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LAURENT EXPANSION OF DIRICHLET SERIES

U. BALAKRISHNAN

Let $\langle a_n \rangle$ be an increasing sequence of real numbers and $\langle b_n \rangle$ a sequence of positive real numbers. We deal here with the Dirichlet series $f(s) = \sum b_n a_n^{-s}$ and its Laurent expansion at the abscissa of convergence, λ, say. When a_n and b_n behave like

$$\sum_{a_n \leqslant N} b_n a_n^{-\lambda} \log^k a_n = P_2(\log N) + C_k + O(N^{-\varepsilon} \log^k N) ,$$

as $N \to \infty$, where $P_2(x)$ is a certain polynomial, we obtain the Laurent expansion of $f(s)$ at $s = \lambda$, namely

$$f(s) = P_1(s-\lambda) + \sum_{k=0}^{\infty} k!^{-1} C_k (\lambda-s)^k ,$$

where $P_1(x)$ is a polynomial connected with $P_2(x)$ above. Also, the connection between P_1 and P_2 is made intuitively transparent in the proof.

Suppose the Dirichlet series $f(s) = \sum b_n a_n^{-s}$, $s = \sigma + it$, is convergent for $\sigma > \lambda$ (> 0) and has a pole of order $d \geqslant 0$ at $s = \lambda$ and also suppose that $f(s)$ has an analytic continuation to $\sigma > \sigma_0 (< \lambda)$. Then we know that $f(s)$ has the Laurent expansion

$$f(s) = P_1(s-\lambda) + \sum_{k=0}^{\infty} k!^{-1} c_k (\lambda-s)^k \tag{1}$$

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where

(1*) $$P_1(x) = (d-1)!\,\Delta_d\,x^{-d} + (d-2)!\,\Delta_{d-1}\,x^{1-d} + \dots + \Delta_1\,x^{-1},$$

at $s = \lambda$, with some constants c_k and Δ_i. It is conjectured that the c_k's are given by

$$c_k = \lim_{N\to\infty}\Big(\sum_{a_n \leqslant N} b_n\,a_n^{-\lambda} \log^k a_n - P_2(\log N) \Big)$$

where

(2) $$P_2(x) = \Delta_d(k+d)^{-1}x^{k+d} + \Delta_{d-1}(k+d-1)^{-1}x^{k+d-1} + \dots + \Delta_1(k+1)^{-1}x^{k+1}.$$

In special cases of the function $f(s)$ this is known to be true (see [3],[4]). What we deal here is a conditional converse of this. We have the following Tauberian theorem:

THEOREM 1. *Let* $0 < a_1 \leqslant a_2 \leqslant a_3 \dots$ *be an increasing sequence of real numbers and* $0 \leqslant b_n$, $n = 1,2,3,\dots$ *be arbitrary positive real numbers satisfying*

$$\sum_{a_n \leqslant N} b_n a_n^{-\lambda} \log^k a_n = P_2(\log N) + c_k + O(N^{-\varepsilon} \log^k N),$$

for $k = 0,1,2,\dots,[\frac{1}{2}\varepsilon \log N]$, *for all* $N \geqslant N_0$ *with the* O-*constant absolute and with a fixed* ε, $0 < \varepsilon < \frac{1}{2}$, *where* $P_2(x)$ *is given by* (2). *Then the Dirichlet series* $f(s) = \sum b_n a_n^{-s}$ *is convergent for* $\sigma > \lambda$ *and has the Laurent expansion* (1) *at* $s = \lambda$, *with* $P_1(x)$ *as in* (1*), *provided that* $c_k \ll (2k/\varepsilon)^k$, *for* $k \geqslant [\frac{1}{2}\varepsilon \log N_0]$.

REMARKS. 1. It follows that the order of the pole of $f(s)$ at $s = \lambda$ is exactly the largest power of $\log N$ appearing in $\sum b_n a_n^{-\lambda}$.

