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ON BASIS-CONJUGATING AUTOMORPHISMS OF FREE GROUPS

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1. Introduction. Let $X = \{x_1, \dots, x_n\}$ be a free generating set of the free group F_n, and let H be the subgroup of Aut F_n consisting of those automorphisms α such that $\alpha(x_i)$ is conjugate to x_i for each $i = 1, 2, \dots, n$. We call H the X-conjugating subgroup of Aut F_n. In [**1**] Humphries found a generating set for the isomorphic copy H_1 of H consisting of Nielsen transformations

$$\{u_1, \dots, u_n\} \rightarrow \{u_1', \dots, u_n'\},$$

where each u_i' is conjugate to u_i (see remark 1 below). The purpose of this paper is to find a presentation of H (and hence of H_1).

Let $i \neq j$ be elements of $\{1, 2, \dots, n\}$. We denote by $(x_i; x_j)$ the automorphism of F_n which sends x_i to $x_j^{-1}x_ix_j$ and fixes x_k if $k \neq i$. Let S be the set of all such automorphisms. It is easy to check that the following are relations satisfied by the elements of S, provided that, in each case, the subscripts $i, j, k, \dots$ occurring are distinct:

(Z1) $(x_i; x_j)(x_k; x_j) = (x_k; x_j)(x_i; x_j)$

(Z2) $(x_i; x_j)(x_k; x_l) = (x_k; x_l)(x_i; x_j)$

(Z3) $(x_i; x_j)(x_k; x_j)(x_i; x_k) = (x_i; x_k)(x_i; x_j)(x_k; x_j)$.

We denote by Z the set of all relations of the above forms. Our result is

THEOREM. *The group H has presentation $\langle S; Z\rangle$.*

2. Proof of the theorem. We shall assume familiarity with the notation and results of [**5**] (see also [**2**]). We shall use the improved version of Theorem 1 of [**5**], as outlined in Section 4 of that paper. We take U to be the tuple $\{x_1^0, \dots, x_n^0\}$, where x_i^0 denotes the cyclic word (i.e., conjugacy class) determined by x_i. It is clear that U is a minimal tuple. We have to construct the complex K_2 described in Section 4 of [**5**]. The vertices of K_2 will clearly be the $n!2^n$ distinct tuples $U\sigma$, where σ belongs to the extended symmetric group Ω_n. There will be a (directed) edge labelled $(U\sigma, U\sigma\tau; \tau)$ joining $U\sigma$ to $U\sigma\tau$, for each pair $\sigma, \tau \in \Omega_n$. In addition, there will be a number of 'type 2 whitehead edges'. In fact, it is not difficult to see that

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the type 2 edges originating at the vertex V will correspond to the set W of Whitehead automorphisms of the form

$$(x_{i_1}, x_{i_i}^{-1}, \ldots, x_{i_r}, x_{i_r}^{-1}, x_{i_k}^{\epsilon}; x_{i_k}^{\epsilon}),$$

where $\{i_1, \ldots, i_r, k\} \subseteq \{1, 2, \ldots, n\}$, $i_1 \neq i_2 \neq \ldots \neq i_r \neq k$, and $\epsilon = \pm 1$.

We denote this automorphism by $(x_{i_1}, \ldots, x_{i_r}; x_k^{\epsilon})$ and recall that $x_j(x_{i_1}, \ldots, x_{i_r}; x_k^{\epsilon})$ is $x_k^{-\epsilon}x_jx_k^{\epsilon}$ if $j \in \{i_1, \ldots, i_r\}$ and is x_j otherwise. We have $VP = V$ for all vertices V and all $P \in W$. Thus the type 2 edges originating at vertex V consist of the loops $(V, V; P)$, where $P \in W$.

Summarising, the 1-skeleton K_2' of K_2 consists of

(a) the vertices $U\sigma$, $(\sigma \in \Omega_n)$

(b1) the edges $(U\sigma, U\sigma\tau; \tau)$, $(\sigma, \tau \in \Omega_n)$

(b2) the loops $(U\sigma, U\sigma; P)$, $(\sigma \in \Omega_n, P \in W)$.

Following the recipe of Section 4 of [**5**] we must now attach 2-cells to K_2' corresponding to the relations R1-R10 listed in [**4**]. Let C be the complex obtained by adding 2-cells corresponding to the relations R7 to K_2'. Note that if C' is obtained from C by deleting the edges of type (b2) above, then C' is just the Cayley diagram of Ω_n (on the generating set consisting of all elements of Ω_n), with the 2-cells corresponding to R7 added, and hence $\pi_1(C', U)$ is the trivial group.

