; and the *generalised arc length* is defined as the upper Burkill integral UF. We obtain the ordinary arc length when

$$f(y_1, \ldots, y_m) = \left(\sum_{j=1}^{m} y_j^p \right)^{1/p}, \qquad p = 2,$$

but f satisfies the above conditions also for $1 < p < \infty$ by Minkowski's inequality (**1**, §2.11). So does, for instance, the function

$$f(y_1, y_2) = (y_1^2 + k y_1 y_2 + y_2^2)^{\frac{1}{2}}, \qquad 0 < k < 2.$$

By (iv), $f = 0$ for $y_1 = y_2 = \ldots = y_m = 0$, while $f > 0$ otherwise by (iii). Again by (iii) and (iv),

$$f \leqslant f(1, 1, \ldots, 1) \max y_j, f \geqslant y_1 f(1, 0, \ldots, 0), f \geqslant y_2 f(0, 1, 0, \ldots, 0), \ldots.$$

Thus

$$6.2 \qquad \sum_1^m |x_j(I)|f(1,\ldots,1) \geqslant F(I) \geqslant c\sum_1^m |x_j(I)|;\, c = \min\{f(1,0,\ldots),\ f_1(0,1,\ldots),\ldots\}/m,$$

so that $UF < \infty$ if and only if all $x_j(t) \in V$.

Suppose now that 6.1 be satisfied. Take any point $t = i \in R^0$,

$$I_1 = \langle t_1,t\rangle,\ I_2 = \langle t,t_2\rangle,\ I = \langle t_1,t_2\rangle \subset R_j;\ |I| \to 0.$$

$$6.3 \quad F(I_1) + F(I_2) - F(I) \to f(|u_1|,\ldots,|u_m|) + f(|v_1|,\ldots,|v_m|) - f(|w_1|,\ldots,|w_m|),$$

where $u_j = x_j(t) - x_j(t-)$, $v_j = x_j(t+) - x_j(t)$, $w_j = x_j(t+) - x_j(t-)$, and by 6.1, $u_j = (1-\rho_t)w_j$, $v_j = \rho_t w_j$. By the homogeneity of f the expression on the right in 6.3 vanishes. Hence $F(I)$ is infinitesimally additive, therefore $\int_R F(I)$ exists.

Conversely suppose that $F(I)$ is Burkill integrable. Then it must be infinitesimally additive. By 6.3, therefore,

$$6.4 \qquad f(|w_1|,\ldots,|w_m|) = f(|u_1|,\ldots,|u_m|) + f(|v_1|,\ldots,|v_m|).$$

Now $|w_j| \leqslant |u_j| + |v_j|$. By (iii) and (v), therefore, 6.4 remains true when $|w_j|$ is replaced by $|u_j| + |v_j|$. Hence

$$6.5 \qquad f\{(|u_1| + |v_1|),\ (|u_2| + |v_2|),\ldots\} = f(|u_1|,\ldots) + f(|v_1|,\ldots).$$

Any u_j and v_j have equal signs; for if u_jv_j were negative for some j, $|w_j|$ would be $<|u_j| + |v_j|$, and since f is strictly monotone, 6.4 and 6.5 would contradict each other. Hence $u_jv_j \geqslant 0$. By (v), 6.5 implies that the $|u_j|$ and $|v_j|$ be effectively proportional. Thus for some $\sigma > 0$, depending on t only, $v_j = \sigma u_j (j = 1,2,\ldots,m)$ or $u_j = \sigma v_j$. Taking $\rho_t = \sigma(1+\sigma)^{-1}$ or $(1+\sigma)^{-1}$, respectively, we arrive at 6.1. This completes the proof.

Remark 1. Clearly (iv) and (v) imply that f is a convex function.

For $n \leqslant 1$, $U_{I_0}F\ (x_j(I) \epsilon V,\ I \subset I_0 \subset Eu_n)$ is a lower semi-continuous functional.

Remark 2. There are applications of our main theorem to the areas of surfaces $z = f(x,y)$ which are not continuous or even nowhere continuous.

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