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STRUCTURE TOPOLOGY AND EXTREME OPERATORS

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Abstract

We say that a Banach space X is 'nice' if every extreme operator from any Banach space into X is a nice operator (that is, its adjoint preserves extreme points). We prove that if X is a nice almost CL-space, then X is isometrically isomorphic to $c_0(I)$ for some set I. We also show that if X is a nice Banach space whose closed unit ball has the Krein–Milman property, then X is l_∞^n for some $n \in \mathbb{N}$. The proof of our results relies on the structure topology.

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1. Introduction

Nice operators were introduced in [15] as those operators whose adjoint preserves extreme points of the unit ball. As a consequence of the Krein–Milman theorem, every nice operator is an extreme operator. Nice operators have been intensively studied (see, for example, [17] for recent results). Even before its formal definition, the coincidence of extreme and nice operators between spaces of continuous functions was considered in [2]. In subsequent papers, Sharir proved the existence of extreme nonnice operators between spaces of continuous functions and that every extreme operator between L_1-spaces is a nice operator (see [18–20]). Recently, in [16], the authors studied the coincidence of nice operators and surjective isometries in the context of spaces of continuously differentiable functions. In [3], the notion of a nice Banach space was introduced and studied for the first time. A Banach space is said to be nice if every extreme operator into it is a nice operator. The main results in [3] characterise spaces of continuous functions and reflexive Banach spaces which are nice. It was also proved in [3] that, if μ is σ-finite, the only nice $L_1(\mu)$-space is either the scalar field or l_1^2. Later on, in [4] and [5], nice Banach spaces were characterised in the context of special types of L_1-preduals, namely, simplex spaces and G-spaces. Lima in [11] introduced almost CL-spaces, though this class of Banach spaces appears implicitly

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in [12]. We give below the definition of an almost CL-space, but it is worth mentioning that L_1-spaces and their preduals are almost CL-spaces. The aim of this paper is to characterise almost CL-spaces which are nice. We prove that a nice almost CL-space is isometrically isomorphic to $c_0(I)$ for some nonempty set I. We also prove that nice Banach spaces whose closed unit ball satisfies the Krein–Milman property are isometrically isomorphic to l_∞^n for some $n \in \mathbb{N}$. The main tool for getting our results is the structure topology, which we introduce in Section 3.

2. Notation and preliminaries

Throughout this paper we only consider real Banach spaces. Given a Banach space X, B_X, S_X and E_X will stand for the closed unit ball of X, the unit sphere of X and the set of extreme points of B_X, respectively. If A is a nonempty subset of X, $\mathrm{co}(A)$, $\mathrm{lin}(A)$ and $\overline{\mathrm{co}}(A)$ will denote the convex hull of A, the linear span of A and the closed convex hull of A, respectively. The space of all bounded linear operators from a Banach space X into a Banach space Y will be denoted by $\mathcal{L}(X, Y)$, endowed with its usual operator norm. According to the custom, we will write X^* instead of $\mathcal{L}(X, \mathbb{R})$ and the adjoint of an operator T will be represented by T^*. If B is any nonempty subset of X^*, we will denote by $\overline{B}^{w^*}$ and $\overline{\mathrm{co}}^{w^*}(B)$ the closure and the closed convex closure of B in the w^*-topology of X^*. If M is a subspace of X, then $M^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in M\}$.

Next we introduce the class of Banach spaces we are interested in.

Definition 2.1. A Banach space X is said to be an *almost CL-space* if any maximal convex subset of S_X fulfils $B_X = \overline{\mathrm{co}}(F \cup -F)$. If one can omit the closure in the above equality, then X is said to be a *CL-space*.

Fullerton in [8] first introduced CL-spaces, and Lima in [11] defined almost CL-spaces as a generalisation of CL-spaces, although, as far as we know, the existence of an almost CL-space which is not a CL-space is an open question. Examples of CL-spaces are $L_1(\Omega, \mathcal{A}, \mu)$, for any $(\Omega, \mathcal{A}, \mu)$ measure space, and its isometric preduals (see [10, Section 3]).