2. The condition in Theorem 1 could be altered to

$$\sum_{a_n \leqslant N} b_n a_n^{-\lambda} \log^k a_n = P_2(\log M) + c_k + O(M^{-\varepsilon} \log^k M),$$

for $k = 0,1,2,\dots,[\frac{1}{2}\varepsilon \log M]$ for all $N \geqslant N_0$ where $M = M(N) \to \infty$ as $N \to \infty$ and we restrict a_n by $a_n \leqslant (M(n))^{100}$ for $n \geqslant N_0$.

3. The proof of the Theorem reveals explicitly how the powers of $\log N$ in $\sum b_n a_n^{-\lambda}$ are transformed to powers of $(s-\lambda)^{-1}$ in the Laurent expansion of $f(s)$.

4. We have given in Theorem 2 below a class of sequences satisfying the hypothesis of Theorem 1.

Proof of Theorem 1. We write a_n^{-s} in the form

$$a_n^{-s} = a_n^{-\lambda}\,(1 + \eta \log a_n + 2!^{-1}\eta^2 \log^2 a_n + \ldots + t!^{-1}\eta^t \log^t a_n)$$
$$+ O(t!^{-1}\,|\eta|^t\, a_n^{|\eta|-\lambda} \log^t a_n)\ ,$$

where we have denoted $\lambda - s$ by η and have used the fact that for $x \in \mathbb{C}$

$$e^x = 1 + x + 2!^{-1}\,x^2 + \ldots + t!^{-1}\,x^t + O(t!^{-1}\,|x|^t\,e^{|x|})\ .$$

Now we consider a_1^{-s}, a_2^{-s}, ..., a_N^{-s} for the above expansion and by columnwise addition we get

$$\sum_{a_n\leqslant N} b_n a_n^{-s} = \sum b_n a_n^{-\lambda} + \eta\sum b_n a_n^{-\lambda}\log a_n + \ldots + t!^{-1}\,\eta^t \sum b_n a_n^{-\lambda}\log^t a_n$$
$$+ O(t!^{-1}\,|\eta|^t \sum b_n a_n^{-\lambda+|\eta|}\log^t a_n)\ ,$$

and using the hypothesis of the theorem we get for

(3) $$0 \neq |\eta| \leqslant 10^{-6}\,\varepsilon\,\min(1,\lambda) \quad\text{and}\quad t = [\tfrac{1}{4}\,\varepsilon\log N]$$

that

(4) $$\sum_{a_n\leqslant N} b_n a_n^{-s} = \sum_{k=0}^{t} k!^{-1}\,\eta^k \sum_{r=1}^{d} \Delta_r (k+r)^{-1}\,(\log N)^{k+r} + \sum_{k=0}^{t} k!^{-1}\,c_k\eta^k$$
$$+ O(N^{-\varepsilon}\sum_{k=0}^{\infty} k!^{-1}\,|\eta\log N|^k + t!^{-1}|\eta|^t \sum_{a_n\leqslant N} b_n a_n^{-\lambda+|\eta|}\log^t a_n)\ .$$

Now using the hypothesis of the theorem again, we get

(5) $$t!^{-1}|\eta|^t \sum_{a_n\leq N} b_n a_n^{-\lambda+|\eta|}\log^t a_n \ll N^{|\eta|}\log^d N\, t!^{-1}\,|\eta\log N|^t$$
$$\ll N^{|\eta|-\varepsilon}\log^d N\ ,$$

using the choice of t from (3). Now let

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$$(6)\qquad Q = \sum_{k=0}^{t} k!^{-1} \eta^k \sum_{r=1}^{d} \Delta_r (k+r)^{-1} (\log N)^{k+r}$$

$$= \sum_{r=1}^{d} \Delta_r \eta^{-r} \sum_{k=0}^{t} (k+r)!^{-1} (\eta \log N)^{k+r} (k+1)(k+2)\ldots(k+r-1) .$$