We now examine the relations R1-R5 and R8-R10 to see which of these give rise to 2-cells of K_2. A straightforward examination shows that we use the following relations:

From R1:

(Q1) $\quad (x_{i_1}, \ldots, x_{i_r}; x_k^{\epsilon})^{-1} = (x_{i_1}, \ldots, x_{i_r}; x_k^{-\epsilon}).$

From R2:

(Q2) $\quad (x_{i_1}, \ldots, x_{i_r}; x_k^{\epsilon})(x_{j_i}, \ldots, x_{j_s}; x_k^{\epsilon}) = (x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_s}; x_k^{\epsilon})$

if $\{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_s\} = \emptyset$.

From R3:

(Q3) $\quad (x_{i_1}, \ldots, x_{i_r}; x_k^{\epsilon})(x_{j_i}, \ldots, x_{j_s}; x_l^{\eta})$

$$= (x_{j_1}, \ldots, x_{j_s}; x_l^{\eta})(x_{i_1}, \ldots, x_{i_r}; x_k^{\epsilon})$$

if $\{i_1, \ldots, i_r, k\} \cap \{j_1, \ldots, j_s, l\} = \emptyset$.

From R4: no relations arise, since if $(A; a)$, $(B; b) \in W$, where $A \cap B = \emptyset$, $a^{-1} \notin B$ and $b^{-1} \in A$, then $a \neq b^{-1}$ (otherwise $a^{-1} = b \in B$), so $b^{-1} \in A - a$. However $A - a$ is closed under inversion, so that $b \in A - a$, which contradicts $A \cap B = \emptyset$).

From R5: no relations arise, since no type 2 of the form $(A - a + a^{-1}; b)$ with $a \neq b, \bar{b}$ can be in W.

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From R8: we obtain only relations which are consequences of (Q1) and (Q2) above:

From R9:

(Q4) $$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; x_i^{\eta})(x_{j_1}, \ldots, x_{jr}; x_k^{\epsilon}) \times (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; x_i^{-\eta}) = (x_{j_i}, \ldots, x_{j_r}; x_k^{\epsilon})$$

if $i \notin \{j_i, \ldots, j_r, k\}$.

From R10: no relations arise (for the same reason as R4).

We now take each instance of a relation in the set

$$Q = Q1 \cup Q2 \cup Q3 \cup Q4,$$

rewrite it in the form

$$P_1 P_2 \ldots P_s = 1 \quad (P_i \in W),$$

and for each vertex $U\sigma$ of C, attach a 2-cell with boundary

(2.1) $$(U\sigma, U\sigma; P_1)(U\sigma, U\sigma; P_2) \ldots (U\sigma, U\sigma; P_s).$$

Let us denote the resulting complex by K_2'.

To obtain K_2 we add to K_2' 2-cells corresponding to the relations R6. The relations arising here are all of the form

$$\tau^{-1}(A; a)\tau = (A\tau; a\tau),$$

where $(A; a) \in W$ and $\tau \in \Omega_n$. Thus we attach to K_2' all 2-cells with boundaries

(2.2) $$(U\sigma, U\sigma\tau^{-1}; \tau^{-1})(U\sigma\tau^{-1}, U\sigma\tau^{-1}; (A; a)) \times (U\sigma\tau^{-1}, U\sigma; \tau)(U\sigma, U\sigma; (A\tau; a\tau)^{-1}),$$

where $(A; a) \in W$. The resulting complex is K_2.

Let T be a maximal tree in K_2. We compute a presentation of $H = \pi_1(K_2, U)$ using T. Note that T consists of edges of the form $(U\sigma, U\sigma\tau; \tau)$ for $\sigma, \tau \in \Omega_n$. We have a generator $(U\alpha, U\alpha\beta; \beta)$ for each edge of K_2. Since C' is simply connected, we will have $(U\sigma; U\sigma\tau; \tau) = 1$ in H for $\sigma, \tau \in \Omega_n$. Taking $\sigma = \tau$ in (2.2) (and using the fact that

$$(U\alpha, U\alpha\beta; \beta)^{-1} = (U\alpha\beta, U\beta; \beta^{-1}) \text{ in } H \text{ for any } (U\alpha, U\alpha\beta; \beta))$$