For more information about almost CL-spaces, we refer to [14]. In that paper the authors showed several basic facts about maximal convex subsets of S_X, which we recall below for the sake of clarity. It is a consequence of the Hahn–Banach and Krein–Milman theorems that for each maximal convex subset F of S_X, there exists x^* in E_{X^*} such that $F = F(x^*) = \{x \in S_X : x^*(x) = 1\}$. We denote by $mexB_{X^*}$ the set of elements x^* in E_{X^*} such that $F(x^*)$ is a maximal convex subset of S_X. It is easy to prove that, for any x in X, there exists x^* in $mexB_{X^*}$ such that $x^*(x) = \|x\|$. The Hahn–Banach theorem allows us to get $B_{X^*} = \overline{\mathrm{co}}^{w^*}(mexB_{X^*})$ and the reversed Krein–Milman theorem yields $E_{X^*} \subseteq \overline{mexB_{X^*}}^{w^*}$.

Finally, we give the central notion in this paper.

Definition 2.2. A Banach space X is said to be *nice* if for any Banach space Y, every extreme operator T in $\mathcal{L}(Y, X)$ satisfies $T^*(E_{Y^*}) \subseteq E_{X^*}$. That is, every extreme operator in $\mathcal{L}(Y, X)$ is a nice operator.

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3. The structure topology

We need the notion of an L-summand in order to introduce the structure topology. The main reference concerning this concept is [9].

DEFINITION 3.1. A mapping π from X into X is said to be a *semi-L-projection* on X if π satisfies:

(i) $\pi(\alpha x + \pi(y)) = \alpha\pi(x) + \pi(y)$ for all x, y in X and α in $\mathbb{R}$;
(ii) $\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$ for all x in X.

A linear semi-L-projection is called an *L-projection*. The range of a semi-L-projection (L-projection) on X is said to be a *semi-L-summand* (respectively, an *L-summand*) in X.

If π is a semi-L-projection on X and $J = \pi(X)$, it is easy to prove that, for x in X, $\pi(x)$ is the unique best approximant to x in J, that is, $\pi(x)$ is the only element in J which satisfies $\|x + J\| = \|x - \pi(x)\|$. Thus, J is a closed subspace of X and there is a unique semi-L-projection on X with range J.

The notion of L-summand enables us to define a topology on the set E_{X^*} for any Banach space X. This topology was first introduced by Alfsen and Effros in [1] and it will be the main tool for getting our results.

DEFINITION 3.2. Let X be a Banach space. The sets $J \cap E_{X^*}$, where J is a w^*-closed L-summand in X^*, are the closed sets of a topology on E_{X^*}, called *the structure topology*.

Next we give a characterisation of 'isolated points' in the structure topology.

PROPOSITION 3.3. *Let X be a Banach space and let e_0^* be in E_{X^*}. The following assertions are equivalent:*

(i) $X^* \neq \overline{\mathrm{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$;
(ii) $\{\pm e_0^*\}$ *is structurally open.*

PROOF. We prove that (i) implies (ii). The Hahn–Banach theorem provides an element x_0 in S_X such that $e^*(x_0) = 0$ for all $e^* \in E_{X^*} \setminus \{\pm e_0^*\}$. Moreover, since $1 = \|x_0\| = \max\{e^*(x_0) : e^* \in E_{X^*}\}$ (see [7, Fact 3.119]), it follows that $|e_0^*(x_0)| = 1$. We can suppose that $e_0^*(x_0) = 1$. We consider the operator $\pi : X^* \to X^*$ defined by $\pi(x^*) = x^* - x^*(x_0)e_0^*$. It is clear that π is a linear projection and that

$$\pi(X^*) = \{x^* \in X^* : x^*(x_0) = 0\}.$$

Therefore, $J = \pi(X^*)$ is w^*-closed and $J \cap E_{X^*} = E_{X^*} \setminus \{\pm e_0^*\}$. We are going to prove that π is an L-projection. Let x, y be in S_X and let x^* be in X^*. Then

$$\pi(x^*)(x) = x^*(x - e_0^*(x)x_0)$$

and

$$(\mathrm{Id}_{X^*} - \pi)(x^*)(y) = x^*(e_0^*(y)x_0).$$

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Taking into account [7, Fact 3.119],

$$\|x - e_0^*(x)x_0 + e_0^*(y)x_0\| = \max\{e^*(x - e_0^*(x)x_0 + e_0^*(y)x_0) : e^* \in E_{X^*}\}$$
$$= \max\{|\, e_0^*(y)\,|, e^*(x) : e^* \in E_{X^*} \setminus \{\pm e_0^*\}\} \le 1.$$

This yields

$$\pi(x^*)(x) + (\mathrm{Id}_{X^*} - \pi)(x^*)(y) = x^*(x - e_0^*(x)x_0 + e_0^*(y)x_0) \le \|x^*\|.$$

From here it can be easily deduced that

$$\|\pi(x^*)\| + \|(\mathrm{Id}_{X^*} - \pi)(x^*)\| \le \|x^*\|.$$

This finishes the proof of this implication.