We write for a fixed r

$$(k+1)\ldots(k+r-1) \equiv A_1 + A_2(k+r) + A_3(k+r)(k+r-1) + \ldots + A_r(k+r)(k+r-1)\ldots(k+2),$$

as an identity in k. It is easy to check that, for $1 \leqslant i \leqslant r-1$,

$$(7)\quad i!^{-1}A_1 + (i-1)!^{-1}A_2 + \ldots + A_{i+1} = 0;\ A_1 = (-1)^{r-1}(r-1)!\ ;\ A_m = A_m(r).$$

Let us use this expansion in (6) and get

$$(8)\ Q = \sum_{r=1}^{d} \Delta_r \eta^{-r} \sum_{i=1}^{r} \sum_{k=0}^{t} A_i (k+r-i+1)!^{-1} (\eta \log N)^{k+r}$$

$$= \sum_{r=1}^{d} \Delta_r \eta^{-r} \sum_{i=1}^{r} A_i (\eta \log N)^{i-1} \sum_{k=0}^{t} (k+r-i+1)!^{-1} (\eta \log N)^{k+r-i+1}$$

$$= \sum_{r=1}^{d} \Delta_r \eta^{-r} \sum_{i=1}^{r} A_i (\eta \log N)^{i-1} \Big\{ N^\eta - (1 + \eta \log N + 2!^{-1}(\eta \log N)^2 +$$

$$+ \ldots + (r-i)!^{-1}(\eta \log N)^{r-i}) + O(t!^{-1} N^{|\eta|} |\eta \log N|^{t+d}) \Big\}$$

$$= \sum_{r=1}^{d} \Delta_r \eta^{-r} \sum_{i=1}^{r} A_i (\eta \log N)^{i-1} (N^\eta + O\ (N^{|\eta|-\varepsilon} \log^d N)$$

$$- \sum_{r=1}^{d} \Delta_r \eta^{-r} \Big\{ A_1 + (A_1+A_2)\eta \log N + \ldots + (\eta \log N)^{r-1} \sum_{i=1}^{r} (r-i)!^{-1} A_i \Big\} .$$

Using (7), we have all of $A_1 + A_2$, $2!^{-1} A_1 + A_2 + A_3, \ldots, \sum_{i=1}^{r} (r-i)!^{-1} A_i$ are zero. Also by the choice of η and t as in (3) and reading A_1 from (7) we get from (8) that

$$(9)\qquad Q = - \sum_{r=1}^{d} (r-1)!\ (-1)^{r-1} \Delta_r \eta^{-r} + O(N^\eta \log^d N + N^{|\eta|-\varepsilon} \log^{2d} N) \quad .$$

It now follows from (5), (6) and (9) that

$$\sum_{a_n \leqslant N} b_n a_n^{-s} = \sum_{r=1}^{d} (r-1)!\ \Delta_r (-\eta)^{-r} + \sum_{k=0}^{t} t!^{-1} c_k \eta^k + O(N^\eta \log^d N + N^{-\frac{1}{2}\varepsilon}) .$$

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We are in $\sigma > \lambda$ and hence $\operatorname{Re} \eta < 0$ and the truth of the theorem follows, as we let $N \to \infty$.

Now we verify the hypotheses of theorem 1 for the case $f(s) = \zeta(s,a)$, the Hurwitz zeta function. We have

$$\zeta(s,a) = a^{-s} + (1+a)^{-s} + (2+a)^{-s} + \dots, \ \sigma > 1, \ 0 < a \leqslant 1 .$$

We consider the sum

$$(10) \quad \sum_{n=0}^{N} (n+a)^{-1} \log^k (n+a) = \sum_{n=1}^{\infty} \left\{ (n+a)^{-1} \log^k (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right\}$$
$$+ a^{-1} \log^k a + \int_1^N u^{-1} \log^k u \, du + O(N^{-1} \log^k N)$$
$$+ O\left(\sum_{n=N}^{\infty} \left| (n+a)^{-1} \log^k (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right| \right)$$