we obtain

(2.3) $$(U, U; (A; a)) = (U\sigma, U\sigma; (A\sigma; a\sigma))$$

in H, for all $\sigma \in \Omega_n$ and $(A; a) \in W$. It follows that H is generated by the elements $(U, U; P)$ for $P \in W$. This tells us that H is generated by the set W, and that the 2-cells (2.1), interpreted as relations on W, merely give us back the relations Q1-Q4. Any other relation arising in our presentation will be an instance of a relation arising from the 2-cells (2.1). Using (2.3) (and interpreting $(U, U; (A; a))$ as being $(A; a)$), this will yield a relation on the generating set W. However, it is clear that the relation obtained will actually be an instance of one of Q1-Q4 (essentially because in Aut F_n we have $\sigma W \sigma^{-1} = W$ for all $\sigma \in \Omega_n$, and the set $Q = Q1 \cup Q2 \cup Q3 \cup Q4$ is closed under conjugation by $\sigma \in \Omega_n$).

Thus we have established that H has presentation $\langle W; Q\rangle$. To obtain the presentation of the theorem, we proceed as follows:

From Q2 we have

$$(2.4) \quad (x_{i_1}, \dots, x_{i_r}; x_k) = (x_{i_1}; x_k)(x_{i_2}; x_k) \dots (x_{i_r}; x_k),$$

and it follows from Q1 that H is generated by the $(x_i; x_j)$. Starting with the presentation $\langle S; Z\rangle$, it is clear that if we define $(x_{i_1}, \dots, x_{i_r}; x_k)$ by (2.4) and define $(x_{i_1}, \dots, x_{i_r}; x_k^{-1})$ to be $(x_i, \dots, x_{i_r}; x_k)^{-1}$, we can then recover the relations Q2 and Q3. It only remains then to show, with these definitions, that Q4 holds. We may clearly assume in Q4 that $\epsilon = \eta = 1$. We thus consider

$$L = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; x_i)(x_{j_1}, \dots, x_{j_r}; x_k) \\ \times (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; x_i)^{-1}.$$

Using Z2, we have in $\langle S, Z\rangle$,

$$\begin{aligned} L &= (x_{j_1}, \dots, x_{j_r}, x_k; x_i)(x_{j_i}, \dots, x_{j_r}; x_k)(x_{j_1}, \dots, x_{j_r}, x_k; x_i)^{-1} \\ &= (x_{j_2}, \dots, x_{j_r}; x_i)(x_k; x_i)(x_{j_1}; x_i) \\ &\times (x_{j_1}; x_k)(x_{j_2}, \dots, x_{j_r}; x_k)(x_{j_1}, \dots, x_{j_r}, x_k; x_i)^{-1} \\ &= (x_{j_2}, \dots, x_{j_r}; x_i)(x_{j_1}; x_k)(x_k; x_i)(x_{j_1}; x_i) \\ &\times (x_{j_2}, \dots, x_{j_r}; x_k)(x_{j_1}, \dots, x_{j_r}, x_k; x_i)^{-1} \end{aligned}$$

(using Z3)

$$= (x_{j_i}; x_k)(x_{j_2}, \dots, x_{j_r}, x_k; x_i)(x_{j_2}, \dots, x_{j_r}; x_k) \\ \times (x_{j_2}, \dots, x_{j_r}, x_k; x_i)^{-1}$$

and it follows, by induction on r, that

$$L = (x_{j_1}, \dots, x_{j_r}; x_k)$$

in $\langle S, Z\rangle$, as required. Thus the theorem is established.

Remark 1. Let Γ be the set of all n-tuples of F_n, and let $\alpha \in \text{Aut } F_n$. The Nielsen transformation $\bar{\alpha}$ corresponding to α is defined to be the element of the symmetric group $S(\Gamma)$ on Γ given by

$$\bar{\alpha}\{u_1, \dots, u_n\} = \{w_1(u_1, \dots, u_n), \dots, w_n(u_1, \dots, u_n)\},$$

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where

$$(x_i)\alpha = w_i(x_1, \ldots, x_n) \ (1 \leqq i \leqq n).$$

It is then easy to check that $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$, and that we obtain an isomorphism of Aut F_n with a subgroup of $S(\Gamma)$ (note that we apply elements of $S(\Gamma)$ on the left, and elements of Aut F_n on the right, unlike the convention of **[3]** pp. 129-131, where an anti-isomorphism is obtained). This isomorphism then induces an isomorphism from H to the group H_1.

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