To conclude, we show that (ii) implies (i). The hypothesis yields a w^*-closed subspace J of X^* such that $E_{X^*} \setminus \{\pm e_0^*\} = J \cap E_{X^*}$. Thus, $\overline{\mathrm{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*} \subseteq J$ and e_0^* does not belong to J. □

Despite its technical flavour, the following statement will play a key role in the proof of the main result of the paper.

Proposition 3.4. *Let X be a nice Banach space and let G be a structurally open subset of E_{X^*} such that $E_{X^*} \subseteq \overline{G}^{w^*}$. Then $G = E_{X^*}$.*

Proof. We argue by contradiction. Let us suppose that $G \neq E_{X^*}$. Then $E_{X^*} \setminus G$ is a nonempty closed set in the structure topology. Therefore, there exist π, an L-projection in X^*, and M, a closed subspace of X, such that $\pi(X^*) = M^\perp$ and $E_{X^*} \setminus G = M^\perp \cap E_{X^*}$. Let us consider the operator $T : M \to X$ defined by $T(x) = x$ for all x in M. Since $T^*(x^*) = 0$ for all x^* in $E_{X^*} \setminus G$, it follows that T is not nice. By [9, Lemma I.1.5], $G = E_{\ker(\pi)}$ and, for all x^* in $\ker(\pi)$,

$$\|x^*_{|_M}\| = \|x^* + M^\perp\| = \|x^* - \pi(x^*)\| = \|x^*\|.$$

We deduce that the map $T^*_{|_{\ker(\pi)}}$ is a linear isometric bijection from $\ker(\pi)$ onto M^* and, as a consequence, $T^*(x^*)$ belongs to E_{M^*} for all x^* in G. Next we prove that T is an extreme operator. Let S be in $\mathcal{L}(M, X)$ such that $\|T \pm S\| \le 1$. Let x^* be in G. Then $\|T^*(x^*) \pm S^*(x^*)\| \le \|T^* \pm S^*\| \le 1$. Since $T^*(x^*)$ belongs to E_{M^*}, we conclude that $S^*(x^*) = 0$. Taking into account that $E_{X^*} \subseteq \overline{G}^{w^*}$, the Krein–Milman theorem allows us to conclude that $S = 0$. We have proved that T is an extreme nonnice operator and this is a contradiction. □

4. The results

We can now state the main result in this paper.

Theorem 4.1. *Let X be an almost CL-space. Then X is nice if and only if X is isometrically isomorphic to $c_0(I)$ for some nonempty set I.*

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PROOF. In view of [3, Proposition 2.1], we only need to prove the 'only if' part. Let e_0^* be in $mexB_{X^*}$. By [14, Lemma 3] and [11, Theorem 3.1], $\mathbb{R}e_0^*$ is a semi-L-summand in X^*. Let π be the (only) semi-L-projection in X^* such that $\pi(X^*) = \mathbb{R}e_0^*$. We are going to prove that $\pi(e^*) = 0$ for all e^* in $E_{X^*} \setminus \{\pm e_0^*\}$. On the contrary, let e^* be in $E_{X^*} \setminus \{\pm e_0^*\}$ such that $\pi(e^*) \neq 0$. Since $e^* \neq \pm e_0^*$, we can write

$$e^* = \|\pi(e^*)\| \frac{\pi(e^*)}{\|\pi(e^*)\|} + \|e^* - \pi(e^*)\| \frac{e^* - \pi(e^*)}{\|e^* - \pi(e^*)\|}$$

with $\|\pi(e^*)\| + \|e^* - \pi(e^*)\| = 1$, which is a contradiction. We obtain