Now

$$(11) \quad \left| (n+a)^{-1} \log^k (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right| \leqslant 2(n^{-1} \log^k (n+1) - (n+1)^{-1} \log^k n)$$
$$\leqslant 2(n+1)^{-1} (\log^k (n+1) - \log^k n) + 2n^{-2} \log^k (n+1)$$
$$\leqslant 2n^{-2} \{ k(\log (n+1))^{k-1} + \log^k (n+1) \} ,$$

so the first sum on the right side of (10) is absolutely convergent and further, for $k \leqslant \frac{1}{2}\varepsilon \log N$,

$$\sum_{n=N}^{\infty} \left| (n+a)^{-1} \log^k (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right| \leqslant 6 \sum_{n=N}^{\infty} (n+1)^{-2} \log^k (n+1)$$
$$\leqslant 12 \, N^{-1} \log^k N .$$

Evaluating the integral we can write (10) as

$$\sum_{n=0}^{N} (n+a)^{-1} \log^k (n+a) = (k+1)^{-1} (\log N)^{k+1} + c_k(a) + O(N^{-1} \log^k N) ,$$

and so the hypotheses of the Theorem 1 are satisfied, with $\Delta_1 = 1$, $d = 1$ and even $\varepsilon = 1$. Also observe using (11) that

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$$c_k(a) = \sum_{n=1}^{\infty} \left\{ (n+a)^{-1} \log^k (n+a) - \int_n^{n+1} u^{-1} \log^k u \, du \right\} + a^{-1} \log^k a$$

$$\ll k! \, .$$

Now Theorem 1 gives us the Laurent expansion of $\zeta(s,a)$ as

$$\zeta(s,a) = (s-1)^{-1} + b_0(a) + (1-s)\, b_1(a) + (1-s)^2\, b_2(a) + \ldots$$

with

$$b_k(a) = k!^{-1} \lim_{N\to\infty} \left\{ \sum_{n=0}^{N} (n+a)^{-1} \log^k (n+a) - (k+1)^{-1} (\log N)^{k+1} \right\} .$$

Of course, for $a = 1$ we get the Laurent expansion of $\zeta(s)$ at $s = 1$. This expression for $b_k(a)$ is already present in [1] and [2]. We can easily see from the above estimates that $b_k(a) = k!^{-1} c_k(a) \ll 1$, which implies the validity of the Laurent expansion of $\zeta(s,a)$ in $|1-s| < 1$. A better estimation of $b_k(a)$ is given in [1].

Below we include a theorem, without proof, which gives a good degree of freedom in choosing a sequence a_n satisfying the hypotheses of Theorem 1. We restrict ourselves to the special case $d = 1 = \lambda$ and $b_n = 1$ for all n. If S_N denote the number of a_n's in the sequence with $a_n \leqslant N$ we would expect S_N to behave as $S_N = \Delta_1 N + O(N^{1-\varepsilon})$. This is in fact true provided we choose the $\Delta_1 N$ numbers as described below.

THEOREM 2. *Let integer* $G \geqslant 1$, *positive real numbers* A, B, T *and* $0 < \varepsilon \leqslant \frac{1}{2}$ *be fixed. Suppose for each* $n \geqslant n_0$, *we choose* G *real numbers from the interval* $[Tn - An^{1-\varepsilon}, Tn + Bn^{1+\varepsilon}]$ *(the same real number may be chosen for different* n*'s, provided that we pick them from the prescribed interval) and insert into the sequence thus formed any number of positive real numbers subject to the condition that* $S_N = T^{-1} G N + O(N^{1-\varepsilon})$, *for* $N \geqslant N_0$. *Then the sequence thus constructed satisfies the hypotheses of Theorem 1 with* $\Delta_1 = T^{-1} G$ *and* $\lambda = 1 = d$.

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School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road,
Bombay - 5,
India.