$$\|e^* + \mathbb{R}e_0^*\| = \|e^* - \pi(e^*)\| = \|e^*\| = 1$$

for all e^* in $E_{X^*} \setminus \{\pm e_0^*\}$. By [5, Theorem 1 and Proposition 1], $X^* \neq \overline{\operatorname{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$, and Proposition 3.3 shows that $\{\pm e_0^*\}$ is structurally open. From this, we see that $mexB_{X^*}$ is structurally open. Since X is an almost CL-space, $E_{X^*} \subseteq \overline{mexB_{X^*}}^{w^*}$. Proposition 3.4 allows us to conclude that $E_{X^*} = mexB_{X^*}$. Hence, $\{\pm e^*\}$ is structurally open for every e^* in E_{X^*} and [5, Proposition 2] ends the proof. □

The following result improves [3, Corollary 2.6], where it is assumed that the measure space involved is σ-finite.

COROLLARY 4.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that $X = L_1(\Omega, \mathcal{A}, \mu)$ is a nice Banach space. Then $X = \mathbb{R}$ or $X = l_\infty^2$.*

PROOF. As we have said before, X is an CL-space. By the above theorem, X is isometrically isomorphic to $c_0(I)$ for some nonempty set I. Now the result is a consequence of [9, Theorem I.1.9]. □

By using the fact that L_1-preduals are CL-spaces, we obtain the following corollary.

COROLLARY 4.3. *Let X be a nice Banach space such that X^* is isometrically isomorphic to $L_1(\Omega, \mathcal{A}, \mu)$ for some measure space $(\Omega, \mathcal{A}, \mu)$. Then X is isometrically isomorphic to $c_0(I)$ for some nonempty set I.*

The above Corollary improves [5, Theorem 3]. Bearing in mind that G-spaces and simplex spaces are L_1-preduals, this result includes [5, Theorem 2] and [4, Theorem 2.4] as particular cases.

We will now characterise nice spaces in a class of Banach spaces which includes Banach spaces with the Radon–Nikodỳm property (RNP for short; see [6] for information about RNP). The relationship between Banach spaces having RNP and almost CL-spaces was established in [13, Theorem 1].

THEOREM 4.4. *Let X be a nice Banach space such that $B_X = \overline{\text{co}}(E_X)$. Then X is isometrically isomorphic to l_∞^n for some $n \in \mathbb{N}$.*

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PROOF. Fix e_0^* in E_{X^*} and let e^* be in $E_{X^*} \setminus \{\pm e_0^*\}$. Then there exist x, y in E_X such that $e^*(x) \neq e_0^*(x)$ and $e^*(y) \neq -e_0^*(y)$. By [3, Proposition 2.8], $|e^*(x)| = |e_0^*(x)| = 1$ and $|e^*(y)| = |e_0^*(y)| = 1$. We can suppose that $e^*(x) = e^*(y) = 1$. Hence, $e_0^*(x) = -1$ and $e_0^*(y) = 1$. Therefore, $\frac{1}{2}(x+y)$ is an element in B_X which satisfies $e^*(\frac{1}{2}(x+y)) = 1$ and $e_0^*(\frac{1}{2}(x+y)) = 0$. From [3, Theorem 2.2], $X^* \neq \overline{\text{lin}(E_{X^*} \setminus \{\pm e_0^*\})}^{w^*}$. By Proposition 3.3, $\{\pm e_0^*\}$ is open in the structure topology. Once we have proved that the structure topology is 'discrete', we derive from [5, Proposition 2] that X is isometrically isomorphic to $c_0(I)$ for some nonempty set I. To finish the proof, it only remains to take into account that $E_{c_0(I)}$ is nonempty if and only if I is finite. □

It is well known that Banach spaces having RNP satisfy the Krein–Milman property. Whether the Krein–Milman property implies RNP is a long-standing open problem in the theory of Banach spaces.

COROLLARY 4.5. *Let X be a Banach space having RNP. Then X is nice if and only if $X = l_\infty^n$ for some $n \in \mathbb{N}$.*

Infinite-dimensional reflexive Banach spaces cannot be nice (see comments below [3, Proposition 2.8]). Finite-dimensional nice spaces were described in [3, Theorem 2.12]. These results are now obtained as a consequence of the fact that reflexive Banach spaces have RNP.

COROLLARY 4.6. *Let X be a reflexive Banach space. Then X is nice if and only if $X = l_\infty^n$ for some $n \in \mathbb{N}$.*

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